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Muckenhoupt A_p properties of distance functions and applications

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1. Muckenhoupt A_p weights

A_p weights

Let $X = (X, d, \mu)$ be a metric space. (Can think of $\mathbb{R}^n$ with Euclidean distance and Lebesgue measure.)

Function $w \in L^1_{\mathrm{loc}}(X)$ is a **weight** if $w(x) > 0$ for a.e. $x \in X$.

A weight w belongs to the **Muckenhoupt class** A_p, $1 < p < \infty$, if there is $C > 0$ such that

$$\left(\fint_B w(x)\, d\mu \right) \left(\fint_B w(x)^{-\frac{1}{p-1}}\, d\mu \right)^{p-1} \leq C$$

for all balls $B \subset X$. Weight w is in class A_1 if there is $C > 0$ such that

$$\left(\fint_B w(x)\, d\mu \right) \operatorname*{ess\,sup}_{x \in B} \frac{1}{w(x)} \leq C$$

for all balls $B \subset X$.

(Here $\fint_B f(x)\, d\mu = \frac{1}{\mu(B)} \int_B f(x)\, d\mu$ is the mean-value integral.)

Properties of A_p weights

Consequences of the A_p condition (well known):

• $A_1 \subset A_p \subset A_q \subset \widetilde{A}_\infty := \bigcup_{1 \le p < \infty} A_p$ when $1 < p < q < \infty$.

• Duality: If $1 < p < \infty$, then $w \in A_p$ if and only if $w^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}}$.

• In $\mathbb{R}^n$, Hardy–Littlewood maximal operator M is bounded on $L^p(w\,dx)$, $1 < p < \infty$, if and only if $w \in A_p$ [Muckenhoupt, 1972]. This implies a rich theory of harmonic analysis in A_p-weighted spaces.

• A_p weights satisfy a reverse Hölder inequality, and hence the A_p condition is self-improving.

• A_p-weights (in $\mathbb{R}^n$) are p-admissible, that is, they satisfy the basic assumptions of analysis on metric spaces: the doubling property and a p-Poincaré inequality.

A_p properties of distance functions

Concrete examples of A_p weights? The following is well known:

Let $\alpha \in \mathbb{R}$ and write $w(x) = |x|^{-\alpha}$ for $x \in \mathbb{R}^n$. Then

(a) $w \in A_p$, $1 < p < \infty$, if and only if $(1-p)n < \alpha < n$.
(b) $w \in A_1$ if and only if $0 \leq \alpha < n$.

Here $w(x) = |x|^{-\alpha} = \operatorname{dist}(x, \{0\})^{-\alpha}$.

More generally, we are interested in the A_p properties of weights

$$w(x) = \operatorname{dist}(x, E)^{-\alpha} \quad \text{for (closed) } E \subset X,\ \alpha \in \mathbb{R}.$$

These have been studied e.g. in [Aikawa, 1991], [Horiuchi, 1989, 1991], [Durán–López García, 2010] and [Aimar–Carena–Durán–Toschi, 2014].

A characterization in $\mathbb{R}^n$

The following is the Euclidean special case of the characterization from

Bartłomiej Dyda, Lizaveta Ihnatsyeva, Juha Lehrbäck, Heli Tuominen, Antti V. Vähäkangas: *Muckenhoupt A_p-properties of distance functions and applications to Hardy-Sobolev -type inequalities*, Potential Anal. 50 (2019), 83–105.

Theorem 1 (DILTV, 2019).

Assume that a closed set $E \subset \mathbb{R}^n$ is porous (equivalently $\dim_A(E) < n$). Let $\alpha \in \mathbb{R}$ and write $w(x) = \operatorname{dist}(x, E)^{-\alpha}$. Then

(a) $w \in A_p$, *for* $1 < p < \infty$, *if and only if*

$$(1 - p)(n - \dim_A(E)) < \alpha < n - \dim_A(E).$$

(b) $w \in A_1$ *if and only if* $0 \le \alpha < n - \dim_A(E)$.

Here $\dim_A(E)$ is the **Assouad dimension** of $E \subset \mathbb{R}^n$.

2. Assouad (co)dimension

Assouad dimension

Definition 2.

Let $E \subset X$. The **Assouad dimension** $\dim_{\mathrm{A}}(E)$ is the infimum of all $\lambda \geq 0$ for which there is $C > 0$ such that $E \cap B(x, R)$ can be covered by **at most** $C(\frac{r}{R})^{-\lambda}$ balls of radius r for all $x \in E$ and all $0 < r < R$ ($< \operatorname{diam}(E)$).

[Sometimes this is called the upper Assouad dimension $\overline{\dim}_{\mathrm{A}}(E)$. The natural dual ("how many balls are needed **at least**") can then be called the lower (Assouad) dimension $\underline{\dim}_{\mathrm{A}}(E)$.]

Assouad dimension was introduced by P. Assouad around 1980 in connection to bi-Lipschitz embedding problem between metric and Euclidean spaces; see e.g. [Assouad, 1983]. Equivalent (or closely related) concepts have appeared under different names, e.g. (uniform) metric dimension.

Assouad and Minkowski

So, $\dim_{\mathrm{A}}(E) = \overline{\dim}_{\mathrm{A}}(E)$ is the infimum of $\lambda \geq 0$ such that $E \cap B(x,R)$ can always be covered by at most $C(\frac{r}{R})^{-\lambda}$ balls of radius $0 < r < R < d(E)$.

For comparison, the **upper Minkowski dimension** (or box dimension) $\overline{\dim}_{\mathrm{M}}(E)$ of a bounded set $E \subset X$ is the infimum of $\lambda \geq 0$ such that E can be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < d(E)$.

Thus

$$\dim_{\mathrm{H}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \dim_{\mathrm{A}}(E) \quad \text{for all } E \subset X,$$

where $\dim_{\mathrm{H}}(E)$ is the Hausdorff dimension. All these inequalities can be strict.

For instance, let $E = \left\{\frac{1}{j} : j \in \mathbb{N}\right\} \cup \{0\} \subset \mathbb{R}$.

Then $\dim_{\mathrm{H}}(E) = 0$, $\overline{\dim}_{\mathrm{M}}(E) = \frac{1}{2}$, and $\dim_{\mathrm{A}}(E) = 1$.

Assouad codimension

However, when examining A_p classes in the space (X, d, μ), we also need to take into account the effect of the measure μ. Then the following is in many instances a more suitable concept.

Definition 3.

Let $E \subset X$. The Assouad codimension $\operatorname{co}\dim_{\mathrm{A}}(E)$ is the supremum of all $\rho \geq 0$ for which there is $C \geq 1$ such that

$$\frac{\mu(E_r \cap B(x, R))}{\mu(B(x, R))} \leq C\left(\frac{r}{R}\right)^{\rho}$$

for all $x \in E$ and all $0 < r < R < 2\operatorname{diam}(X)$.

Here $E_r = \{x \in X : \operatorname{dist}(x, E) < r\}$ is the open r-neighborhood of $E \subset X$.

Note that if $\operatorname{co}\dim_{\mathrm{A}}(E) > 0$, then $\mu(E) = 0$ by the Lebesgue density theorem.

The Ahlfors regular case

The space $X = (X, d, \mu)$ is **(Ahlfors)** ***Q*-regular**, for $Q > 0$, if there is $C \geq 1$ such that

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q$$

for all $x \in X$ and all $0 < r < \operatorname{diam}(X)$. This can be equivalently required to hold for $\mu = \mathcal{H}^Q$, the Q-dimensional Hausdorff measure.

If X is Q-regular, then it is not hard to see that

$$\dim_{\mathrm{A}}(E) = Q - \operatorname{co}\dim_{\mathrm{A}}(E) \quad \text{for all } E \subset X.$$

On the other hand, if $E \subset X$ is Ahlfors λ-regular, then

$$\dim_{\mathrm{H}}(E) = \dim_{\mathrm{A}}(E) = \lambda.$$

Doubling

If the space X is not Ahlfors regular, we still need to assume the weaker condition that μ is **doubling**: there is a $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\,\mu(B(x, r)) \quad \text{for all } x \in X, r > 0. \tag{1}$$

Iteration of (1) shows that there are $\sigma > 0$ and $C > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\Big(\frac{r}{R}\Big)^{\sigma} \quad \text{whenever } B(y,r) \subset B(x,R) \subset X.$$

In some of our applications we also need the **reverse doubling condition** that there are $\eta > 0$ and $C > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq C\Big(\frac{r}{R}\Big)^{\eta} \quad \text{whenever } B(y,r) \subset B(x,R) \subset X. \tag{2}$$

Note that (2) implies $\mu(\{x\}) = 0$ for all $x \in X$.
If X is unbounded and connected and μ is doubling, then there is some $\eta > 0$ such that (2) holds.

Aikawa condition

The following property provides a link between the A_p condition and the Assouad dimension.

Definition 4.

A closed set $E \subset X$ satisfies the **Aikawa condition** for $\alpha \in \mathbb{R}$, if there is $C \geq 1$ such that

$$\fint_{B(x,r)} \operatorname{dist}(y,E)^{-\alpha} \, d\mu(y) \leq C r^{-\alpha}$$

for all $x \in E$ and all $r > 0$.

Let $w(x) = \operatorname{dist}(x,E)^{-\alpha}$. It is easy to show (using doubling) that if the Aikawa condition holds with $\alpha \geq 0$, then $w \in A_1 \subset A_p$ for all $1 \leq p < \infty$.

On the other hand, by duality, if $\alpha < 0$ and $1 < p < \infty$ are such that the Aikawa condition holds with $\frac{-\alpha}{p-1}$, then $w \in A_p$.

Assouad and Aikawa

Connecting the Assouad dimension and the Aikawa condition, we have the following result, essentially from [L–Tuominen, 2013].

Theorem 5.

Let $E \subset X$ be a closed set and let $\alpha > 0$. Then the Aikawa condition holds with α if and only if $\operatorname{co\,dim}_{\mathrm{A}}(E) > \alpha$.

The proof is based on simple covering arguments, but to get the strict inequality also the **self-improvement** of the Aikawa condition is needed.

Self-improvement follows from the so-called Gehring Lemma, since the Aikawa condition for $\alpha > 0$ implies a reverse Hölder inequality.

(This is due to the fact that for any $\beta > 0$

$$r^{-\beta} \leq \fint_{B(x,r)} \operatorname{dist}(y, E)^{-\beta}\, d\mu(y) \qquad \text{for all } x \in E,\ r > 0.)$$

3. A_p distance weights in metric measure spaces

Sufficiency

Theorem 6 (DILTV, 2019).

Let $E \subset X$ be a closed (non-empty) set and let $\alpha \in \mathbb{R}$. Then the following statements hold for $w(x) = \operatorname{dist}(x, E)^{-\alpha}$.

(a) *If* $\operatorname{co\,dim}_A(E) > \alpha \geq 0$, *then* $w \in A_p$ *for all* $1 \leq p < \infty$.

(b) *If* $\alpha < 0$ *and* $1 < p < \infty$ *are such that* $\operatorname{co\,dim}_A(E) > \dfrac{\alpha}{1-p}$, *then* $w \in A_p$.

(c) *If* $\operatorname{co\,dim}_A(E) > \max\{0, \alpha\}$, *then* $w \in \widetilde{A}_\infty$.

(a) follows using Theorem 5 and the Aikawa condition.
(b) follows from (a) by the A_p duality.
(c) follows from (a) and (b).

Note: Here it is important that in the definition of $\operatorname{co\,dim}_A(E)$ all radii $0 < r < R < 2\operatorname{diam}(X)$ are considered.

Necessity for porous sets

For porous sets, we have also a converse to Theorem 6.

Recall that $E \subset X$ is **porous**, if there is $0 < c < 1$ such that for all $x \in E$ and all $0 < r < 2\operatorname{diam}(X)$ there is $y \in X$ satisfying $B(y, cr) \subset B(x, r) \setminus E$. (If X is Q-regular, then $E \subset X$ is porous if and only if $\dim_{\mathrm{A}}(E) < Q$.)

Theorem 7 (DILTV, 2019).

Let $E \subset X$ be a closed and porous (non-empty) set and let $\alpha \in \mathbb{R}$. Then the following statements hold for $w(x) = \operatorname{dist}(x, E)^{-\alpha}$.

(a) *If $\alpha > 0$ and $w \in A_p$, for some $1 \le p < \infty$, then $w \in A_1$ and $\operatorname{co}\dim_{\mathrm{A}}(E) > \alpha$.*

(b) *If $\alpha < 0$ and $w \in A_p$, for some $1 < p < \infty$, then $\operatorname{co}\dim_{\mathrm{A}}(E) > \dfrac{\alpha}{1-p}$.*

The general characterization

Combining Theorems 6 and 7 we obtain the following characterization for porous sets.

Corollary 8 (DILTV, 2019).

Let $E \subset X$ be a closed and porous (non-empty) set, let $\alpha \in \mathbb{R}$, and write $w(x) = \operatorname{dist}(x, E)^{-\alpha}$. Then

(a) *$w \in A_p$, for $1 < p < \infty$, if and only if*

$$(1-p) \operatorname{co\,dim}_A(E) < \alpha < \operatorname{co\,dim}_A(E).$$

(b) *$w \in A_1$ if and only if $0 \leq \alpha < \operatorname{co\,dim}_A(E)$.*

Theorem 1 follows from this, since $E \subset \mathbb{R}^n$ is porous if and only if $\dim_A(E) < n$, and $\operatorname{co\,dim}_A(E) = n - \dim_A(E)$ for all $E \subset \mathbb{R}^n$.

[For a non-porous sets $E \subset X$, both $\operatorname{dist}(x, E)^{-\alpha} \in A_1$ and $\operatorname{dist}(x, E)^{-\alpha} \notin \widetilde{A}_\infty$ are possible, when $\alpha > 0$.]

4. Applications for Hardy–Sobolev inequalities

Hardy–Sobolev inequalities

We say that a global (q, p, β)-**Hardy–Sobolev inequality** holds with respect to a closed set $E \subset \mathbb{R}^n$, with $|E| = 0$, if there is $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \operatorname{dist}(x, E)^{\frac{q}{p}(n-p+\beta)-n} \, dx \right)^{\frac{p}{q}} \leq C \int_{\mathbb{R}^n} |\nabla u(x)|^p \operatorname{dist}(x, E)^{\beta} \, dx$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Here $\beta \in \mathbb{R}$, and the natural range of exponents is $1 \leq p \leq q \leq \frac{np}{n-p} = p^*$.

For $\beta = 0$, these inequalities form a natural interpolating scale between the Sobolev (case $q = p^* = \frac{np}{n-p}$) and Hardy inequalities (case $q = p$).

For $q = p$, the global (p, β)-Hardy inequality reads as

$$\int_{\mathbb{R}^n} |u(x)|^p \operatorname{dist}(x, E)^{\beta-p} \, dx \leq C \int_{\mathbb{R}^n} |\nabla u(x)|^p \operatorname{dist}(x, E)^{\beta} \, dx.$$

Hardy–Sobolev inequalities in metric spaces

In general metric spaces we consider compactly supported Lipschitz functions $u \in \mathrm{Lip}_0(X)$ instead of C_0^∞ functions. The norm of the gradient $|\nabla u|$ is replaced by an **upper gradient** g of u.

A Borel measurable function $g \geq 0$ is an upper gradient of u, if

$$|u(y) - u(x)| \leq \int_\gamma g\, ds$$

for all rectifiable curves γ joining x and y.

In addition to doubling, we assume for the rest of the talk that $\mu(\{x\}) = 0$ and $B(x, R) \setminus B(x, r) \neq \emptyset$ for all $x \in X$ and all $0 < r < R < \infty$.

Riesz potential

When $s > 0$, the Riesz potential $\mathcal{I}_s(f)$ of a measurable function $f \geq 0$ is defined by

$$\mathcal{I}_s(f)(x) = \int_X \frac{f(y)d(x,y)^s}{\mu(B(x,d(x,y)))}\,d\mu(y), \quad x \in X.$$

Since $\mu(\{x\}) = 0$ for each $x \in X$, we can restrict the integration to the set $X \setminus \{x\}$.

The following Theorem 9 yields an abstract two-weight embedding for the Riesz potential. This is a reformulation of the results from [Perez–Wheeden, 2003].

Here we use the notation $w(B) = \int_B w\,d\mu$.

General weighted embeddings

Theorem 9.

Let $s > 0$. Assume that the reverse doubling condition (2) holds with the exponent $\eta = s$ and that there is $Q > s$ such that $\mu(B(x,r)) \geq Cr^Q$ for all $x \in X$ and all $r \geq 1$. Let $1 < p \leq q < \infty$, and let w, v be weights such that

$$w \in \widetilde{A}_\infty \quad \text{and} \quad h = v^{\frac{1}{1-p}} \in \widetilde{A}_\infty.$$

If there is $K > 0$ such that

$$\frac{r^s\, w(B(x,r))^{\frac{1}{q}}\, h(B(x,r))^{\frac{p-1}{p}}}{\mu(B(x,r))} \leq K$$

for all $x \in X$ and all $r > 0$, then the Riesz potential $\mathcal{I}_s$ is bounded from $L^p(v\,d\mu)$ to $L^q(w\,d\mu)$.

Poincaré inequalities

We say that the space X supports 1-**Poincaré inequality** if there are $C > 0$ and $\tau \geq 1$ such that if $u \in \mathrm{Lip}(X)$ and g is an upper gradient of u, then

$$\fint_{B(x,r)} |u - u_B| \, d\mu \leq Cr \fint_{B(x,\tau r)} g \, d\mu$$

for all $x \in X$ and all $r > 0$.

It follows from the Poincaré inequality and a chaining argument that if $u \in \mathrm{Lip}_0(X)$ and g is an upper gradient of u, then

$$|u(x)| \leq C\mathcal{I}_1(g)(x)$$

for all $x \in X$; here $C > 0$ is independent of u and g.

This pointwise estimate, together with Theorem 9 and the Aikawa condition, implies the validity of Hardy–Sobolev inequalities.

A sufficient condition for Hardy–Sobolev inequalities

Theorem 10.

Assume that X supports a 1-Poincaré inequality, that the reverse doubling condition (2) holds with $\eta = 1$, and that there is $Q > 1$ such that $\mu(B(x,r)) \geq cr^Q$ for all $x \in X$ and $r > 0$.
Let $E \subset X$ be a closed set, and let $1 < p \leq q \leq \frac{Qp}{Q-p} < \infty$ and $\beta \in \mathbb{R}$ be such that

$$\operatorname{co\,dim}_A(E) > \max\Big\{Q - \frac{q}{p}(Q - p + \beta)\,,\ \frac{\beta}{p-1}\Big\}.$$

Then there is $C > 0$ such that the weighted Hardy–Sobolev inequality

$$\left(\int_X |u(x)|^q d(x,E)^{\frac{q}{p}(Q-p+\beta)-Q}\, d\mu(x)\right)^{\frac{p}{q}} \leq C \int_X g(x)^p d(x,E)^{\beta}\, d\mu(x)$$

holds whenever $u \in \mathrm{Lip}_0(X)$ and g is an upper gradient of u.

Idea of the proof 1

Assumption $\operatorname{co}\dim_{\mathrm{A}}(E) > \max\big\{Q - \frac{q}{p}(Q-p+\beta)\,,\, \frac{\beta}{p-1}\big\}$ implies that the weights

$$w(x) = d(x,E)^{\frac{q}{p}(Q-p+\beta)-Q}, \quad v(x) = d(x,E)^{\beta}, \quad h(x) = d(x,E)^{\frac{-\beta}{p-1}}$$

are in $\widetilde{A}_\infty$. Moreover, if $B = B(x,r) \subset B(z,3r)$ for some $z \in E$, the Aikawa condition implies

$$w(B)^{\frac{p}{q}} \le C r^{Q-p+\beta-Q\frac{p}{q}} \mu(B)^{\frac{p}{q}}$$

and $h(B)^{p-1} \le C r^{-\beta} \mu(B)^{p-1}$.

Hence

$$w(B)^{\frac{p}{q}} h(B)^{p-1} \le C r^{Q-p+\beta-Q\frac{p}{q}} \mu(B)^{\frac{p}{q}} r^{-\beta} \mu(B)^{p-1} = C\left(\frac{r^Q}{\mu(B)}\right)^{1-\frac{p}{q}} \left(\frac{\mu(B)}{r}\right)^{p}.$$

If B is far from E, this is easy to show.

Idea of the proof 2

So, $w(B)^{\frac{p}{q}} h(B)^{p-1} \leq C\left(\frac{r^Q}{\mu(B)}\right)^{1-\frac{p}{q}} \left(\frac{\mu(B)}{r}\right)^p$ for all balls. Since $\mu(B(x,r)) \geq cr^Q$ and $p \leq q$, the assumption in Theorem 9, with $s = 1$, is satisfied.

Hence

$$\left(\int_X |u(x)|^q d(x,E)^{\frac{q}{p}(Q-p+\beta)-Q}\, d\mu(x)\right)^{\frac{p}{q}}$$

$$\leq C\left(\int_X \mathcal{I}_1(g)(x)^q d(x,E)^{\frac{q}{p}(Q-p+\beta)-Q}\, d\mu(x)\right)^{\frac{p}{q}}$$

$$\leq C \int_X g(x)^p d(x,E)^{\beta}\, d\mu(x),$$

proving the claim.

Characterization for Hardy–Sobolev inequalities

If X is Q-regular, we have even a characterization in the unbounded case $\beta = 0$. The necessity was shown in the case of $\mathbb{R}^n$ in [L–Vähäkangas, 2016], but the proof works in any Q-regular space.

Theorem 11.

Assume that X is Q-regular and supports a 1-Poincaré inequality. Let $1 < p \leq q < \frac{Qp}{Q-p} < \infty$ and let $E \subset X$ be a closed set. Then the global (q,p)-Hardy–Sobolev inequality

$$\left(\int_X |u(x)|^q \operatorname{dist}(x,E)^{\frac{q}{p}(Q-p)-Q}\,d\mu\right)^{\frac{1}{q}} \leq C\left(\int_X g_u(x)^p\,d\mu\right)^{\frac{1}{p}}$$

holds for all $u \in \operatorname{Lip}_0(X)$ **if and only if** $\operatorname{co\,dim}_A(E) > Q - \frac{q}{p}(Q-p)$, *that is* $\dim_A(E) < \frac{q}{p}(Q-p)$.

Fractional Hardy–Sobolev inequalities

Theorem 12 (DILTV, Fractional case).

Assume that X is connected, that the reverse doubling (2) holds with $\eta = s \in (0,1)$, and that there is $Q > 1$ such that $\mu(B(x,r)) \geq cr^Q$ for all $x \in X$ and $r > 0$. Let $E \subset X$ be closed, and let $1 < p \leq q \leq \frac{Qp}{Q-p} < \infty$ and $\beta \in \mathbb{R}$ be such that

$$\operatorname{co}\dim_{\mathrm{A}}(E) > \max\left\{Q - \tfrac{q}{p}(Q - sp + \beta)\,,\ \tfrac{\beta}{p-1}\right\}.$$

Then, if $1 \leq t < \infty$, there is $C > 0$ such that for all $f \in \operatorname{Lip}_0(X)$

$$\left(\int_X |f(x)|^q \operatorname{dist}(x,E)^{\frac{q}{p}(Q-sp+\beta)-Q}\,d\mu(x)\right)^{\frac{p}{q}} \leq C\int_X \left(\int_X \frac{|f(y)-f(z)|^t}{d(y,z)^{st}\mu(B(y,d(y,z)))}\,d\mu(z)\right)^{\frac{p}{t}} \operatorname{dist}(y,E)^{\beta}\,d\mu(y)$$

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