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Geometric conditions for fractional Hardy and Hardy-Sobolev inequalities

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1. Fractional Hardy inequalities

Hardy inequalities

Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set. The p-Hardy inequality in Ω reads as

$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, \partial\Omega)^p}\, dx \leq C \int_\Omega |\nabla u(x)|^p\, dx,$$

where (usually) $u \in C_0^\infty(\Omega)$ (or $u \in W_0^{1,p}(\Omega)$).
(However, in some cases the zero boundary values can be omitted.)

In this talk we discuss fractional variants of Hardy inequalities and their relation to the geometry of Ω, and also metric space versions of such inequalities. My contributions are from joint works with Bartłomiej Dyda, Lizaveta Ihnazyeva, Heli Tuominen, and Antti Vähäkangas.

Fractional Hardy inequalities

Let $0 < s < 1$ and $1 \leq p < \infty$. We say that an open set $\Omega \subset \mathbb{R}^n$ admits an (s,p)-Hardy inequality if there is $C > 0$ such that

$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{sp}}\,dx \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx$$

holds for every $u \in C_0(\Omega)$.

By [Dyda, 2004], a bounded Lipschitz domain Ω admits an (s,p)-Hardy inequality if and only if $sp > 1$. This should be contrasted with the result of [Nečas, 1962]: if $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, there is $C > 0$ such that the p-Hardy inequality

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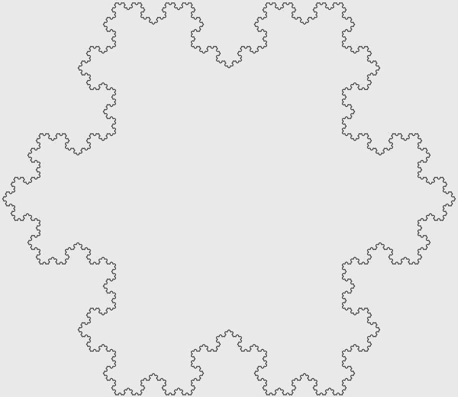
$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{p-\beta}}\, dx \leq C \int_\Omega |\nabla u(x)|^p \operatorname{dist}(x, \partial\Omega)^\beta\, dx$$

holds for every $u \in C_0^\infty(\Omega)$ if (and only if) $p - \beta > 1$.

Motivation: snowflake

The situation changes, if instead of a Lipschitz domain (say a ball) we consider e.g. a snowflake domain $\Omega \subset \mathbb{R}^n$ with $\dim(\partial\Omega) = \lambda \in \,]n-1, n[$.



Motivation: snowflake

For a Lipschitz domain the critical bound for the (p, β)-Hardy inequality is

$$p - \beta > 1 = n - (n - 1) = n - \dim(\partial\Omega)$$

and for the fractional (s, p)-Hardy inequality

$$sp > 1 = n - (n - 1) = n - \dim(\partial\Omega).$$

Similarly, it turns out that for a snowflake domain $\Omega \subset \mathbb{R}^n$ with $\dim(\partial\Omega) = \lambda \in \,]n - 1, n[$, the bound for the (p, β)-Hardy inequality is

$$p - \beta > n - \lambda = n - \dim(\partial\Omega)$$

(based on [Koskela–L, 2009]). and for the fractional (s, p)-Hardy

$$sp > n - \lambda = n - \dim(\partial\Omega)$$

(based on [Ihnatsyeva – L – Tuominen – Vähäkangas, 2014] or [Dyda – Vähäkangas, 2014]). Here both ”thickness” and ”accessibility” of the boundary are needed. More details will be discussed soon.

2. Assouad dimensions

Hausdorff measure and dimension

We need to introduce some notions of dimension. First we recall the usual Hausdorff dimension.

Let $E \subset \mathbb{R}^n$, $n \geq 1$, and let $\lambda \geq 0$ and $0 < \delta \leq \infty$.
The λ-dimensional **Hausdorff** (δ-)**content** of E is

$$\mathcal{H}^\lambda_\delta(E) = \inf\left\{ \sum_{k=1}^{\infty} r_k^\lambda : E \subset \bigcup_{k=1}^{\infty} B(x_k, r_k),\ 0 < r_k \leq \delta \right\}.$$

The λ-dimensional **Hausdorff measure** of E is $\mathcal{H}^\lambda(E) = \lim_{\delta \to 0} \mathcal{H}^\lambda_\delta(E)$.

The **Hausdorff dimension** of E is

$$\begin{aligned} \dim_{\mathrm{H}}(E) &= \inf\{\lambda \geq 0 : \mathcal{H}^\lambda(E) = 0\} \\ &= \inf\{\lambda \geq 0 : \mathcal{H}^\lambda_\infty(E) = 0\}. \end{aligned}$$

Assouad dimensions

Let $E \subset \mathbb{R}^n$ and write $d(E) = \mathrm{diam}(E)$.

Consider all exponents $\lambda \geq 0$ for which there is $C > 0$ such that $E \cap B(x,R)$ can be covered by at most $C(\frac{r}{R})^{-\lambda}$ balls of radius r for all $0 < r < R$ $(< d(E))$ and all $x \in E$.

The infimum of such λ is the **Assouad dimension** $\dim_{\mathrm{A}}(E)$.

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The infimum of such λ is the **(upper) Assouad dimension** $\overline{\dim}_{\mathrm{A}}(E)$.

Conversely: Consider all $\lambda \geq 0$ for which there is $C > 0$ such that if $0 < r < R < d(E)$, then for every $x \in E$ at least $C(\frac{r}{R})^{-\lambda}$ balls of radius r are needed to cover $E \cap B(x, R)$.

The supremum of such λ is the **lower (Assouad) dimension** $\underline{\dim}_{\mathrm{A}}(E)$.

It always holds that $\dim_{\mathrm{H}}(E) \leq \overline{\dim}_{\mathrm{A}}(E)$.

If E is closed, then $\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}(E)$ (this is not immediate), but for instance $\underline{\dim}_{\mathrm{A}}(\mathbb{Q}) = 1 > 0 = \dim_{\mathrm{H}}(\mathbb{Q})$.

Minkowski and Assouad

Recall:
$\overline{\dim}_{\mathrm{A}}(E)$ is the infimum of $\lambda \geq 0$ s.t. $E \cap B(x,R)$ can always be covered by at most $C(\frac{r}{R})^{-\lambda}$ balls of radius $0 < r < R < d(E)$

$\underline{\dim}_{\mathrm{A}}(E)$ is the supremum of $\lambda \geq 0$ s.t. always at least $C(\frac{r}{R})^{-\lambda}$ balls of radius $0 < r < R < d(E)$ are needed to cover $E \cap B(x,R)$.

For comparison, the **upper** and **lower Minkowski** (or **box**) **dimensions** of a bounded set $E \subset \mathbb{R}^n$ can be defined as follows:

$\overline{\dim}_{\mathrm{M}}(E)$ is the infimum of $\lambda \geq 0$ s.t. E can always be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < d(E)$

$\underline{\dim}_{\mathrm{M}}(E)$ is the supremum of $\lambda \geq 0$ s.t. at least $Cr^{-\lambda}$ balls of radius $0 < r < d(E)$ are always needed to cover E.

Thus $\underline{\dim}_{\mathrm{A}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E)$.

Examples

General idea: Assouad dimensions reflect the extreme behavior of sets and take into account all scales $0 < r < d(E)$.

- Let $E = \{0\} \cup [1,2] \subset \mathbb{R}$. Then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ ($\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = 1$).
- $\underline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 0$ and $\overline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 1$ (Minkowski not defined).
- Let $E = \{(\frac{1}{j}, 0, \ldots, 0) : j \in \mathbb{N}\} \cup \{(0,0,\ldots,0)\} \subset \mathbb{R}^n$.

Then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ $\left(\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = \frac{1}{2} \right)$.

Let $E \subset \mathbb{R}^n$ be a closed set, and let $0 < \alpha \leq n$. Then the following conditions are equivalent:

- $\underline{\dim}_A(E) > n - \alpha$.
- There are $\lambda > n - \alpha$ and $C > 0$ such that

$$\mathcal{H}^\lambda_\infty\big(E \cap B(x,r)\big) \geq C r^\lambda$$

for every $x \in E$ and $0 < r < \operatorname{diam}(E)$.

- E satisfies an (s,p)-capacity density condition (uniform (s,p)-fatness), for every $x \in E$ and $0 < r < \operatorname{diam}(E)$, whenever $0 < s < 1$ and $1 \leq p < \infty$ are such that $sp = \alpha$.

3. Fractional Hardy inequalities in $\mathbb{R}^n$

Accessibility

Often the ”thickness” of the boundary (given by the previous equivalent conditions) is not alone sufficient for the fractional (s, p)-Hardy inequalities. A possible additional condition is the following ”accessibility”.

When $x \in \Omega$, we write $B_x = B(x, 2\,\mathrm{dist}(x, \partial\Omega))$. In the accessibility condition we require that for every $z \in \partial\Omega \cap B_x$ there is an arc-length parameterized curve $\gamma\colon [0, \ell] \to \Omega$ (John curve), such that $\gamma(0) = z$, $\gamma(\ell) = x$, and

$$\mathrm{dist}(\gamma(t), \partial\Omega) \geq ct$$

for all $t \in [0, \ell]$. Here $c > 0$ is a uniform constant.

By a recent result in [Azzam, 2018], if $0 \leq \lambda \leq n - 1$ and

$$\mathcal{H}^{\lambda}_{\infty}(\partial\Omega \cap B_x) \geq Cr^{\lambda}$$

for every $x \in \Omega$, then the set of accessible points on $\partial\Omega \cap B_x$ satisfies the same condition for every $0 < \lambda' < \lambda$ (the bound $\lambda \leq n - 1$ is essential).

Fractional Hardy inequalities, thick case

Theorem 1 ((essentially) ILTV, 2014).

Let $0 < s < 1$ *and* $1 < p < \infty$ *satisfy* $0 < sp < n$*, and let* $\Omega \subset \mathbb{R}^n$ *be an open set. Assume that there are* $n - sp < \lambda \leq n$ *and* $C > 0$ *such that, for every* $x \in \Omega$,

$$\mathcal{H}^{\lambda}_{\infty}(\partial\Omega \cap B_x) \geq C \operatorname{dist}(x, \partial\Omega)^{\lambda}$$

and either
(i) $\partial\Omega \cap B_x$ *is (uniformly)* **accessible** *from* x*, or*
(ii) $|\Omega^c \cap B_x| = 0$.
Then Ω *admits an* (s, p)*-Hardy inequality.*

In particular, if $\underline{\dim}_A(\partial\Omega) > n - sp$ and each $x \in \Omega$ satisfies one of (i) and (ii), then Ω admits an (s, p)-Hardy inequality.

The proof of Theorem 1 is based upon a chaining argument along the John-curves and the use of maximal functions.

A counterexample

The condition $\underline{\dim}_A(\partial\Omega) > n - sp$ alone is not sufficient for (s,p)-Hardy inequality in Ω when $0 < sp \leq 1$. We state and explain this in the case $n = 2$, but similar examples exist in higher dimensions as well.

Theorem 2.

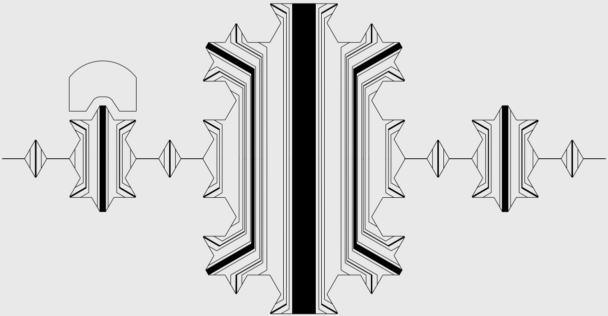
Let $1 < p < \infty$ and $0 < s < 1$ be such that $0 < sp \leq 1$. Then there exists a bounded domain $\Omega \subset \mathbb{R}^2$ such that $\underline{\dim}_A(\partial\Omega) > 2 - sp$, but the (s,p)-Hardy inequality does not hold in Ω.

The idea is to let Ω_0 be the domain inside a snowflake curve with $\underline{\dim}_A(\partial\Omega) = \lambda > 2 - sp$ (for this the (s,p)-Hardy inequality holds). In the case $sp = 1$, we remove inside the domain a fat Cantor set C having positive Lebesgue measure. Then $\underline{\dim}_A(C) = 2$, and for $\Omega = \Omega_0 \setminus C$ we have $\underline{\dim}_A(\partial\Omega) = \lambda > 2 - sp = 1$.

However, in this case the (s,p)-Hardy inequality fails ($sp = 1$). (Based on ideas in [Dyda, 2004])

Snowflaked counterexample

If we want to break an (s,p)-Hardy inequality with $0 < sp < 1$, then instead of the Cantor set we remove a "fat snowflake with tunnels", where the dimension of the snowflake is $2 - sp$.



(The support of one test function showing the failure of the (s,p)-Hardy inequality is also seen in the figure.)

Modified inequality

For comparison: Combination of results from [Edmunds – Hurri-Syrjänen – Vähäkangas, 2014] and [Ihnatsyeva – Vähäkangas, 2013] shows that if $\underline{\dim}_A(\partial\Omega) > n - sp$, without any additional conditions, then inequality

$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{sp}}\,dx \le C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx$$

holds for every $u \in C_0(\Omega)$.

Note that on the right-hand side the integrals are over the whole $\mathbb{R}^n$. Here we understand that functions in $C_0(\Omega)$ are extended as 0 outside Ω.

(Of course, for the usual Hardy inequalities involving the gradient, this would not make a difference.)

4. Metric spaces

Metric space version of fractional Hardy

More generally, we consider variants of the fractional Hardy–Sobolev inequalities in an open set Ω in a metric measure space $X = (X, d, \mu)$. One natural form of such an inequality is

$$\int_\Omega \frac{|u(x)|^p}{d(x,\Omega^c)^{sp}}\,dx \le C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{d(x,y)^{sp}\mu(B(x,d(x,y)))}\,dy\,dx,$$

for functions $u \in \mathrm{Lip}_0(\Omega)$. (Here we write $dx = d\mu(x)$.)

(Compare to $\displaystyle\int_\Omega \frac{|u(x)|^p}{d(x,\partial\Omega)^{sp}}\,dx \le C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx$ when $\Omega \subset \mathbb{R}^n$.)

In [Dyda – L – Vähäkangas, ongoing] we are also interested in the validity of a “localized” version

$$\int_{\Omega\cap B(z,r)} \frac{|u(x)|^p}{d(x,\Omega^c)^{sp}}\,dx \le C \int_{B(z,3r)} \int_{B(z,3r)} \frac{|u(x)-u(y)|^p}{d(x,y)^{sp}\mu(B(x,d(x,y)))}\,dy\,dx,$$

whenever $z \in \Omega^c$, $r > 0$, and $u \in \mathrm{Lip}_0(\Omega)$.

Lower Assouad codimension

When examining dimensional conditions for Hardy inequalities in a metric space (X, d, μ), we also need to take into account the effect of the measure μ. Thus we need different variants of the Assouad dimensions.

Definition 3.

Let $E \subset X$. The lower Assouad codimension $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ is the supremum of all $\rho \geq 0$ for which there is $C > 0$ such that

$$\frac{\mu(E_r \cap B(x,R))}{\mu(B(x,R))} \leq C\Big(\frac{r}{R}\Big)^{\rho}$$

for all $x \in E$ and all $0 < r < R < 2\operatorname{diam}(X)$.

Here $E_r = \{x \in X : \operatorname{dist}(x,E) < r\}$ is the open r-neighborhood of $E \subset X$.

Note that if $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E) > 0$, then $\mu(E) = 0$ by the Lebesgue density theorem.

Upper Assouad codimension

Conversely, we have:

Definition 4.

Let $E \subset X$. The upper Assouad codimension $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$ is the infimum of all $\rho \geq 0$ for which there is $C > 0$ such that

$$\frac{\mu(E_r \cap B(x,R))}{\mu(B(x,R))} \geq C\Big(\frac{r}{R}\Big)^{\rho}$$

for all $x \in E$ and all $0 < r < R < 2\operatorname{diam}(X)$.

The Ahlfors regular case

The space $X = (X, d, \mu)$ is **(Ahlfors)** ***Q*-regular**, for $Q > 0$, if there is $C \geq 1$ such that

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q$$

for all $x \in X$ and all $0 < r < \operatorname{diam}(X)$. This can be equivalently required to hold for $\mu = \mathcal{H}^Q$, the Q-dimensional Hausdorff measure.

If X is Q-regular, then it is not hard to see that

$$\overline{\dim}_{\mathrm{A}}(E) = Q - \underline{\operatorname{co}\dim}_{\mathrm{A}}(E) \quad \text{for all } E \subset X$$

and

$$\underline{\dim}_{\mathrm{A}}(E) = Q - \overline{\operatorname{co}\dim}_{\mathrm{A}}(E) \quad \text{for all } E \subset X.$$

On the other hand, if $E \subset X$ is Ahlfors λ-regular (i.e. a λ-set), for instance a subspace of $X = \mathbb{R}^n$ or a self-similar fractal, then

$$\overline{\dim}_{\mathrm{A}}(E) = \underline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{H}}(E) = \lambda.$$

Doubling

If the space X is not Ahlfors regular, we still need to assume the weaker condition that μ is **doubling**: there is $C > 0$ such that

$$\mu(B(x,2r)) \le C\,\mu(B(x,r)) \quad \text{for all } x \in X, r > 0. \tag{1}$$

Iteration of (1) shows that there are $\sigma > 0$ and $C > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge C\left(\frac{r}{R}\right)^{\sigma} \quad \text{whenever } B(y,r) \subset B(x,R) \subset X.$$

Conversely, we say that μ is **reverse doubling**, if there are $\eta > 0$ and $C > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \le C\left(\frac{r}{R}\right)^{\eta} \quad \text{whenever } B(y,r) \subset B(x,R) \subset X. \tag{2}$$

If X is unbounded and connected and μ is doubling, then there is some $\eta > 0$ such that (2) holds.

Fractional Poincaré inequalities

Let $1 \le p < \infty$. Fix a ball $B = B(x_0, r) \subset X$ and a Lipschitz function $u \in Lip(X)$. Then a simple calculation using only the doubling condition (and Hölder) yields a fractional p-Poincaré inequality:

$$\begin{aligned}
\int_B |u(x) - u_B|^p \, dx &\le \int_B \fint_B |u(x) - u(y)|^p \, dy \, dx \\
&\le r^{sp} \int_B \int_B \frac{|u(x) - u(y)|^p}{r^{sp} \mu(B)} \, dy \, dx \\
&\le C r^{sp} \int_B \int_B \frac{|u(x) - u(y)|^p}{d(x,y)^{sp} \mu(4B)} \, dy \, dx \\
&\le C r^{sp} \int_B \int_B \frac{|u(x) - u(y)|^p}{d(x,y)^{sp} \mu(B(x, d(x,y)))} \, dy \, dx.
\end{aligned}$$

With the help of the quantitative doubling and reverse doubling conditions (1) and (2), this can then be improved into a fractional (q, p)-Poincaré inequality, with some $q > p$, where on the left-hand side we have $\left(\int_B |u(x) - u_B|^q \, dx\right)^{p/q}$.

Localized fractional Hardy inequality

We have the following **localized** fractional Hardy inequalities.

Theorem 5 (DLV, ongoing).

Let $0 < s < 1$ and $1 < p < \infty$. Assume that X is unbounded, and that μ is doubling and reverse doubling. Let Ω be an open set such that Ω^c is unbounded and $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < sp$. Then there is $C > 0$ such that

$$\int_{B(z,r)\setminus E} \frac{|u(x)|^p}{d(x,E)^{sp}}\,dx \le C \int_{B(z,3r)} \int_{B(z,3r)} \frac{|u(x)-u(y)|^p}{d(x,y)^{sp}\mu(B(x,d(x,y)))}\,dy\,dx$$

whenever $z \in E$, $r > 0$, and $u \in \operatorname{Lip}_0(\Omega)$.

This implies that there is also $C > 0$ such that

$$\int_{\Omega} \frac{|u(x)|^p}{d(x,\Omega^c)^{sp}}\,dx \le C \int_X \int_X \frac{|u(x)-u(y)|^p}{d(x,y)^{sp}\mu(B(x,d(x,y)))}\,dy\,dx,$$

for every $u \in \operatorname{Lip}_0(\Omega)$.

Necessity of the dimensional condition

We can also show that the dimensional condition $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(\Omega^c) < sp$ in Theorem 5 is (almost) necessary.

Theorem 6 (DLV, ongoing).

Let $0 < s < 1$ and $1 < p < \infty$. Assume that $\Omega \subset X$ is an open set and that there is $C > 0$ such that

$$\int_{B(z,r)\setminus E} \frac{|u(x)|^p}{d(x,E)^{sp}}\,dx \le C \int_{B(z,3r)} \int_{B(z,3r)} \frac{|u(x)-u(y)|^p}{d(x,y)^{sp}\mu(B(x,d(x,y)))}\,dy\,dx$$

whenever $z \in E$, $r > 0$, and $u \in C_0(\Omega)$.
Then $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) \le sp$.

Global fractional Hardy–Sobolev inequalities

Conversely, we have the following global fractional Hardy–Sobolev inequalities in the metric setting.

Theorem 7 (DILTV, 2019).

Assume that X is connected, that the reverse doubling (2) *holds with $\eta = s \in (0,1)$, and that there is $Q > 1$ such that $\mu(B(x,r)) \geq cr^Q$ for all $x \in X$ and $r > 0$. Let $E \subset X$ be closed, and let $1 < p \leq q \leq \frac{Qp}{Q-sp} < \infty$ be such that $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E) > Q - \frac{q}{p}(Q - sp)$.*
Then there is $C > 0$ such that for all $u \in \operatorname{Lip}_0(X)$

$$\left(\int_X |u(x)|^q d(x,E)^{\frac{q}{p}(Q-sp)-Q} \, dx \right)^{\frac{p}{q}} \leq C \int_X \int_X \frac{|u(x) - u(y)|^p \; dy \, dx}{d(x,y)^{sp} \mu(B(x, d(x,y)))}$$

At least under Ahlfors Q-regularity, the dimensional condition is also necessary. Note that when $q = p$, this condition reduces to $\underline{\operatorname{co\,dim}}_{\mathrm{A}}(E) > sp$. (More of these things in the talk of Bartek Dyda).

Global fractional Hardy–Sobolev inequalities in $\mathbb{R}^n$

Finally, in $\mathbb{R}^n$ the previous Theorem 7 implies the following Hardy–Sobolev inequalities.

Let $E \subset \mathbb{R}^n$ be closed, and let $1 < p \leq q \leq \frac{np}{n-sp} < \infty$ be such that $\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n - sp)$.

Then there is $C > 0$ such that for all $u \in \mathrm{Lip}_0(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |u(x)|^q d(x,E)^{\frac{q}{p}(n-sp)-n}\, dx\right)^{\frac{p}{q}} \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\, dx.$$

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