FRACTIONAL HARDY–SOBOLEV TYPE INEQUALITIES FOR HALF SPACES AND JOHN DOMAINS

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Abstract. As our main result we prove a variant of the fractional Hardy–Sobolev–Maz'ya inequality for half spaces. This result contains a complete answer to a recent open question by Musina and Nazarov. In the proof we apply a new version of the fractional Hardy–Sobolev inequality that we establish also for more general unbounded John domains than half spaces.

1. Introduction

The main result in this note is the following *fractional Hardy–Sobolev–Maz'ya inequality* for functions $u \in C_0^\infty(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+$ is the upper half space of $\mathbb{R}^n$ with $n \geq 2$:

$$(1) \qquad \iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx - \mathcal{D}\int_{\mathbb{R}^n_+} |u(x)|^p x_n^{-sp}\,dx \geq \sigma \left(\int_{\mathbb{R}^n_+} |u(x)|^q x_n^{-bq}\,dx \right)^{p/q}.$$

Here $2 \leq p, q < \infty$ and $0 < s < 1$ are such that $sp < n$ and $p < q \leq np/(n-sp)$, and $b = n(1/q - 1/p) + s$; notice that then

$$\frac{b}{n} = \frac{1}{q} - \frac{n-sp}{np} \qquad \text{and} \qquad -bq = \frac{q}{p}(n-sp) - n\,.$$

The constant $\sigma = \sigma(n,p,q,s) > 0$ in (1) is independent of u, and $\mathcal{D} = \mathcal{D}(n,p,s) \geq 0$ is the optimal constant for which the left-hand side of (1) is non-negative for all $u \in C_0^\infty(\mathbb{R}^n_+)$; see (7) in Section 4 for an explicit expression of this constant. By approximation, inequality (1) holds for all functions in the associated fractional Sobolev space $\mathcal{W}_0^{s,p}(\mathbb{R}^n_+)$; cf. Theorem 4.2.

The validity of inequality (1) completely solves the Open Problem 1 posed by Musina and Nazarov at the end of the paper [14], where actually only the case $p = 2$ was under consideration. Our results also extend the validity of [14, Theorem 3.1] to the case $0 < s < 1/2$.

When $sp = 1$, the constant $\mathcal{D}$ equals zero, and then for $q = np/(n-sp) = np/(n-1)$ inequality (1) is the usual Sobolev inequality. For $sp \neq 1$ it holds that $\mathcal{D} > 0$. When $sp > 1$, then in the special case $q = np/(n-sp)$ the validity of inequality (1) was proved in [16] for $p = 2$ and in [2] for general $p \geq 2$; see also [6] for similar results. On the other hand, when $sp < 1$, the validity of inequality (1) seems to be completely new.

In the proof of inequality (1), we bring together in a novel way adaptations of some recent results related to fractional inequalities. We begin in Section 2 by extending a fractional Riesz potential estimate from [10] to the case of unbounded John domains, including the upper half-space $\mathbb{R}^n_+$. The definition and some important properties of John domains are recalled at the beginning of that section. In Section 3, we establish the *weighted fractional Hardy–Sobolev inequality*

$$(2) \qquad \int_D \int_{B(x,\tau\delta_{\partial D}(x))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,\delta_{\partial D}^{\beta}(x)\,dx \geq C\left(\int_D |u(x)|^q \delta_{\partial D}^{(q/p)(n-sp+\beta)-n}(x)\,dx \right)^{p/q}$$

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for functions $u \in C_0^\infty(D)$, where $0 < \tau < 1$ and D is an unbounded John domain satisfying the additional assumption that the Assouad dimension of the boundary ∂D is small enough; we use here the notation $\delta_{\partial D}(x) = \operatorname{dist}(x, \partial D)$. The proof of inequality (2) is based on the Riesz potential estimate from Section 2 and general two weight inequalities for Riesz potentials from [3]. An important feature in inequality (2) is that, due to the parameter $0 < \tau < 1$, the inner integral in the left-hand side is taken over a ball which is not too close to the boundary ∂D. This crucial fact allows some flexibility to modify the weight functions that are powers of the distance-to-boundary function $\delta_{\partial D}$; cf. estimate (5). The fractional Hardy–Sobolev–Maz'ya inequality (1) is then proved in Section 4, relying on the Hardy–Sobolev inequality (2) and a sharp fractional Hardy inequality with a reminder term from [9, Theorem 1.2]. Section 4 also contains discussion related to the space $\mathcal{W}_0^{s,p}(\mathbb{R}^n_+)$ and the approximation argument that allows us to extend the validity of inequality (1) for the functions belonging to this space.

Notation. Throughout this note, we work in the n-dimensional Euclidean space $\mathbb{R}^n$, with $n \geq 2$. We write $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$, and denote by $C_0^\infty(\mathbb{R}^n_+)$ the space of smooth functions whose support is a compact set in $\mathbb{R}^n_+$. The open ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ is denoted $B(x, r)$. When $E \neq \emptyset$ is a set in $\mathbb{R}^n$, the Euclidean distance from $x \in \mathbb{R}^n$ to E is written as $\operatorname{dist}(x, E) = \delta_E(x)$, the diameter of E is $\operatorname{diam}(E)$, and we write χ_E for the characteristic function of E; that is, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. In addition, $\overline{E}$ denotes the closure of E. The Lebesgue n-measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$, and if $0 < |E| < \infty$ and u is an integrable function on E, we use the notation

$$u_E = \frac{1}{|E|} \int_E u(y)\, dy\,.$$

The letter C is used to denote positive constants whose values are not necessarily the same at each occurrence. We also write $C = C(*, \cdots, *)$ to indicate that the constant C depends (at most) on the quantities appearing in the parentheses.

2. A fractional potential estimate on John domains

In this section we prove Theorem 2.4, which provides a fractional potential estimate for unbounded John domains; recall that a domain is an open and connected set. Following [17], we will first define John domains in such a way that unbounded domains are allowed. Several equivalent definitions for John domains can be found in [17]. When $D \subset \mathbb{R}^n$ is a domain and $x_1, x_2 \in D$, we say that a curve $\gamma\colon [0, \ell] \to D$ joins x_1 to x_2 if $\gamma(0) = x_1$ and $\gamma(\ell) = x_2$.

Definition 2.1. A domain $D \subsetneq \mathbb{R}^n$, with $n \geq 2$, is a *c-John domain*, for $c \geq 1$, if each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc length parametrized curve $\gamma\colon [0, \ell] \to D$ satisfying $\operatorname{dist}(\gamma(t), \partial D) \geq \min\{t, \ell - t\}/c$ for every $t \in [0, \ell]$.

It is clear that for example the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$ is an unbounded John domain, but it is also easy to come up with more irregular examples, since the class of John domains is quite flexible. For instance, the unbounded domain whose boundary is the usual von Koch snowflake curve is an unbounded John domain in $\mathbb{R}^2$.

The next lemma recalls a useful property which can actually be used to characterize bounded John domains. See [17, Theorem 3.6] for more details and a proof of this result.

Lemma 2.2. *Assume that $D \subset \mathbb{R}^n$ is a bounded c_1-John domain, $n \geq 2$. Then there is a point $x_0 \in D$ such that each $x \in D$ can be joined to x_0 by a rectifiable arc length parametrized curve $\gamma\colon [0, \ell] \to D$ satisfying* $\operatorname{dist}(\gamma(t), \partial D) \geq t/(4c_1^2)$ *for every $t \in [0, \ell]$.*

The point x_0 appearing in Lemma 2.2 is called a *John center of D*. The following engulfing property of John domains can be found in [17, Theorem 4.6].

Lemma 2.3. *A c-John domain $D \subsetneq \mathbb{R}^n$ can be written as the union of c_1-John domains $D_1, D_2, \ldots$, where $c_1 = c_1(c, n)$ and $\overline{D_i}$ is compact in D_{i+1} for each $i = 1, 2, \ldots$.*

We now turn to the potential estimate, which is given in terms of Riesz potentials. Recall that the *Riesz potential* $\mathcal{I}_\alpha(f)$ of a measurable function $f\colon \mathbb{R}^n \to [0,\infty]$, for $0<\alpha<n$, is defined as

$$\mathcal{I}_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy\,, \qquad x\in\mathbb{R}^n\,.$$

Theorem 2.4. *Assume that $D\subsetneq\mathbb{R}^n$ is an unbounded c-John domain, and let $0<\tau,s<1$ and $1\le p<\infty$. Then there is a constant $C=C(\tau,n,c,s,p)>0$ such that the inequality*

$$|u(x)| \le C\int_D \frac{g(y)}{|x-y|^{n-s}}\,dy = C\,\mathcal{I}_s(\chi_D g)(x)$$

holds whenever $u\in\bigcup_{1\le r<\infty}L^r(D)$ and $x\in D$ is Lebesgue point of u, where we have denoted

$$g(y) := \left(\int_{B(y,\tau\delta_{\partial D}(y))} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}}\,dz\right)^{1/p} \tag{3}$$

for every $y\in D$.

For the proof of Theorem 2.4, we first need a a fractional potential estimate for bounded John domains, which is stated in Proposition 2.5 below. For a simple proof of Proposition 2.5, we refer to the proof of [10, Theorem 4.10]; see formula (4.13) therein. For our purposes, we actually need to track the constants a bit more carefully than what is done in [10], but an inspection of the proof in [10] shows that the constants depend on the c_1-John domain D only through c_1; we omit further details. We also remark that while the statement of [10, Theorem 4.10] contains the assumption $sp<n$, this is not needed for [10, formula (4.13)] to hold.

Proposition 2.5. *Assume that $D\subset\mathbb{R}^n$ is a bounded c_1-John domain, and let $0<\tau,s<1$ and $1\le p<\infty$. Let $x_0\in D$ be a John center of D as in Lemma 2.2, let $M>2/\tau$, and denote*

$$B = B\left(x_0, \frac{\delta_{\partial D}(x_0)}{16Mc_1^2}\right).$$

Then there is a constant $C=C(M,n,c_1,s,p)>0$ such that

$$|u(x)-u_B| \le C\int_D \frac{g(y)}{|x-y|^{n-s}}\,dy$$

whenever $u\in L^1_{\mathrm{loc}}(D)$, $x\in D$ is a Lebesgue point of u, and g is as in (3) *with respect to the bounded c_1-John domain D.*

We are now ready for the proof of Theorem 2.4.

Proof of Theorem 2.4. Assume that $u\in L^r(D)$ for some $1\le r<\infty$ and choose $M=3/\tau$. By Lemma 2.3, there are bounded c_1-John domains D_i with $c_1=c_1(c,n)\ge 1$ such that

$$D_i\subset\overline{D_i}\subset D_{i+1}\,,\qquad \text{for all } i=1,2,\dots\,,$$

and $D=\bigcup_{i=1}^\infty D_i$. Let $x_i\in D_i$ be a John center of D_i given by Lemma 2.2, and write

$$B_i := B\left(x_i, \frac{\delta_{\partial D_i}(x_i)}{16Mc_1^2}\right)\subset D_i\subset D\,.$$

By Lemma 2.2 we have $\delta_{\partial D_i}(x_i)\ge(12c_1^2)^{-1}\operatorname{diam}(D_i)$. Observe that the numbers $\operatorname{diam}(D_i)$ converge to ∞ as $i\to\infty$, and thus $\lim_{i\to\infty}|B_i|=\infty$. In particular, by Hölder's inequality,

$$|u_{B_i}| \le \frac{1}{|B_i|}\int_{B_i}|u(x)|\,dx \le \frac{\|u\|_{L^r(D)}}{|B_i|^{1/r}} \xrightarrow{i\to\infty} 0\,.$$

Let us denote by g_i the function defined as in (3), but with respect to the bounded c_1-John domain D_i, and let $x\in D$ be a Lebesgue point of u. Since $x\in D_i$ for all sufficiently large indices

i and $u|_{D_i} \in L^1_{\mathrm{loc}}(D_i)$, we find, by an application of Proposition 2.5 and monotone convergence, that

$$|u(x)| = \lim_{i\to\infty} |u(x) - u_{B_i}| \le C(M,n,c_1,s,p) \limsup_{i\to\infty} \int_{D_i} \frac{g_i(y)}{|x-y|^{n-s}}\, dy$$
$$\le C(M,n,c_1,s,p) \int_D \frac{g(y)}{|x-y|^{n-s}}\, dy\,.$$

This concludes the proof of the theorem. □

3. Weighted fractional Hardy–Sobolev inequalities

In this section we establish weighted fractional inequalities of the general form

$$\left(\int_D |u(x)|^q \delta_{\partial D}^{(q/p)(n-sp+\beta)-n}(x)\, dx\right)^{p/q} \le C \int_D \int_{B(x,\tau\delta_{\partial D}(x))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\, \delta_{\partial D}^{\beta}(x)\, dx\,, \tag{4}$$

where $u \in L^r(D)$ for some $1 \le r < \infty$ and $D \subsetneq \mathbb{R}^n$ is an unbounded John domain satisfying the dimensional condition (6) below. Recall that we write $\delta_{\partial D}(x) = \operatorname{dist}(x, \partial D)$. As was already mentioned in the Introduction, an important feature here is that we obtain inequality (4) with a parameter $0 < \tau < 1$. This allows us to use in applications of (4) estimates of the type

$$\delta_{\partial D}^{\beta}(x) \le C \delta_{\partial D}^{\beta_1}(y) \delta_{\partial D}^{\beta_2}(x) \tag{5}$$

for $x \in D$ and $y \in B(x, \tau\delta_{\partial D}(x))$, where and $\beta_1 + \beta_2 = \beta$ and $C = C(\tau, \beta_1, \beta_2)$.

When $E \subset \mathbb{R}^n$, the *Assouad dimension* denoted $\dim_{\mathrm{A}}(E)$ is the infimum of exponents $\alpha \ge 0$ for which there is a constant $C \ge 1$ such that for each $x \in E$ and every $0 < r < R$, the set $E \cap B(x,R)$ can be covered by at most $C(r/R)^{-\alpha}$ balls of radius r. For example, the Assouad dimension of the boudary of the half-space $\mathbb{R}^n_+$ is $\dim_{\mathrm{A}}(\partial \mathbb{R}^n_+) = n-1$, and more generally, if $E \subset \mathbb{R}^n$ is an m-dimensional subspace, then $\dim_{\mathrm{A}}(E) = m$. See e.g. [13] for more details, properties, and examples related to the Assouad dimension.

The following Theorem 3.1 is a partial generalization of [5, Theorem 1], where a weighted fractional Hardy-type inequality (the case $q = p$) is addressed, and it extends [11, Theorem 5.2], where a fractional Sobolev inequality is obtained in the case when $\beta = 0$ and $q = np/(n-sp)$. Theorem 3.1 is an improved version of the recent metric space result [3, Theorem 5.3], where all the integrals were taken over the whole space.

Theorem 3.1. *Assume that $D \subsetneq \mathbb{R}^n$ is an unbounded c-John domain and that $0 < s < 1$,*

$$1 < p \le q \le \frac{np}{n-sp} < \infty\,,$$

and $\beta \in \mathbb{R}$ are such that

$$\dim_{\mathrm{A}}(\partial D) < \min\left\{ \frac{q}{p}(n-sp+\beta)\,,\, n - \frac{\beta}{p-1} \right\}. \tag{6}$$

In addition, let $\tau \in (0,1)$. Then there is a constant $C = C(\beta,\tau,n,c,s,p,q) > 0$ such that inequality (4) holds for all $u \in \bigcup_{1\le r<\infty} L^r(D)$.

Proof. Fix a function $u \in \bigcup_{1\le r<\infty} L^r(D)$ and write, as in (3), for every $y \in D$,

$$g(y) = \left(\int_{B(y,\tau\delta_{\partial D}(y))} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}}\, dz\right)^{1/p}.$$

Also denote, for every $x \in \mathbb{R}^n \setminus \partial D$,

$$w(x) = \delta_{\partial D}^{(q/p)(n-sp+\beta)-n}(x)\,.$$

We remark that w is defined and positive almost everywhere in $\mathbb{R}^n$. Indeed, notice first that $\dim_{\mathrm{A}}(\partial D) < n$ by the assumption (6), and thus $|\partial D| = 0$; we refer to [3, Remark 3.2].

By Theorem 2.4, there is a constant $C = C(\tau, n, c, s, p) > 0$ such that inequality

$$|u(x)|^q w(x) \le C\,\mathcal{I}_s(\chi_D g)(x)^q w(x)$$

holds for every Lebesgue point $x \in D$ of u. In particular, since almost every point $x \in D$ is a Lebesgue point of u, we obtain that

$$\begin{aligned}\bigg(\int_D |u(x)|^q w(x)\,dx\bigg)^{p/q} &\le C\bigg(\int_D \mathcal{I}_s(\chi_D g)(x)^q w(x)\,dx\bigg)^{p/q}\\ &\le C\bigg(\int_{\mathbb{R}^n} \mathcal{I}_s(\chi_D g)(x)^q w(x)\,dx\bigg)^{p/q}.\end{aligned}$$

Next we apply [3, Theorem 4.1], which yields two weight inequalities for the Riesz potentials, where the weights are powers of the distance function $\delta_{\partial D}$. In [3] the result is formulated in a general metric space, but it is straightforward to see that in $\mathbb{R}^n$ the dimensional condition in [3, Theorem 4.1] coincides with (6). We remark that the proof of [3, Theorem 4.1] is based on the Muckenhoupt A_p-properties of the powers of $\delta_{\partial D}$ and general A_p-weighted inequalities; the Euclidean space versions of the latter are originally due to Pérez [15]. From [3, Theorem 4.1] it follows that

$$\begin{aligned}\bigg(\int_{\mathbb{R}^n} \mathcal{I}_s(\chi_D g)(x)^q w(x)\,dx\bigg)^{p/q} &\le C\int_{\mathbb{R}^n} \chi_D(y) g(y)^p\,\delta_{\partial D}^{\beta}(y)\,dy\\ &= C\int_D \int_{B(y,\tau\delta_{\partial D}(y))} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}}\,dz\,\delta_{\partial D}^{\beta}(y)\,dy\,.\end{aligned}$$

Here the constant $C > 0$ is independent of u and g, and so the desired inequality (4) follows by combining the two estimates above. □

Remark 3.2. In the case $D = \mathbb{R}^n_+$ we have $\dim_{\mathrm{A}}(\partial\mathbb{R}^n_+) = n-1$. Then the bounds in (6) are equivalent to

$$\frac{p}{q}(n-1) - n + sp < \beta < p-1.$$

From this we see that the lower bound for β is strictly decreasing in terms of q. For $q = p$ the lower bound is $sp-1$ and for $q = np/(n-sp)$ the lower bound is $sp/n-1$. In particular, the value $\beta = sp-1$, which will be used in the following Section 4 while proving our main inequality (1), is allowed in (4) for $D = \mathbb{R}^n_+$ whenever $0 < s < 1$ and $1 < p < q \le np/(n-sp) < \infty$.

Let us however point out that we do not know if the above bounds for β are optimal in $\mathbb{R}^n_+$ or in more general unbounded c-John domains; in particular, the necessity of the upper bound $\beta < p-1$ is questionable.

Remark 3.3. Assume that $D \subset \mathbb{R}^n$ is a bounded c_1-John domain such that (6) holds, where s, p, q, β are as in Theorem 3.1, and let $u \in L^1_{\mathrm{loc}}(D)$. For each $y \in D$, we write

$$g(y) = \bigg(\int_{B(y,\tau\delta_{\partial D}(y))} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}}\,dz\bigg)^{1/p}.$$

Then it follows from Proposition 2.5 that

$$|u(x) - u_B|^q w(x) \le C\,\mathcal{I}_s(\chi_D g)(x)^q w(x)$$

for every Lebesgue point $x \in D$ of u, where B is as in Proposition 2.5 and w is as in the proof of Theorem 3.1. We can then repeat the rest of the proof of Theorem 3.1, and conclude that

$$\bigg(\int_D |u(x)-u_B|^q \delta_{\partial D}^{(q/p)(n-sp+\beta)-n}(x)\,dx\bigg)^{p/q} \le C\int_D \int_{B(x,\tau\delta_{\partial D}(x))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,\delta_{\partial D}^{\beta}(x)\,dx\,.$$

Remark 3.4. We note that both Theorem 2.4 and Theorem 3.1 hold for every $u \in L^1_{\mathrm{loc}}(D)$ sastisfying $u_{B_i} \to 0$ whenever $B_i \subset D$ is a sequence of balls with $\operatorname{diam}(B_i) \to \infty$. For example, it is enough that $u \in L^1_{\mathrm{loc}}(D)$ and $u(x) \to 0$ as $|x| \to \infty$.

4. Fractional Hardy–Sobolev–Maz'ya inequality on half spaces

We are now prepared to prove the fractional Hardy–Sobolev–Maz'ya inequality (1) in the half space $\mathbb{R}^n_+$. As we will see, this inequality, which is reformulated in Theorem 4.2 below, is a rather immediate consequence of Theorem 3.1 and the fractional Hardy inequality with the best constant $\mathcal{D}$ and a remainder term, [9, Theorem 1.2]. This constant $\mathcal{D}$ has the explicit form

$$\mathcal{D} = \mathcal{D}(n,p,s) = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+sp}{2})}{\Gamma(\frac{n+sp}{2})} \int_0^1 \left|1 - r^{(sp-1)/p}\right|^p \frac{dr}{(1-r)^{1+sp}}\,, \tag{7}$$

where Γ denotes the usual gamma function. In particular, $\mathcal{D}$ is the largest number for which the left-hand side of the Hardy–Sobolev–Maz'ya inequality (1) is non-negative for every $u \in C_0^\infty(\mathbb{R}^n_+)$; see [9, Theorem 1.1]. We also refer to [1, 7, 8, 12] for more results concerning fractional Hardy inequalities with best constants.

We actually prove inequality (1) in Theorem 4.2 for functions belonging to space $\mathcal{W}^{s,p}_0(\mathbb{R}^n_+)$, which is defined as follows. When $1 \le p < \infty$ and $0 < s < 1$, the fractional Sobolev seminorm $|u|_{W^{s,p}(\mathbb{R}^n_+)}$ of a measurable function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ is

$$|u|_{W^{s,p}(\mathbb{R}^n_+)} = \left(\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx\right)^{1/p}.$$

The space $\mathcal{W}^{s,p}_0(\mathbb{R}^n_+)$ is then the completion of $C_0^\infty(\mathbb{R}^n_+)$ with respect to the seminorm $|\cdot|_{W^{s,p}(\mathbb{R}^n_+)}$.

Remark 4.1. Assume that $1 \le p < n/s$, where $n \ge 2$, $1 \le p < \infty$ and $0 < s < 1$. Then the space $\mathcal{W}^{s,p}_0(\mathbb{R}^n_+)$ can be identified as a subspace of $L^{np/(n-sp)}(\mathbb{R}^n_+)$ using the following reasoning. First, by the Sobolev Embedding Theorem [11, Theorem 5.2], there exists a constant $C > 0$ such that $\|u\|_{L^{np/(n-sp)}(\mathbb{R}^n_+)} \le C|u|_{W^{s,p}(\mathbb{R}^n_+)}$ for all $u \in C_0^\infty(\mathbb{R}^n_+)$. Therefore, if $(u_j)_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n_+)$ is a Cauchy sequence with respect to the seminorm $|\cdot|_{W^{s,p}(\mathbb{R}^n_+)}$, then there exists $u \in L^{np/(n-sp)}(\mathbb{R}^n_+)$ such that $\lim_{j\to\infty}\|u_j - u\|_{L^{np/(n-sp)}(\mathbb{R}^n_+)} = 0$. A straightforward adaptation of [4, Proposition 7] then shows that $\lim_{j\to\infty}|u-u_j|_{W^{s,p}(\mathbb{R}^n_+)} = 0$.

Theorem 4.2. *Let $n \ge 2$ and assume that $2 \le p, q < \infty$ and $0 < s < 1$ are such that $sp < n$ and $p < q \le np/(n-sp)$, and write $b = n(1/q - 1/p) + s$. Then there is a constant $\sigma = \sigma(n,p,q,s) > 0$ such that*

$$\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx - \mathcal{D}\int_{\mathbb{R}^n_+} |u(x)|^p x_n^{-sp}\,dx \ge \sigma\left(\int_{\mathbb{R}^n_+} |u(x)|^q x_n^{-bq}\,dx\right)^{p/q}$$

for all $u \in \mathcal{W}^{s,p}_0(\mathbb{R}^n_+)$, where the constant $\mathcal{D} = \mathcal{D}(n,p,s)$ is as in (7).

Proof. The proof follows the ideas presented in [2, Section 2], but instead of using the Sobolev inequality as in [2], we will use the more general inequality (4).

We consider first the case $u \in C_0^\infty(\mathbb{R}^n_+)$. Our starting point is the inequality

$$\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx - \mathcal{D}\int_{\mathbb{R}^n_+} |u(x)|^p x_n^{-sp}\,dx \ge c_p J[v]\,, \tag{8}$$

where $c_p > 0$ is an explicit constant (for $p = 2$, (8) is an identity with $c_2 = 1$),

$$J[v] := \iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}(x_n y_n)^{(sp-1)/2}\,dy\,dx\,,$$

and $v(x) := x_n^{-(sp-1)/p}u(x)$ for each $x \in \mathbb{R}^n_+$. Notice that $v \in C_0^\infty(\mathbb{R}^n_+)$ and $x_n = \delta_{\partial\mathbb{R}^n_+}(x)$ if $x \in \mathbb{R}^n_+$. Inequality (8) was derived in [9, Theorem 1.2], using the 'ground state representation' method from [8].

We apply Theorem 3.1 for $D = \mathbb{R}^n_+$, $\beta = sp-1$ and a fixed $0 < \tau < 1$; recall here Remark 3.2. Then we use estimate (5) with $\beta_1 = \beta_2 = \beta/2$, and obtain that

$$\begin{aligned}\left(\int_{\mathbb{R}^n_+} |v(x)|^q x_n^{(q/p)(n-sp+\beta)-n}(x)\,dx\right)^{p/q} &\le C\int_{\mathbb{R}^n_+}\int_{B(x,\tau x_n)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,dy\,x_n^\beta\,dx\\ &\le C\int_{\mathbb{R}^n_+}\int_{B(x,\tau x_n)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} y_n^{\beta/2}\,dy\,x_n^{\beta/2}\,dx\\ &\le CJ[v].\end{aligned}$$

Combining the above inequality with (8) and the fact that

$$|v(x)|^q x_n^{(q/p)(n-sp+\beta)-n} = |u(x)|^q x_n^{(q/p)(n-sp)-n} = |u(x)|^q x_n^{-bq}$$

proves the claim for functions $u \in C_0^\infty(\mathbb{R}^n_+)$.

In the general case $u \in \mathcal{W}_0^{s,p}(\mathbb{R}^n_+) \subset L^{np/(n-sp)}(\mathbb{R}^n_+)$, it suffices to consider a sequence $(u_j)_{j\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R}^n_+)$ functions, which is Cauchy with respect to the seminorm $|\cdot|_{W^{s,p}(\mathbb{R}^n_+)}$ and which converges to u in $L^{np/(n-sp)}(\mathbb{R}^n_+)$. Then $\lim_{j\to\infty}|u-u_j|_{W^{s,p}(\mathbb{R}^n_+)} = 0$; cf. Remark 4.1. By taking a subsequence, if necessary, we may also assume that $\lim_{j\to\infty} u_j(x) = u(x)$ for almost every $x \in \mathbb{R}^n_+$. By Fatou's lemma, and the already proved inequality (1) for $C_0^\infty(\mathbb{R}^n_+)$ functions,

$$\begin{aligned}\sigma\left(\int_{\mathbb{R}^n_+} |u(x)|^q x_n^{-bq}\,dx\right)^{p/q} &\le \liminf_{j\to\infty}\sigma\left(\int_{\mathbb{R}^n_+} |u_j(x)|^q x_n^{-bq}\,dx\right)^{p/q}\\ &\le \liminf_{j\to\infty}\left(\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{n+sp}}\,dy\,dx - \mathcal{D}\int_{\mathbb{R}^n_+}|u_j(x)|^p x_n^{-sp}\,dx\right).\end{aligned}$$

When $\mathcal{D} = \mathcal{D}(n,p,s) \neq 0$, we have $sp \neq 1$, and therefore, by the fractional Hardy inequality in [9, Theorem 1.1],

$$\left(\mathcal{D}\int_{\mathbb{R}^n_+}|u_j(x)|^p x_n^{-sp}\,dx\right)^{1/p} + \left(\mathcal{D}\int_{\mathbb{R}^n_+}|u(x)|^p x_n^{-sp}\,dx\right)^{1/p} \le |u_j|_{W^{s,p}(\mathbb{R}^n_+)} + |u|_{W^{s,p}(\mathbb{R}^n_+)} < \infty\,,$$

and furthermore

$$\begin{aligned}&\left|\left(\mathcal{D}\int_{\mathbb{R}^n_+}|u_j(x)|^p x_n^{-sp}\,dx\right)^{1/p} - \left(\mathcal{D}\int_{\mathbb{R}^n_+}|u(x)|^p x_n^{-sp}\,dx\right)^{1/p}\right|\\ &\qquad\le \left(\mathcal{D}\int_{\mathbb{R}^n_+}|u(x)-u_j(x)|^p x_n^{-sp}\,dx\right)^{1/p} \le |u-u_j|_{W^{s,p}(\mathbb{R}^n_+)} \xrightarrow{j\to\infty} 0\,.\end{aligned}$$

Since $\lim_{j\to\infty}|u-u_j|_{W^{s,p}(\mathbb{R}^n_+)} = 0$, we find that $\lim_{j\to\infty}|u_j|_{W^{s,p}(\mathbb{R}^n_+)} = |u|_{W^{s,p}(\mathbb{R}^n_+)}$. Hence,

$$\begin{aligned}&\liminf_{j\to\infty}\left(\iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{n+sp}}\,dy\,dx - \mathcal{D}\int_{\mathbb{R}^n_+}|u_j(x)|^p x_n^{-sp}\,dx\right)\\ &\qquad= \iint_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx - \mathcal{D}\int_{\mathbb{R}^n_+}|u(x)|^p x_n^{-sp}\,dx\,.\end{aligned}$$

The claim follows from the above estimates. $\square$

References

[1] K. Bogdan and B. Dyda. The best constant in a fractional Hardy inequality. *Math. Nachr.*, 284(5-6):629–638, 2011.

[2] B. Dyda and R. L. Frank. Fractional Hardy-Sobolev-Maz'ya inequality for domains. *Studia Math.*, 208(2):151–166, 2012.

[3] B. Dyda, L. Ihnatsyeva, J. Lehrbäck, H. Tuominen, and A. V. Vähäkangas. Muckenhoupt A_p-properties of distance functions and applications to Hardy–Sobolev -type inequalities. arXiv:1705.01360 [math.CA], 2017.

[4] B. Dyda and M. Kassmann. Function spaces and extension results for nonlocal Dirichlet problems. arXiv:1612.01628 [math.AP], 2016.

[5] B. Dyda and A. V. Vähäkangas. A framework for fractional Hardy inequalities. *Ann. Acad. Sci. Fenn. Math.*, 39(2):675–689, 2014.

[6] S. Filippas, L. Moschini, and A. Tertikas. Sharp trace Hardy-Sobolev-Maz'ya inequalities and the fractional Laplacian. *Arch. Ration. Mech. Anal.*, 208(1):109–161, 2013.

[7] R. L. Frank, E. H. Lieb, and R. Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc.*, 21(4):925–950, 2008.

[8] R. L. Frank and R. Seiringer. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.*, 255(12):3407–3430, 2008.

[9] R. L. Frank and R. Seiringer. Sharp fractional Hardy inequalities in half-spaces. In *Around the research of Vladimir Maz'ya. I*, volume 11 of *Int. Math. Ser. (N. Y.)*, pages 161–167. Springer, New York, 2010.

[10] R. Hurri-Syrjänen and A. V. Vähäkangas. On fractional Poincaré inequalities. *J. Anal. Math.*, 120:85–104, 2013.

[11] R. Hurri-Syrjänen and A. V. Vähäkangas. Fractional Sobolev-Poincaré and fractional Hardy inequalities in unbounded John domains. *Mathematika*, 61(2):385–401, 2015.

[12] M. Loss and C. Sloane. Hardy inequalities for fractional integrals on general domains. *J. Funct. Anal.*, 259(6):1369–1379, 2010.

[13] J. Luukkainen. Assouad dimension: antifractal metrization, porous sets, and homogeneous measures. *J. Korean Math. Soc.*, 35(1):23–76, 1998.

[14] R. Musina and A. I. Nazarov. Fractional Hardy–Sobolev inequalities on half spaces. arXiv:1707.02710 [math.AP], 2017.

[15] C. Pérez. Two weighted norm inequalities for Riesz potentials and uniform L^p-weighted Sobolev inequalities. *Indiana Univ. Math. J.*, 39(1):31–44, 1990.

[16] C. A. Sloane. A fractional Hardy-Sobolev-Maz'ya inequality on the upper halfspace. *Proc. Amer. Math. Soc.*, 139(11):4003–4016, 2011.

[17] J. Väisälä. Exhaustions of John domains. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 19(1):47–57, 1994.

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