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# Assouad type dimensions: Examples and applications

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# 1. Assouad dimensions

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Let $E \subset \mathbb{R}^n$ and write $d(E) = \mathrm{diam}(E)$.

Consider all exponents $\lambda \geq 0$ for which there is $C > 0$ such that $E \cap B(x, R)$ can be covered by at most $C(\frac{r}{R})^{-\lambda}$ balls of radius $r$ for all $0 < r < R$ $(< d(E))$ and all $x \in E$.

The infimum of such $\lambda$ is the **Assouad dimension** $\dim_{\mathrm{A}}(E)$.

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The infimum of such $\lambda$ is the **(upper) Assouad dimension** $\overline{\dim}_{\mathrm{A}}(E)$.

Conversely: Consider all $\lambda \geq 0$ for which there is $C > 0$ such that if $0 < r < R < d(E)$, then for every $x \in E$ at least $C(\frac{r}{R})^{-\lambda}$ balls of radius $r$ are needed to cover $E \cap B(x, R)$.

The supremum of such $\lambda$ is the **lower (Assouad) dimension** $\underline{\dim}_{\mathrm{A}}(E)$.

# Some comments on Assouad dimensions

(Upper) Assouad dimension was introduced by P. Assouad around 1980 in connection to bi-Lipschitz embedding problem between metric and Euclidean spaces. Equivalent (or closely related) concepts have appeared under different names, e.g. (uniform) metric dimension.

A nice account on the basic properties and history of (upper) Assouad dimension is given in [Luukkainen 1998].

Lower Assouad dimension has (essentially) appeared under names lower dimension, minimal dimensional number, and uniformity dimension. An early reference is [Larman 1967].

Some basic properties of this dimension have been discussed in [Fraser 2014] and [Käenmäki–L–Vuorinen 2013].

# Minkowski and Assouad

Recall:

$\overline{\dim}_{\mathrm{A}}(E)$ is the infimum of $\lambda \geq 0$ s.t. $E \cap B(x, R)$ can always be covered by at most $C(\frac{r}{R})^{-\lambda}$ balls of radius $0 < r < R < d(E)$

$\underline{\dim}_{\mathrm{A}}(E)$ is the supremum of $\lambda \geq 0$ s.t. always at least $C(\frac{r}{R})^{-\lambda}$ balls of radius $0 < r < R < d(E)$ are needed to cover $E \cap B(x, R)$.

For comparison, the **upper** and **lower Minkowski** (or **box**) **dimensions** of a bounded set $E \subset \mathbb{R}^n$ can be defined as follows:

$\overline{\dim}_{\mathrm{M}}(E)$ is the infimum of $\lambda \geq 0$ s.t. $E$ can be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < d(E)$

$\underline{\dim}_{\mathrm{M}}(E)$ is the supremum of $\lambda \geq 0$ s.t. at least $Cr^{-\lambda}$ balls of radius $0 < r < d(E)$ are needed to cover $E$.

Thus $\quad \underline{\dim}_{\mathrm{A}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E).$

# Examples

General idea: Assouad dimensions reflect the extreme behavior of sets and take into account all scales $0 < r < d(E)$.

- Let $E = \{0\} \cup [1,2] \subset \mathbb{R}$. Then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ ( $\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = 1$ ).

- $\underline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 0$ and $\overline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 1$.

- Let
$$E = \{(\tfrac{1}{j}, 0, \ldots, 0) : j \in \mathbb{N}\} \cup \{(0,0,\ldots,0)\} \subset \mathbb{R}^n.$$

Then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ $\left( \underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = \frac{1}{2} \right)$.

# 2. Assouad dimensions of inhomogeneous self-similar sets

# Self-similar sets with condensation

Let $\{\varphi_i\}_{i=1}^N$ be an iterated function system (IFS) of contractive similitudes on $\mathbb{R}^n$ and let $E$ be the assiociated self-similar set,

$$E = \bigcup_{i=1}^N \varphi_i(E).$$

If $C \subset \mathbb{R}^n$ is compact, then there is unique compact set $E_C \subset \mathbb{R}^n$, called the **inhomogeneous self-similar set with condensation** $C$, such that

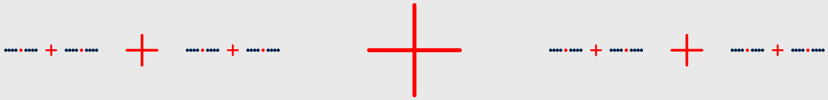
$$E_C = \bigcup_{i=1}^N \varphi_i(E_C) \cup C.$$

# Example

Let $\varphi_1, \varphi_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$\varphi_1(x) = \tfrac{1}{3}x, \qquad \varphi_2(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, 0).$$

Then $E$ is the usual $\frac{1}{3}$-Cantor set (as a subset of the plane $\mathbb{R}^2$).



If $C = +$, centered at $(\frac{1}{2}, 0)$, we obtain an inhomogeneous self-similar set $E_C$ as above.

# Dimensions of self-similar sets with condensation

Typically $\dim(E_C) = \max\{\dim(E), \dim(C)\}$.

For instance, by the countable stability it always holds that

$$\dim_{\mathrm{H}}(E_C) = \max\{\dim_{\mathrm{H}}(E), \dim_{\mathrm{H}}(C)\}.$$

Olsen and Snigireva [2007] showed that under a strong separation condition

$$\overline{\dim}_{\mathrm{M}}(E_C) = \max\{\overline{\dim}_{\mathrm{M}}(E), \overline{\dim}_{\mathrm{M}}(C)\}.$$

Fraser [2012] proved this under the open set condition (OSC) for the IFS, and in [Baker–Fraser–Máthé 2017] the same was obtained under a weak separation property.

Fraser [2012] also showed that under OSC

$$\max\{\underline{\dim}_{\mathrm{M}}(E), \underline{\dim}_{\mathrm{M}}(C)\} \le \underline{\dim}_{\mathrm{M}}(E_C) \le \max\{\underline{\dim}_{\mathrm{M}}(E), \overline{\dim}_{\mathrm{M}}(C)\},$$

but that there is no hope of an equality under any separation condition.

# COSC

The inhomogeneous self-similar set $E_C$ satisfies the **condensation open set condition** (COSC), if there is an open set $U$ such that $\varphi_i(U) \subset U$ for all $i$,

$$\varphi_i(U) \cap \varphi_j(U) = \emptyset \text{ whenever } i \neq j,$$

and

$$C \subset U \setminus \bigcup_{i=1}^{N} \overline{\varphi_i(U)}.$$

Without the last condition, this is the usual open set condition (OSC).

Roughly speaking, COSC guarantees that there is a lot of separation.

# Assouad dimensions of sets with condensation

Under COSC we have:

**Theorem 1 (Käenmäki–L, 2017).**

*Let $\{\varphi_i\}$ be a similitude IFS satisfying the COSC with condensation $C$, and let $E$ be the associated self-similar set. Then*

$$\overline{\dim}_A(E_C) = \max\{\overline{\dim}_A(E), \overline{\dim}_A(C)\}.$$

**Theorem 2 (Käenmäki–L, 2017).**

*Let $\{\varphi_i\}$ be a similitude IFS satisfying the COSC with non-empty condensation $C$, and let $E$ be the associated self-similar set. Then*

$$\underline{\dim}_A(E_C) = \underline{\dim}_A(C).$$

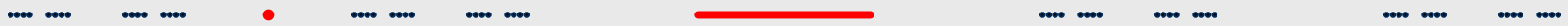
# Necessity of the COSC (Thm 2)

**Theorem 2**: $\underline{\dim}_{\mathrm{A}}(E_C) = \underline{\dim}_{\mathrm{A}}(C)$.

Inequality $\underline{\dim}_{\mathrm{A}}(E_C) \geq \underline{\dim}_{\mathrm{A}}(C)$ is always valid.
(Note that this is not completely trivial since $\underline{\dim}_{\mathrm{A}}$ is not monotone).

For the converse, COSC can not be removed (or replaced with OSC).

$E \cup C$



Indeed, let $E \subset [0,1]$ be the usual $\frac{1}{3}$-Cantor set with $\dim(E) = \frac{\log 2}{\log 3}$, corresponding to IFS $\varphi_1, \varphi_2 \colon \mathbb{R} \to \mathbb{R}$, $\varphi_1(x) = \frac{1}{3}x$, $\varphi_2(x) = \frac{1}{3}x + \frac{2}{3}$.

Let $C = [\frac{4}{9}, \frac{5}{9}] \cup \{\frac{1}{6}\}$. Then $\underline{\dim}_{\mathrm{A}}(C) = 0$.

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Let $C = [\frac{4}{9}, \frac{5}{9}] \cup \{\frac{1}{6}\}$. Then $\underline{\dim}_{\mathrm{A}}(C) = 0$. Note that $E_C$ does not satisfy the COSC due to overlaps. When $x \in E_C$ and $0 < R \leq 1$, the ball $B(x,R)$ contains an interval of length $cR$ (with $c$ independent of $x$ and $R$), and hence $\underline{\dim}_{\mathrm{A}}(E_C) = 1 > 0 = \underline{\dim}_{\mathrm{A}}(C)$.

# Necessity of the COSC (Thm 1)

**Theorem 1**: $\overline{\dim}_{\mathrm{A}}(E_C) = \max\{\overline{\dim}_{\mathrm{A}}(E), \overline{\dim}_{\mathrm{A}}(C)\}$.

Inequality $\overline{\dim}_{\mathrm{A}}(E_C) \geq \max\{\overline{\dim}_{\mathrm{A}}(E), \overline{\dim}_{\mathrm{A}}(C)\}$ is always valid, by the monotonicity of $\overline{\dim}_{\mathrm{A}}$.


For the converse, COSC can not be removed (or replaced with OSC). Indeed, let $E$ be the $\frac{1}{3}$-Cantor set as before, corresponding to $\varphi_1, \varphi_2$, and let

$$C = \left\{ \left(1 + \tfrac{1}{j+1}\right) 3^{-j} : j \in \mathbb{N} \right\} \cup \{0\}.$$

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Then $E_C$ does not satisfy the COSC. It is easy to see that $\overline{\dim}_{\mathrm{A}}(C) = 0$.
For each $k \geq 2$, $E_C$ contains a dilated copy of the set
$W_k = \{\frac{1}{j} : j \in \{2, 3, \ldots, k\}\}$, whence $\overline{\dim}_{\mathrm{A}}(E_C) \geq \overline{\dim}_{\mathrm{A}}\left(\bigcup_{k=2}^{\infty} W_k\right) = 1$.
Thus $\overline{\dim}_{\mathrm{A}}(E_C) = 1 > \frac{\log 2}{\log 3} = \max\{\overline{\dim}_{\mathrm{A}}(E), \overline{\dim}_{\mathrm{A}}(C)\}$.

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Thus $\overline{\dim}_{\mathrm{A}}(E_C) = 1 > \frac{\log 2}{\log 3} = \max\{\overline{\dim}_{\mathrm{A}}(E), \overline{\dim}_{\mathrm{A}}(C)\}$.

## 3. Applications of Assouad dimensions

# $A_p$ weights

Function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a weight if $w(x) > 0$ for a.e. $x \in \mathbb{R}^n$.

A weight $w$ belongs to the Muckenhoupt class $A_p$, $1 < p < \infty$, if there is $C > 0$ such that

$$\left( \fint_B w(x)\,dx \right) \left( \fint_B w(x)^{-\frac{1}{p-1}}\,dx \right)^{p-1} \le C$$

for all balls $B \subset \mathbb{R}^n$, and $w$ is in $A_1$ if there is $C > 0$ such that

$$\left( \fint_B w(x)\,dx \right) \operatorname*{ess\,sup}_{x \in B} \frac{1}{w(x)} \le C$$

for all balls $B \subset \mathbb{R}^n$.

(Here $\fint_B f(x)\,dx = \frac{1}{|B|} \int_B f(x)\,dx$ is the mean-value integral.)

# Properties of $A_p$ weights

Consequences of the $A_p$ condition (well known):

- $A_1 \subset A_p \subset A_q \subset A_\infty = \bigcup_{1\le p<\infty} A_p$ when $1 < p < q < \infty$.

- Duality: If $1 < p < \infty$, then $w \in A_p$ if and only if $w^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}}$.

- Hardy–Littlewood maximal operator $M$ is bounded on $L^p(w\,dx)$, $1 < p < \infty$, if and only if $w \in A_p$. This implies a rich theory of harmonic analysis in $A_p$-weighted spaces.

- $A_p$ weights satisfy a reverse Hölder inequality, and hence the $A_p$ condition is self-improving.

- $A_p$-weights are $p$-admissible, that is, they satisfy the basic assumptions of analysis on metric spaces: the doubling property and a $p$-Poincaré inequality.

# $A_p$ properties of distance functions

Concrete examples of $A_p$ weights? The following is well known:

Let $w(x) = |x|^{-\alpha}$ in $\mathbb{R}^n$, for $\alpha \in \mathbb{R}$.

Then

(a) $w \in A_p$, $1 < p < \infty$, if and only if $(1-p)n < \alpha < n$.
(b) $w \in A_1$ if and only if $0 \leq \alpha < n$.

Here $w(x) = |x|^{-\alpha} = \operatorname{dist}(x, \{0\})^{-\alpha}$.

More generally, we are interested in the $A_p$ properties of weights

$$w(x) = \operatorname{dist}(x, E)^{-\alpha} \quad \text{for (closed) } E \subset \mathbb{R}^n,\ \alpha \in \mathbb{R}.$$

# Aikawa condition

The following property provides a link between the $A_p$ condition and the Assouad dimension:

A (closed) set $E \subset \mathbb{R}^n$ satisfies the **Aikawa condition** for $\alpha \in \mathbb{R}$, if there is $C \geq 1$ such that

$$\fint_{B(x,r)} \operatorname{dist}(y,E)^{-\alpha}\,dy \leq C_\alpha r^{-\alpha}$$

for all $x \in E$ and all $r > 0$.

(We use the convention that $0^0 = 1$, and if $\alpha > 0$ we require that $|E| = 0$.)

Let $w(x) = \operatorname{dist}(x,E)^{-\alpha}$. It is easy to see that if the Aikawa condition holds with $\alpha \geq 0$, then $w \in A_1 \subset A_p$ for all $1 \leq p \leq \infty$.

In addition, by duality, if $\alpha < 0$ and $1 < p < \infty$ are such that the Aikawa condition holds with $\frac{-\alpha}{p-1}$, then $w \in A_p$.

# Assouad and Aikawa

On the other hand, we have:

**Theorem 3 (L–Tuominen, 2013).**

*Let $E \subset \mathbb{R}^n$ be a closed set and let $\alpha > 0$. Then the Aikawa condition holds for $\alpha$ if and only if $\overline{\dim}_A(E) < n - \alpha$.*

The proof follows from rather simple covering arguments, but to get the strict inequality also the self-improvement of the Aikawa condition is needed.

# A characterization

The connection between Assouad and Aikawa implies sufficient conditions for the inclusion $w \in A_p$ in terms of the (upper) Assouad dimension.

Moreover, for porous sets we obtain even characterizations.

**Theorem 4 (Dyda–Ihnatsyeva–L–Tuominen–Vähäkangas, 2017).**

*Assume that a closed set $E \subset \mathbb{R}^n$ is porous (equivalently $\overline{\dim}_A(E) < n$). Let $\alpha \in \mathbb{R}$ and write $w(x) = \operatorname{dist}(x, E)^{-\alpha}$. Then*

*(a)* $w \in A_p$, *for* $1 < p < \infty$, *if and only if*

$$(1-p)(n - \overline{\dim}_A(E)) < \alpha < n - \overline{\dim}_A(E).$$

*(b)* $w \in A_1$ *if and only if* $0 \le \alpha < n - \overline{\dim}_A(E)$.

Note: For instance here it is important that in the definition of $\overline{\dim}_A(E)$ all radii $0 < r < R < \infty$ are considered if $E$ is unbounded.

# Hardy–Sobolev inequalities

We say that an open set $\Omega \subsetneq \mathbb{R}^n$ admits a $(q,p)$-**Hardy–Sobolev inequality** if there is $C > 0$ such that

$$\left( \int_\Omega |u(x)|^q \operatorname{dist}(x, \partial\Omega)^{\frac{q}{p}(n-p)-n} \, dx \right)^{1/q} \leq C \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}$$

for all $u \in C_0^\infty(\Omega)$.

These inequalities form a natural interpolating scale between the Sobolev (case $q = p^* = \frac{np}{n-p}$) and Hardy inequalities (case $q = p$).

The $p$-Hardy inequality reads as

$$\int_\Omega |u(x)|^p \operatorname{dist}(x, \partial\Omega)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx.$$

# Sufficient conditions for Hardy–Sobolev inequalities

Using general machinery of $A_p$-weighted embeddings, due to [Muckenhoupt–Wheeden 1974] and [Pérez 1990], together with the above $A_p$-properties of distance functions, we obtain:

**Theorem 5 (L–Vähäkangas, 2016).**

*Let $1 \le p \le q < np/(n-p) < \infty$ and let $E \subset \mathbb{R}^n$ be a closed set. Then the global $(q,p)$-Hardy–Sobolev inequality*

$$\left( \int_{\mathbb{R}^n} |u(x)|^q \operatorname{dist}(x,E)^{\frac{q}{p}(n-p)-n} \, dx \right)^{1/q} \le C \left( \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \right)^{1/p}$$

*holds for all $u \in C_0^\infty(\mathbb{R}^n)$ if and only if $\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n-p)$.*

In particular, the open set $\Omega = \mathbb{R}^n \setminus E$ admits a $p$-Hardy inequality if $\overline{\dim}_{\mathrm{A}}(E) < n-p$.

# Sufficient conditions for Hardy inequalities

On the other hand, known sufficient conditions for the $p$-Hardy inequality, such as **uniform $p$-fatness** of $\Omega^c = \mathbb{R}^n \setminus \Omega$, can be formulated in terms of the lower Assouad dimension. Indeed, a set $E \subset \mathbb{R}^n$ is uniformly $p$-fat if and only if $E$ is unbounded and $\underline{\dim}_A(E) > n - p$ (observed in [Käenmäki–L–Vuorinen 2013]).

Together with the sufficient condition $\overline{\dim}_A(\Omega^c) < n - p$ from the previous slide we thus have

**Theorem 6 (L, 2017).**

*Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set. If*

$$\overline{\dim}_A(\Omega^c) < n - p \quad \text{or} \quad \underline{\dim}_A(\Omega^c) > n - p,$$

*then $\Omega$ admits a p-Hardy inequality;*
*in the latter case, if $\Omega$ is unbounded, then also $\Omega^c$ has to be unbounded.*

# Necessary conditions for Hardy inequalities

There are also necessary conditions complementing the above sufficient conditions for $p$-Hardy inequalities. These are (essentially) due to [Koskela–Zhong, 2003]; see also [L. 2008].

**Theorem 7.**

*Let $1 < p < \infty$ and assume that $\Omega \subset \mathbb{R}^n$ admits a p-Hardy inequality. Then*

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < n - p \quad \text{or} \quad \dim_{\mathrm{H}}(\Omega^c) > n - p.$$

Recall that, by the previous theorem, $\underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p$ is sufficient for the $p$-Hardy inequality, and that $\underline{\dim}_{\mathrm{A}}(\Omega^c) \leq \dim_{\mathrm{H}}(\Omega^c)$ since $\Omega^c$ is closed.

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(However, it is not possible to change $\underline{\dim}_{\mathrm{A}}(\Omega^c) \leftrightarrow \dim_{\mathrm{H}}(\Omega^c)$ in either of these two, so we do not have a characterization.)

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