FRACTIONAL MAXIMAL FUNCTIONS
IN METRIC MEASURE SPACES

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ABSTRACT. We study the mapping properties of fractional maximal operators in Sobolev and Campanato spaces in metric measure spaces. We show that, under certain restrictions on the underlying metric measure space, fractional maximal operators improve the Sobolev regularity of functions and map functions in Campanato spaces to Hölder continuous functions. We also give an example of a space where fractional maximal function of a Lipschitz function fails to be continuous.

1. INTRODUCTION

Fractional maximal operators are standard tools in partial differential equations, potential theory and harmonic analysis. In the Euclidean case they are studied for example in [1], [2], [3], [17], [18], [19], [22], and in the metric setting in [8], [9], [10], [15], [16], [24], [26], [27]. In the Euclidean case, the fractional maximal operator $\mathcal{M}_\alpha$, defined by

$$\mathcal{M}_\alpha u(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |u(y)| \, dy,$$

has similar smoothing properties as the Riesz potential, see Kinnunen and Saksman [17]. In this paper, we generalize the Euclidean results of [17] to the metric setting under an annular decay property, which places certain restrictions on the geometry of the space X. More precisely, in Theorem 4.2 we show that the fractional maximal function of an L^p-function, $p > 1$, belongs to a (pointwise fractional) Sobolev space $M^{s,p}(X)$, where s depends on the annular decay. Another example of a smoothing property is given in Theorem 4.5, where we show that the fractional maximal operator maps the Sobolev space $M^{1,p}(X)$ to a slightly better Sobolev space $M^{1,p^*}(X)$, where p^* is the conjugate exponent of p. The proof of this result is based on an unpublished proof of MacManus [21] for the Sobolev boundedness of the usual Hardy-Littlewood maximal operator.

We also study the action of the fractional maximal function in Campanato spaces. In this context our main result is Theorem 3.1, which states that the fractional maximal function maps functions in Campanato spaces to Hölder continuous functions, provided that the space satisfies the annular decay property. The result is new even in the Euclidean case. This is again analogous to the well known properties of the Riesz potential, studied in the metric setting for example in [6] and [7]. Note

here that the Campanato estimates for the Riesz potential do not immediately imply the corresponding oscillation estimates for the fractional maximal function. In particular, Theorem 3.1 shows that the fractional maximal operator maps Hölder continuous functions to Hölder continuous functions with a better exponent, and functions of bounded mean oscillation to Hölder continuous functions.

A part of the motivation for our work comes from [15], where it was shown that similar mapping properties hold for the so called discrete fractional maximal operator even without the annular decay property. In contrast to those results, we conclude this paper with two examples which verify that there is a real obstruction in the study of fractional maximal functions in metric measure spaces. We modify the example given by Buckley in [4] for the standard Hardy-Littlewood maximal function and show that the fractional maximal function of a Lipschitz continuous function may fail to be continuous if the space does not satisfy annular decay. The same is actually true also for the noncentered fractional maximal function, for which the main results given in Sections 3 and 4 hold as well, as is briefly noted at the end of each theorem.

2. Notation and preliminaries

We assume that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular, doubling outer measure μ, for which the measure of every open set is positive and the measure of each bounded set is finite. The doubling property means that there exists a fixed constant $c_d > 0$, called the doubling constant, such that

$$(2.1) \qquad \mu(B(x, 2r)) \leq c_d \mu(B(x, r))$$

for every ball $B(x, r) = \{y \in X : d(y, x) < r\}$.

We say that the measure μ satisfies a lower bound condition if there exist constants $Q \geq 1$ and $c_l > 0$ such that

$$(2.2) \qquad \mu(B(x, r)) \geq c_l r^Q$$

for all $x \in X$ and $r > 0$.

We follow the standard procedure that the letter C denotes a positive constant whose value is not necessarily the same at each occurrence.

The fractional maximal function. Let $\alpha \geq 0$. The fractional maximal function of a locally integrable function u is

$$(2.3) \qquad \mathcal{M}_\alpha u(x) = \sup_{r > 0} r^\alpha \fint_{B(x,r)} |u| \, d\mu,$$

where $u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu$ is the integral average of u over B. For $\alpha = 0$, we have the usual Hardy-Littlewood maximal function, $\mathcal{M} u = \mathcal{M}_0 u$,

$$\mathcal{M} u(x) = \sup_{r > 0} \fint_{B(x,r)} |u| \, d\mu.$$

Annular decay properties. Let $0 < \delta \leq 1$. We say that the metric measure space X satisfies the δ-annular decay property, if there exists a constant $C > 0$ such that for all $x \in X$, $R > 0$, and $0 < h < R$, we have

$$(2.4) \qquad \mu\big(B(x,R) \setminus B(x, R-h)\big) \leq C\Big(\frac{h}{R}\Big)^\delta \mu(B(x,R)).$$

Similarly, we say that X satisfies the *relative* δ-annular decay property, if there exist a constant $C > 0$ such that for all $x \in X$, $R > 0$, and $0 < h < R$, we have

$$(2.5) \qquad \mu\big(B \cap (B(x,R) \setminus B(x, R-h))\big) \leq C\Big(\frac{h}{r_B}\Big)^\delta \mu(B)$$

for all balls B with radius $r_B \leq 3R$. Note that the relative condition (2.5) implies the δ-annular decay property (2.4).

If X is a geodesic space (or more generally a length space), then X satisfies the relative annular decay for some $\delta > 0$. See for instance [4], [14, Chapter 9] and [25] for this fact and for more information on these and related conditions.

Buckley studied in [4] the action of the usual maximal operator $\mathcal{M}$ on $C^{0,\beta}(X)$, the space of β-Hölder continuous functions equipped with the seminorm

$$\|u\|_{C^{0,\beta}(X)} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x,y)^\beta}.$$

He showed that if a doubling space X satisfies the δ-annular decay property, then

$$(2.6) \qquad \mathcal{M} \colon C^{0,\beta}(X) \to C^{0,\beta}(X)$$

is bounded whenever $0 < \beta \leq \delta$. He also gave an example of a doubling space where the maximal function of a Lipschitz function is not continuous; see Section 5.

3. CAMPANATO SPACES

In this section, we study the action of the fractional maximal operator $\mathcal{M}_\alpha$ on Campanato spaces $\mathcal{L}^{p,\beta}(X)$. A locally integrable function u belongs to $\mathcal{L}^{p,\beta}(X)$, if

$$\|u\|_{\mathcal{L}^{p,\beta}(X)} = \sup r^{-\beta} \Big(\fint_{B(x,r)} |u - u_{B(x,r)}|^p \, d\mu \Big)^{1/p} < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$. It is known that

$$(3.1) \qquad \|u\|_{\mathrm{BMO}(X)} \leq \|u\|_{\mathcal{L}^{p,0}(X)} \leq C\|u\|_{\mathrm{BMO}(X)}$$

and

$$(3.2) \qquad C^{-1}\|u\|_{C^{0,\beta}(X)} \leq \|u\|_{\mathcal{L}^{p,\beta}(X)} \leq \|u\|_{C^{0,\beta}(X)},$$

if $\beta > 0$, where C is independent of p, see for example [23]. Here $\mathrm{BMO}(X) = L^{1,0}(X)$ is the space of functions of bounded mean oscillation.

Theorem 3.1. *Assume that X satisfies the δ-annular decay property (2.4) and that $0 < \alpha \leq \delta$ and $0 \leq \alpha + \beta \leq \delta$. If $u \in \mathcal{L}^{p,\beta}(X)$ and $\mathcal{M}_\alpha u \not\equiv \infty$, then $\mathcal{M}_\alpha u \in C^{0,\alpha+\beta}(X)$, and there is a constant $C > 0$, depending only on the doubling constant and constant of the annular decay, such that*

$$(3.3) \qquad \|\mathcal{M}_\alpha u\|_{C^{0,\alpha+\beta}(X)} \leq C\|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Note that Theorem 3.1 implies that

$$(3.4) \qquad \mathcal{M}_\alpha \colon C^{0,\beta}(X) \to C^{0,\alpha+\beta}(X)$$

and

$$(3.5) \qquad \mathcal{M}_\alpha \colon \mathrm{BMO}(X) \to C^{0,\alpha}(X)$$

are bounded operators, when restricted to functions satisfying $\mathcal{M}_\alpha u \not\equiv \infty$. If the measure μ satisfies the lower bound condition (2.2) and $\alpha - Q/p \geq 0$, then also

$$(3.6) \qquad \mathcal{M}_\alpha \colon L^p(X) \to C^{0,\alpha-Q/p}(X).$$

Notice that (3.4), (3.5) and (3.6), where $\alpha - Q/p > 0$, are analogous to the well known properties of the Riesz potential, studied in the metric setting for example in [6] and [7].

For the proof Theorem 3.1, we need a lemma which is proved using a chaining argument.

Lemma 3.2 ([15], Lemma 7.1.). *Let $x \in X$, $0 < r \leq R$ and $y \in B(x, C_0 R)$, and let $u \in \mathcal{L}^{p,\beta}(X)$. If $\beta < 0$, then*

$$(3.7) \qquad \left|u_{B(y,r)} - u_{B(x,R)}\right| \leq C r^\beta \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

If $\beta = 0$, then

$$(3.8) \qquad \left|u_{B(y,r)} - u_{B(x,R)}\right| \leq C \log \frac{CR}{r} \|u\|_{\mathcal{L}^{p,0}(X)}.$$

The constant C depends only on the doubling constant and C_0.

Proof of Theorem 3.1. Since $|u| \in \mathcal{L}^{p,\beta}(X)$ and $\||u|\|_{\mathcal{L}^{p,\beta}(X)} \leq C\|u\|_{\mathcal{L}^{p,\beta}(X)}$, we may assume that $u \geq 0$. Let $r > 0$ and define $v \colon X \to [0,\infty)$ as $v(x) = r^\alpha u_{B(x,r)}$. We begin by proving the claim for v. Let $x, y \in X$.
Case 1. Assume first that $r \leq 2d(x,y)$. Let $B = B(x, 2d(x,y))$, $B_x = B(x,r)$ and $B_y = B(y,r)$. If $\beta < 0$, then, by (3.7),

$$\begin{aligned}
|v(x) - v(y)| = |r^\alpha u_{B_x} - r^\alpha u_{B_y}| &\leq r^\alpha \left(|u_{B_x} - u_B| + |u_B - u_{B_y}|\right) \\
&\leq C r^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)} \\
&\leq C d(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}.
\end{aligned}$$

Similarly, if $\beta = 0$, estimate (3.8) implies that

$$\begin{aligned}
\left|r^\alpha u_{B_x} - r^\alpha u_{B_y}\right| &\leq C r^\alpha \log \frac{Cd(x,y)}{r} \|u\|_{\mathcal{L}^{p,0}(X)} \\
&= C d(x,y)^\alpha \left(\frac{r}{Cd(x,y)}\right)^\alpha \log \frac{Cd(x,y)}{r} \|u\|_{\mathcal{L}^{p,0}(X)} \\
&\leq C d(x,y)^\alpha \|u\|_{\mathcal{L}^{p,0}(X)},
\end{aligned}$$

where the last inequality follows by the boundedness of the function $f(t) = t^{-\alpha} \log t$ for $t \geq 1$.

If $\beta > 0$, then u is β-Hölder continuous and so

$$\left| r^\alpha u_{B_x} - r^\alpha u_{B_y} \right| \le r^\alpha \fint_{B_x} \fint_{B_y} |u(z) - u(w)| \, dz \, dw$$

$$\le C d(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Case 2. Suppose then that $r > 2d(x,y)$. Let $\Delta_x = B_x \setminus B_y$, $\Delta_y = B_y \setminus B_x$ and $\Delta = \Delta_x \cup \Delta_y$. As in [20], we write

$$\fint_{B_x} u \, d\mu - \fint_{B_y} u \, d\mu = \frac{1}{\mu(B_x)} \left(\int_{B_x} u \, d\mu - \int_{B_y} u \, d\mu + \left(\mu(B_y) - \mu(B_x) \right) u_{B_y} \right)$$

$$= \frac{1}{\mu(B_x)} \left(\int_{\Delta_x} u \, d\mu - \int_{\Delta_y} u \, d\mu + \left(\mu(\Delta_y) - \mu(\Delta_x) \right) u_{B_y} \right)$$

$$= \frac{1}{\mu(B_x)} \left(\int_{\Delta_x} (u - u_{B_y}) \, d\mu - \int_{\Delta_y} (u - u_{B_y}) \, d\mu \right),$$

which implies that

(3.9)
$$\left| \fint_{B_x} u \, d\mu - \fint_{B_y} u \, d\mu \right| \le C \frac{1}{\mu(B_x)} \int_\Delta |u - u_{B_y}| \, d\mu.$$

Suppose that $\beta < 0$. Let $B_1, B_2, \ldots, B_k$ be a maximal collection of disjoint balls of radius $d(x,y)$ centered at Δ. Then $\Delta \subset \cup_i 2B_i$ and $\cup_i B_i \subset \Delta'$, where

$$\Delta' = B(x, r + d(x,y)) \setminus B(x, r - 2d(x,y)) \cup B(y, r + d(x,y)) \setminus B(y, r - 2d(x,y)).$$

For each i, by (3.7) and the Hölder inequality, we have

(3.10)
$$\fint_{2B_i} |u - u_{B_y}| \, d\mu \le |u_{2B_i} - u_{B_y}| + \fint_{2B_i} |u - u_{2B_i}| \, d\mu$$

$$\le C d(x,y)^\beta \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

To estimate the measure of Δ', we first use the annular decay property and doubling to obtain

$$\mu(B(x, r + d(x,y)) \setminus B(x, r - 2d(x,y))) \le C \left(\frac{3d(x,y)}{r + d(x,y)} \right)^\delta \mu(B(x, r + d(x,y)))$$

$$\le C \left(\frac{d(x,y)}{r} \right)^\delta \mu(B_x),$$

and similarly, because $\mu(B_x)$ is comparable with $\mu(B_y)$ by the doubling property and the assumption $r > 2d(x,y)$,

$$\mu(B(y, r + d(x,y)) \setminus B(y, r - 2d(x,y))) \le C \left(\frac{d(x,y)}{r} \right)^\delta \mu(B_x).$$

Thus

(3.11)
$$\mu(\Delta') \le C \left(\frac{d(x,y)}{r} \right)^\delta \mu(B_x).$$

Using (3.9), the facts that $\Delta \subset \cup_i 2B_i$ and $B_i \subset \Delta'$ for all i, (3.10), disjointedness of balls B_i and (3.11), we obtain

$$|v(x) - v(y)| = \left| r^\alpha \fint_{B_x} u \, d\mu - r^\alpha \fint_{B_y} u \, d\mu \right| \leq C \frac{r^\alpha}{\mu(B_x)} \int_\Delta |u - u_{B_y}| \, d\mu$$

$$\leq C \frac{r^\alpha}{\mu(B_x)} \sum_i \mu(2B_i) \fint_{2B_i} |u - u_{B_y}| \, d\mu$$

$$\leq C \frac{r^\alpha d(x,y)^\beta}{\mu(B_x)} \|u\|_{\mathcal{L}^{p,\beta}(X)} \sum_i \mu(B_i)$$

$$\leq C r^\alpha d(x,y)^\beta \frac{\mu(\Delta')}{\mu(B_x)} \|u\|_{\mathcal{L}^{p,\beta}(X)}$$

$$\leq C r^\alpha d(x,y)^\beta \left(\frac{d(x,y)}{r} \right)^\delta \|u\|_{\mathcal{L}^{p,\beta}(X)}$$

$$\leq C d(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)},$$

where the last inequality follows because $r > d(x,y)$ and $0 < \alpha < \delta$.

Assume then that $\beta \geq 0$ and $s > 1$. Let $q = s/(s-1)$. By (3.9), the Hölder inequality, the facts that $\Delta \subset 2B_y$ and $\mu(B_x)$ is comparable with $\mu(B_y)$, and (3.11), we have

$$\left| \fint_{B_x} u \, d\mu - \fint_{B_y} u \, d\mu \right| \leq C \frac{\mu(\Delta)^{1/s}}{\mu(B_x)} \left(\int_\Delta |u - u_{B_y}|^q \, d\mu \right)^{1/q}$$

$$\leq C \left(\frac{\mu(\Delta)}{\mu(B_x)} \right)^{1/s} \left(\fint_{2B_y} |u - u_{2B_y}|^q \, d\mu \right)^{1/q}$$

$$\leq C \left(\frac{d(x,y)}{r} \right)^{\delta/s} r^\beta \|u\|_{\mathcal{L}^{q,\beta}(X)}.$$

Since, by (3.1) and (3.2), $\|u\|_{\mathcal{L}^{q,\beta}(X)} \leq C \|u\|_{\mathcal{L}^{1,\beta}(X)}$, where C is independent of q, letting $s \to 1$, it follows that

$$\left| \fint_{B_x} u \, d\mu - \fint_{B_y} u \, d\mu \right| \leq C \left(\frac{d(x,y)}{r} \right)^\delta r^\beta \|u\|_{\mathcal{L}^{1,\beta}(X)}.$$

Since $r > d(x,y)$ and $0 < \alpha + \beta \leq \delta$, the estimate above implies

$$|v(x) - v(y)| = \left| r^\alpha \fint_{B_x} u \, d\mu - r^\alpha \fint_{B_y} u \, d\mu \right| \leq C r^{\alpha+\beta-\delta} d(x,y)^\delta \|u\|_{\mathcal{L}^{1,\beta}(X)}$$

$$\leq C d(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Finally, we prove the claim for $\mathcal{M}_\alpha u$. We may assume that $\mathcal{M}_\alpha u(x) \geq \mathcal{M}_\alpha u(y)$. Let $\varepsilon > 0$, and let $r > 0$ be such that $r^\alpha u_{B(x,r)} > \mathcal{M}_\alpha u(x) - \varepsilon$. Then, by the first part of the proof,

$$\mathcal{M}_\alpha u(x) - \mathcal{M}_\alpha u(y) \leq r^\alpha u_{B(x,r)} - r^\alpha u_{B(y,r)} + \varepsilon = v(x) - v(y) + \varepsilon$$

$$\leq C d(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)} + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$. $\qquad\square$

Remark 3.3. A modification of the proof above shows that the result holds also for the noncentered fractional maximal function

$$\widetilde{\mathcal{M}}_\alpha u(x) = \sup_{B(z,r) \ni x} r^\alpha \fint_{B(z,r)} |u| \, d\mu.$$

Let $x, y \in X$. We may assume that $u \geq 0$ and $\widetilde{\mathcal{M}}_\alpha u(x) \geq \widetilde{\mathcal{M}}_\alpha u(y)$. Let $\varepsilon > 0$. Then there exists a ball $B(z,r)$ containing x such that

$$\widetilde{\mathcal{M}}_\alpha u(x) < r_\alpha u_{B(z,r)} + \varepsilon.$$

Since $y \in B(z, r + d(x,y))$, we have that

$$\widetilde{\mathcal{M}}_\alpha u(x) - \widetilde{\mathcal{M}}_\alpha u(y) \leq r^\alpha (u_{B(z,r)} - u_{B(z,r+d(x,y))}) + \varepsilon.$$

Arguments similar to those in the proof of Theorem 3.1 imply that

$$r^\alpha (u_{B(z,r)} - u_{B(z,r+d(x,y))}) \leq Cd(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

The claim follows by letting $\varepsilon \to 0$.

4. SOBOLEV SPACES

In this section, we show that the fractional maximal operator $\mathcal{M}_\alpha$ maps L^p-spaces, $p > 1$, to Sobolev spaces, and Sobolev spaces to slightly better Sobolev spaces. We prove the results for Sobolev spaces $M^{s,p}(X)$, defined by a pointwise equation. These spaces were introduced by Hajłasz in [12] for $s = 1$, and the fractional versions by Yang in [28].

Let $s > 0$. We say that a measurable function $g \geq 0$ is *a generalized s-gradient* of a measurable function u, $g \in D^s(u)$, if there is a set $E \subset X$ with $\mu(E) = 0$ such that

$$(4.1) \qquad |u(x) - u(y)| \leq d(x,y)^s \big(g(x) + g(y)\big)$$

for all $x, y \in X \setminus E$. The Sobolev space $M^{s,p}(X)$, $1 \leq p < \infty$, consists of functions $u \in L^p(X)$ for which there exists a function $g \in L^p(X) \cap D^s(u)$. The space $M^{s,p}(X)$, equipped with the norm

$$(4.2) \qquad \|u\|_{M^{s,p}(X)} = \big(\|u\|_{L^p(X)}^p + \inf \|g\|_{L^p(X)}^p\big)^{1/p},$$

where the infimum is taken over all functions $g \in L^p(X) \cap D^s(u)$, is a Banach space [13, Theorem 8.3].

It follows from (4.1) that every $u \in M^{s,p}(X)$ and $g \in D^s(u)$ satisfy the Poincaré inequality

$$(4.3) \qquad \fint_{B(x,r)} |u - u_{B(x,r)}| \leq Cr^s \fint_{B(x,r)} g \, d\mu,$$

where C depends only on s. Hence the assumption that X satisfies a Poincaré inequality is not needed.

We will use the following Sobolev type theorem for the fractional maximal operator. As in [11], it can be proven easily using the Hardy–Littlewood maximal function theorem; see also [5] or [8].

Theorem 4.1. *Assume that the measure μ satisfies the lower bound condition (2.2). Let $p > 1$ and $0 < \alpha < Q/p$. There is a constant $C > 0$, depending only on the doubling constant, constant in the measure lower bound, p and α, such that*

$$\| \mathcal{M}_\alpha u \|_{L^{p^*}(X)} \leq C \|u\|_{L^p(X)},$$

for every $u \in L^p(X)$ with $p^* = Qp/(Q - \alpha p)$.

The following theorem is a generalization of the main result of [17] to the metric setting. It shows that the fractional maximal operator is a smoothing operator. More precisely, the fractional maximal function of an L^p-function u has a generalized gradient, and both $\mathcal{M}_\alpha u$ and the generalized gradient belong to a higher Lebesgue space than u.

Theorem 4.2. *Assume that the measure μ satisfies the lower bound condition (2.2) and that X satisfies the δ-annular decay property (2.4). Assume that $u \in L^p(X)$ with $1 < p < Q$. Let*

$$\delta \leq \alpha < Q/p, \quad p^* = Qp/(Q - \alpha p) \quad and \quad q = Qp/(Q - (\alpha - \delta)p).$$

Then there is a constant $C > 0$ depending only on the doubling constant and the constant of the δ-annular decay property such that $C \mathcal{M}_{\alpha-\delta} u$ is a generalized δ-gradient of $\mathcal{M}_\alpha u$. Moreover,

$$\| \mathcal{M}_\alpha u \|_{L^{p^*}(X)} \leq C \|u\|_{L^p(X)} \quad and \quad \| \mathcal{M}_{\alpha-\delta} u \|_{L^q(X)} \leq C \|u\|_{L^p(X)},$$

where C depends only on the doubling constant, the constant in the measure lower bound, p and α.

Proof. We may assume that $u \geq 0$. Let $x, y \in X$. Assume that $\mathcal{M}_\alpha u(x) \geq \mathcal{M}_\alpha u(y)$. Let $\varepsilon > 0$ and let $r > 0$ such that

$$r^\alpha \fint_{B(x,r)} u \, d\mu > \mathcal{M}_\alpha u(x) - \varepsilon.$$

If $r \leq d(x,y)$, then, as $u \geq 0$,

$$\mathcal{M}_\alpha u(x) - \mathcal{M}_\alpha u(y) \leq r^\delta r^{\alpha-\delta} \fint_{B(x,r)} u \, d\mu + \varepsilon \leq d(x,y)^\delta \mathcal{M}_{\alpha-\delta} u(x) + \varepsilon.$$

If $r > d(x,y)$, we write $a = d(x,y)$ and use the doubling property to obtain

$$\mathcal{M}_\alpha u(x) - \mathcal{M}_\alpha u(y) < r^\alpha \fint_{B(x,r)} u \, d\mu + \varepsilon - (r+a)^\alpha \fint_{B(y,r+a)} u \, d\mu$$

$$\leq r^\alpha \left(\frac{1}{\mu(B(x,r))} - \frac{1}{\mu(B(y,r+a))} \right) \int_{B(x,r)} u \, d\mu + \varepsilon$$

$$= r^\alpha \frac{\mu(B(y,r+a)) - \mu(B(x,r))}{\mu(B(y,r+a))} \fint_{B(x,r)} u \, d\mu + \varepsilon$$

$$\leq C r^\alpha \frac{\mu(B(x,r+2a) \setminus B(x,r))}{\mu(B(x,r+2a))} \fint_{B(x,r)} u \, d\mu + \varepsilon.$$

The δ-annular decay property together with the assumption $r > d(x, y)$ implies that

$$\frac{\mu(B(x, r + 2a) \setminus B(x, r))}{\mu(B(x, r + 2a))} \leq K \left(\frac{a}{r + 2a}\right)^{\delta} \leq K \left(\frac{d(x, y)}{r}\right)^{\delta},$$

and hence

$$\mathcal{M}_{\alpha} u(x) - \mathcal{M}_{\alpha} u(y) \leq Cd(x, y)^{\delta} \mathcal{M}_{\alpha - \delta} u(x) + \varepsilon.$$

By letting $\varepsilon \to 0$ and changing the roles of x and y, we have that

$$|\mathcal{M}_{\alpha} u(x) - \mathcal{M}_{\alpha} u(y)| \leq Cd(x, y)^{\delta} \left(\mathcal{M}_{\alpha - \delta} u(x) + \mathcal{M}_{\alpha - \delta} u(y)\right).$$

Hence $\mathcal{M}_{\alpha - \delta} u$ is a generalized δ-gradient of $\mathcal{M}_{\alpha} u$. The norm estimates for $\mathcal{M}_{\alpha} u$ and $\mathcal{M}_{\alpha - \delta} u$ follow from Theorem 4.1. $\square$

Remark 4.3. With the assumptions of Theorem 4.2, $\mathcal{M}_{\alpha} u \in M_{\mathrm{loc}}^{\delta, q}(X)$ and

$$\|\mathcal{M}_{\alpha} u\|_{M^{\delta, q}(A)} \leq \mu(A)^{1/q - 1/p^*} \|u\|_{L^p(A)}$$

for all open sets $A \subset X$ with $\mu(A) < \infty$.

Remark 4.4. The above proof with the ball $B(x, r)$ replaced by a ball $B(z, r)$ for which $\widetilde{\mathcal{M}} u(x) < r^{\alpha} u_{B(z, r)} + \varepsilon$ and $B(y, r + a)$ replaced by the ball $B(z, r + a)$ shows that the result holds also for the noncentered fractional maximal function.

In the next theorem, we show that if u is a Sobolev function, then its fractional maximal function belongs to a Sobolev space with the Sobolev conjugate exponent. The proof is a modification of the result that the usual Hardy–Littlewood maximal operator is bounded in Sobolev spaces if the underlying space satisfies the relative 1-annular decay property. Since the original proof of MacManus [21] (for $\alpha = 0$) is unpublished, we give in the below proof all the details.

Theorem 4.5. *Assume that the measure μ satisfies the lower bound condition and X satisfies the relative 1-annular decay property (2.5). Let $p > 1$, $u \in M^{1,p}(X)$ and $0 < \alpha < Q/p$. Then $\mathcal{M}_{\alpha} u \in M^{1,p^*}(X)$ with $p^* = Qp/(Q - \alpha p)$ and there is a constant $C > 0$, depending only on the doubling constant, the constant in the measure lower bound, p and α, such that*

(4.4) $$\|\mathcal{M}_{\alpha} u\|_{M^{1,p^*}(X)} \leq C \|u\|_{M^{1,p}(X)}.$$

Proof. Let $u \in M^{1,p}(X)$ and let $g \in L^p(X)$ be a generalized gradient of u. We may assume that $u \geq 0$ since $|u| \in M^{1,p}(X)$ and g is a generalized gradient of $|u|$. Fix $1 < q < p$ and define

$$\tilde{g} = \left(\mathcal{M}_{\alpha q}(g^q)\right)^{1/q}.$$

Since $p/q > 1$, Theorem 4.1 implies that

$$\|\tilde{g}\|_{p^*} = \|\mathcal{M}_{\alpha q}(g^q)\|_{Q \frac{p}{q} / (Q - (\alpha q) \frac{p}{q})}^{1/q} \leq C \|g^q\|_{\frac{p}{q}}^{1/q} = C \|g\|_p.$$

If we can show that $\tilde{g}$ is a generalized gradient of $\mathcal{M}_{\alpha} u$, then this together with Theorem 4.1 for u implies norm estimate (4.4).

We are going to show that

$$(4.5) \qquad \left| r^\alpha \fint_{B(x,r)} u \, d\mu - r^\alpha \fint_{B(y,r)} u \, d\mu \right| \le C \, d(x,y) \, (\tilde{g}(x) + \tilde{g}(y))$$

for almost all $x, y \in X$ and all $r > 0$. This implies that

$$(4.6) \qquad |\mathcal{M}_\alpha u(x) - \mathcal{M}_\alpha u(y)| \le C \, d(x,y) \, (\tilde{g}(x) + \tilde{g}(y))$$

for almost every $x, y \in X$, which proves our theorem.

The proof of (4.5) is easy if $r \le 3 \, d(x,y)$: Since g is a generalized gradient of u,

$$|u(z) - u(w)| \le C \, d(z,w) \, (g(z) + g(w)) \le C \, d(x,y) \, (g(z) + g(w))$$

for almost all $z \in B(x,r)$ and $w \in B(y,r)$. By integrating both sides with respect to z and w and using the Hölder inequality, we obtain

$$\left| \fint_{B(x,r)} u \, d\mu - \fint_{B(y,r)} u \, d\mu \right| \le C \, d(x,y) \left(\left(\fint_{B(x,r)} g^q \, d\mu \right)^{1/q} + \left(\fint_{B(y,r)} g^q \, d\mu \right)^{1/q} \right).$$

Now (4.5) follows by multiplying both sides by r^α and using the definition of $\tilde{g}$.

Suppose then that $r > 3 \, d(x,y)$. Let $\Delta_y = B(y,r) \setminus B(x,r)$, $\Delta_x = B(x,r) \setminus B(y,r)$ and $\Delta = \Delta_y \cup \Delta_x$. As in the proof of Theorem 3.1, we have

$$\fint_{B(y,r)} u \, d\mu - \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(y,r))} \int_{\Delta_y} (u - u_{B(x,r)}) \, d\mu - \int_{\Delta_x} (u - u_{B(x,r)}) \, d\mu,$$

and hence

$$\left| \fint_{B(x,r)} u \, d\mu - \fint_{B(y,r)} u \, d\mu \right| \le \frac{1}{\mu(B(y,r))} \int_\Delta |u - u_{B(x,r)}| \, d\mu.$$

Now, let

$$A = B(x, r + d(x,y)) \setminus B(x, r - d(x,y)).$$

Since the balls $B(x,r)$ and $B(y,r)$ have comparable measures and $\Delta \subset A$,

$$\left| \fint_{B(x,r)} u \, d\mu - \fint_{B(y,r)} u \, d\mu \right| \le \frac{C}{\mu(B(x,r))} \int_A |u - u_{B(x,r)}| \, d\mu.$$

We want to show that the right-hand side is bounded by $C \, d(x,y) \, (\fint_{B(x,5r)} g^q \, d\mu)^{1/q}$, which implies (4.5). For that, we prove the following estimate for the integral over the annulus.

Claim:

$$\frac{1}{\mu(B(x,r))} \int_A |u - u_{B(x,r)}| \, d\mu \le C \, d(x,y) \fint_{B(x,5r)} g(w) \log \frac{5 \, r}{|r - d(w,x)|} \, d\mu$$

To prove the claim, define $r_k = 3^k r$ and $B_k(z) = B(z, r_k)$, $k \in \mathbb{Z}$.

We use a standard chaining argument, Poincaré inequality (4.3), and the fact that $B(x,r) \subset B_1(z)$ for each $z \in A$ to see that

$$|u(z) - u_{B(x,r)}| \le |u(z) - u_{B_1(z)}| + |u_{B_1(z)} - u_{B(x,r)}| \le C \sum_{k \le 1} r_k \fint_{B_k(z)} g \, d\mu.$$

for all Lebesgue points $z \in A$. (Since almost every point is a Lebesgue point of u, this holds for almost all $z \in A$.)

Integration of both sides over A and a use of the Fubini theorem yield

$$\int_A |u(z) - u_{B(x,r)}| \, d\mu(z) \le C \int_X g(w) \, K(w) \, d\mu(w),$$

where

$$K(w) = \int_A \left(\sum_{k \le 1} \frac{r_k}{\mu(B_k(z))} \chi_{B_k(z)}(w) \right) d\mu(z).$$

Since $k \le 1$ we have that $K(w) = 0$ when $w \notin B(x, 5r)$. This implies that

$$\int_A |u(z) - u_{B(x,r)}| \, d\mu(z) \le C \int_{B(x,5r)} g(w) \, K(w) \, d\mu(w),$$

where, by that fact that $\chi_{B_k(z)}(w) = \chi_{B_k(w)}(z)$,

$$K(w) = \int_A \left(\sum_{k \le 1} \frac{r_k}{\mu(B_k(z))} \chi_{B_k(w)}(z) \right) d\mu(z).$$

If $z \in B_k(w)$, then the balls $B_k(z)$ and $B_k(w)$ have comparable measures. Thus

$$\int_A \frac{r_k}{\mu(B_k(z))} \chi_{B_k(w)}(z) \, d\mu(z) \le C r_k \frac{\mu(A \cap B_k(w))}{\mu(B_k(w))}$$

from which we obtain that

$$K(w) \le C \sum_{k \le 1} r_k \frac{\mu(A \cap B_k(w))}{\mu(B_k(w))}.$$

It follows from the relative 1-annular decay that each term in the above sum is bounded by $C \min\{r_k, d(x,y)\}$. Moreover, for the indices $k \le 1$ for which $B_k(w) \cap A = \emptyset$, the terms are zero. Now

$$K(w) \le C \sum_{k \in \mathcal{K}_1} r_k + C \sum_{k \in \mathcal{K}_2} d(x,y),$$

where $\mathcal{K}_1 = \{k \le 1 : r_k \le d(x,y)\}$ and $\mathcal{K}_2 = \{k \le 1 : r_k > d(x,y), B_k(w) \cap A \ne \emptyset\}$. The first term at most $Cd(x,y)$. An upper bound for the second term is

$$(4.7) \qquad\qquad Cd(x,y) \log \frac{5r}{|r - d(w,x)|}.$$

To see this, we consider two cases. Assume first that $|r - d(w,x)| < 3d(x,y)$. Since $r_k > d(x,y)$, we have that

$$k > \frac{1}{\log 3} \log \frac{d(x,y)}{r} > \frac{1}{\log 3} \log \frac{|r - d(w,x)|}{3r},$$

which implies that the amount of such indices in the sum is less than

$$C \log \frac{5r}{|r - d(w,x)|}.$$

The second case, $|r - d(w,x)| \geq 3\,d(x,y)$ can be split to two parts. When $w \in B(x, r - 3d(x,y))$, the requirement that $B_k(w) \cap A \neq \emptyset$ implies that

$$3^k r = r_k \geq r - d(x,y) - d(w,x),$$

and hence we have at most

$$C \log \frac{r}{r - d(x,y) - d(w,x)} \leq C \log \frac{2r}{|r - d(w,x)|}$$

such terms. (Note that the condition $B_k(w) \cap A \neq \emptyset$ gives the number of terms, not the condition $r_k > h$.) Similarly, when $w \in 5B(x,r) \setminus B(x, r + 3d(x,y))$, we must have

$$3^k r = r_k \geq d(w,x) - (r + d(x,y)),$$

and the upper bound for the number of terms follows just as in the first case. This implies that the second term is bounded above by (4.7). Since $0 \leq d(w,x) < 5r$, we have that $\log \frac{5r}{|r - d(w,x)|} \geq \log \frac{5}{4}$, and hence the claim follows.

Now, by the claim and the Hölder inequality,

$$\frac{1}{\mu(B(x,r))} \int_A |u - u_{B(x,r)}| \, d\mu \leq C d(x,y) \left(\fint_{B(x,5r)} g^q \, d\mu \right)^{1/q} \left(\fint_{B(x,5r)} L^{q'} \, d\mu \right)^{1/q'},$$

where q' is the conjugate exponent of q and

$$L(w) = \log \frac{5r}{|r - d(w,x)|}.$$

To estimate the integral of $L^{q'}$ over $B(x, 5r)$, we define for each $i = 0, 1, \ldots$

$$A_i = \{ w \in B(x,5r) : 4^{-i} r \leq |r - d(w,x)| < 4^{-i+1} r \}.$$

The sets A_i are disjoint, and on each A_i we have

$$L(w) \leq C(1 + i).$$

Moreover, the set $\{ w \in B(x,5r) : |r - d(w,x)| = 0 \}$ has measure zero by the relative annular decay, and so

$$\mu \left(B(x, 5r) \setminus \bigcup_{i=0}^{\infty} A_i \right) = 0.$$

It follows that

$$\fint_{B(x,5r)} L^{q'} \, d\mu \leq C \sum_{i=0}^{\infty} \frac{(1+i)^{q'} \mu(A_i)}{\mu(B(x,r))}.$$

For $i \geq 2$, the set A_i consists of two annuli of thickness $4^{-i} r$ centered at x. The inner and outer radii of these annuli are comparable to r. Thus the relative 1-annular decay implies that

$$\mu(A_i) \leq C \, 4^{-i} \mu(B(x,r)).$$

The same estimate is trivial when $i = 0$ or 1. It follows that

$$\fint_{B(x,5r)} L^{q'} \, d\mu \leq C \sum_{i=0}^{\infty} \frac{(1+i)^{q'}}{4^i}.$$

This sum converges, which can be seen for example by a ratio test, and we have that

$$\frac{1}{\mu(B(x,r))} \int_A |u - u_{B(x,r)}| \, d\mu \leq C \, d(x,y) \left(\fint_{B(x,5r)} g^q \, d\mu \right)^{1/q}.$$

Estimate (4.5) and hence the theorem follows from this. $\qquad\square$

We close this section by considering the more general case where u belongs to a fractional Sobolev space and X satisfies the relative δ-annular decay property. Using similar arguments as in the proof above, we obtain the following results.

Theorem 4.6. *Assume that X satisfies the relative δ-annular decay property* (2.5). *Let $1 < q < p$, $s > 0$, $u \in M^{s,p}(X)$ and $g \in \mathrm{D}^s(u)$. Then there is a constant $C > 0$ such that the following holds.*

    a) *If $s < \delta$, then*

$$\tilde{g} = C \, \mathcal{M}_\alpha \, g$$

       is a generalized s-gradient of $\mathcal{M}_\alpha \, u$.

    b) *If $s = \delta$, then*

$$\tilde{g} = C \, (\mathcal{M}_{\alpha q} \, (g^q))^{1/q}$$

       is a generalized s-gradient of $\mathcal{M}_\alpha \, u$.

    c) *If $s > \delta$, then*

$$\tilde{g} = C \, \mathcal{M}_{\alpha+s-\delta} \, g$$

       is a generalized δ-gradient of $\mathcal{M}_\alpha \, u$.

Proof. It suffices to prove the claim for the functions $x \mapsto r^\alpha u_{B(x,r)}$, $r > 0$. Fix $x, y \in X$ and $r > 0$. Suppose first that $r \leq 3d(x,y)$. If $s \leq \delta$, the desired estimate follows as in the proof of Theorem 4.5. If $s > \delta$, we need a simple chaining argument. Let k be the smallest integer such that $2^k r \geq 4d(x,y)$ and let $z \in \{x,y\}$. Then the doubling property, the fact that $B(x, 3d(x,y)) \subset B(z, 2^k r)$ and the Poincaré inequality (4.3) imply that

$$
\left| u_{B(z,r)} - u_{B(x,3d(x,y))} \right| \leq \sum_{i=0}^{k-1} \left| u_{B(z,2^i r)} - u_{B(z,2^{i+1}r)} \right| + \left| u_{B(z,2^k r)} - u_{B(x,3d(x,y))} \right|
$$

$$
\leq C \sum_{i=1}^{k} \fint_{B(z,2^i r)} |u - u_{B(z,2^i r)}| \, d\mu
$$

$$
\leq C \sum_{i=1}^{k} (2^i r)^s \fint_{B(z,2^i r)} g \, d\mu.
$$

Hence, using the selection of k, we have that

$$r^\alpha |u_{B(z,r)} - u_{B(x,3d(x,y))}| \leq C \sum_{i=1}^{k} (2^i r)^{\alpha+s} \fint_{B(z,2^i r)} g \, d\mu$$

$$\leq C \sum_{i=1}^{k} (2^i r)^\delta \, \mathcal{M}_{\alpha+s-\delta} \, g(z)$$

$$\leq C d(x,y)^\delta \, \mathcal{M}_{\alpha+s-\delta} \, g(z),$$

and so

$$|r^\alpha u_{B(x,r)} - r^\alpha u_{B(y,r)}| \leq r^\alpha |u_{B(x,r)} - u_{B(x,3d(x,y))}| + r^\alpha |u_{B(y,r)} - u_{B(x,3d(x,y))}|$$

$$\leq C d(x,y)^\delta \left(\mathcal{M}_{\alpha+s-\delta} \, g(x) + \mathcal{M}_{\alpha+s-\delta} \, g(y) \right).$$

Suppose then that $r > 3d(x,y)$. By following the proof of Theorem 4.5, we obtain estimate

(4.8)
$$|u_{B(x,r)} - u_{B(y,r)}| \leq C \fint_{B(x,5r)} g(w) K(w) \, d\mu(w),$$

where

$$K(w) \leq C \sum_{k \leq 1} r_k^s \, \frac{\mu \left(A \cap B_k(w) \right)}{\mu \left(B_k(w) \right)}.$$

The relative δ-annular decay implies that

$$K(w) \leq C \Big(\sum_{k \in \mathcal{K}_1} r_k^s + d(x,y)^\delta \sum_{k \in \mathcal{K}_2} r_k^{s-\delta} \Big),$$

where $\mathcal{K}_1 = \{k \leq 1 : r_k \leq d(x,y)\}$ and $\mathcal{K}_2 = \{k \leq 1 : r_k > d(x,y), \, B_k(w) \cap A \neq \emptyset\}$. It follows that

$$K(w) \leq C \begin{cases} d(x,y)^s, & \text{if } s < \delta \\ d(x,y)^s \log \frac{5r}{|r-d(w,x)|}, & \text{if } s = \delta \\ d(x,y)^\delta r^{s-\delta}, & \text{if } s > \delta \end{cases}.$$

If $s < \delta$ or $s > \delta$, the claim follows by combining the above estimate with (4.8) and multiplying the resulting inequality by r^α. In the case $s = \delta$, we argue as in the proof of Theorem 4.5. $\qquad\square$

Corollary 4.7. *Assume that the measure μ satisfies the lower bound condition and X satisfies the relative δ-annular decay property (2.5). Let $p > 1$, $s > 0$, $u \in M^{s,p}(X)$ and $0 < \alpha < Q/p$.*

If $s \leq \delta$, then $\mathcal{M}_\alpha u \in M^{s,p^*}(X)$ with $p^* = Qp/(Q - \alpha p)$ and there is a constant $C > 0$, independent of u, such that

$$\| \mathcal{M}_\alpha u \|_{M^{s,p^*}(X)} \leq C \| u \|_{M^{s,p}(X)}.$$

If $s \geq \delta$, then $\mathcal{M}_\alpha u \in \dot{M}^{\delta,q}(X) \cap L^{p^*}(X)$, where $q = Qp/(Q - (\alpha + s - \delta)p)$ and $\dot{M}^{\delta,q}(X)$ is the homogeneous Hajlasz space equipped with the seminorm $\|u\|_{\dot{M}^{\delta,q}(X)} = \inf_{g \in D^\delta(u)} \|g\|_{L^q(X)}$. Moreover, there is a constant C such that

$$\| \mathcal{M}_\alpha u \|_{\dot{M}^{\delta,q}(X)} + \| \mathcal{M}_\alpha u \|_{L^{p^*}(X)} \leq C \| u \|_{M^{s,p}(X)}.$$

Proof. The claim follows from Theorems 4.6 and 4.1. □

For related results concerning the discrete fractional maximal operator, see [16].

Remark 4.8. The usual modifications in the proofs show that Theorems 4.5 and 4.6 remain true for the noncentered fractional maximal function.

5. EXAMPLES

We modify the example given by Buckley [4] a little bit and show that the fractional maximal function of a Lipschitz function may fail to be continuous.

Example 5.1. First we recall Buckley's example. Let X be the subset of the complex plane consisting of the real line and the points x on the unit circle whose argument θ lies in the interval $[0, \frac{\pi}{2}]$. Equip X with the Euclidean metric and the 1-dimensional Hausdorff measure. Let $u \colon X \to [0, 1]$ be a Lipschitz function such that $u(x) = 0$, if $x \in \mathbb{R}$ or $\mathrm{Arg}(x) \leq \pi/5$, and $u(x) = 1$, if $\mathrm{Arg}(x) \geq \pi/4$. Then $\mathcal{M} u$ has a jump discontinuity at the origin. Indeed, since

$$\mathcal{M} u(0) = \lim_{r \to 1+} \fint_{B(0,r)} u \, d\mu = \frac{1}{2 + \frac{\pi}{2}} \int_{\overline{B}(0,1)} |u| \, d\mu,$$

we have that

$$\mathcal{M} u(0) \leq \frac{\frac{\pi}{2} - \frac{\pi}{5}}{2 + \frac{\pi}{2}} = \frac{3\pi}{20 + 5\pi}.$$

If $x < 0$, then $B(x, r(x))$, where $r(x) = d(x, e^{i\pi/4})$, includes points on the arc if and only if their argument exceeds $\frac{\pi}{4}$. It follows that

$$\lim_{x \to 0-} \mathcal{M} u(x) \geq \lim_{x \to 0-} \fint_{B(x,r(x))} |u| \, d\mu = \frac{\frac{\pi}{4}}{2 + \frac{\pi}{4}} = \frac{\pi}{8 + \pi} > \mathcal{M} u(0).$$

The first part of the above argument does not work for $\mathcal{M}_\alpha$, because for α large enough,

$$r^\alpha \fint_{B(0,r)} |u| \, d\mu = \frac{r^\alpha}{2r + \frac{\pi}{2}} \int_{\overline{B}(0,1)} |u| \, d\mu$$

no longer maximizes as $r \to 1+$. This difficulty can be overcome by modifying the measure. Let $w \colon X \to \mathbb{R}$ be a weight such that $w(x) = 1 + \frac{\pi}{2}$, when $x > 1$ and $w(x) = 1$ otherwise. Then the measure ν, defined by

$$\nu(A) = \int_A w \, d\mu,$$

is doubling and satisfies $\nu(B(0, r)) = (2 + \frac{\pi}{2})r$, for $r > 1$. If $\alpha > 1$, it follows that $\mathcal{M}_\alpha u \equiv \infty$. If $\alpha \leq 1$, then, for $r > 1$,

$$r^\alpha \fint_{B(0,r)} u \, d\nu = \frac{r^\alpha}{(2 + \frac{\pi}{2})r} \int_{\overline{B}(0,1)} |u| \, d\mu \leq \frac{1}{2 + \frac{\pi}{2}} \int_{\overline{B}(0,1)} |u| \, d\mu,$$

which implies that

$$\mathcal{M}_\alpha u(0) = \frac{1}{2 + \frac{\pi}{2}} \int_{\overline{B}(0,1)} |u| \, d\mu.$$

The rest of the argument is the same as above.

The next example shows that also the noncentered fractional maximal function of a Lipschitz function may not be continuous.

Example 5.2. Let $X = (\mathbb{R} \times \{0\}) \cup (\{0\} \times (-\infty, 1]) \subset \mathbb{R}^2$. Equip X with the metric $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ and the 1-dimensional Hausdorff measure μ. Define $u \colon X \to \mathbb{R}$ by setting $u(x) = x_2$ for $0 < x_2 \leq 1$ and $u(x) = 0$ otherwise. Then it is easy to see that $\widetilde{\mathcal{M}}_\alpha u$, $0 \leq \alpha \leq 1$, is not continuous at the origin. If $x_2 > 0$, then clearly $\widetilde{\mathcal{M}}_\alpha u(x) \geq \frac{1}{2}$. We will show that $\widetilde{\mathcal{M}}_\alpha u(0) \leq \frac{1}{3}$. Suppose that $B(x, r)$ contains the origin. If $r \leq \frac{1}{2}$, then

$$r^\alpha \fint_{B(x,r)} u \, d\mu \leq \frac{r^\alpha}{4r} \cdot 2r^2 \leq \frac{1}{4}.$$

If $r > \frac{1}{2}$, then

$$r^\alpha \fint_{B(x,r)} u \, d\mu \leq \frac{r^\alpha}{3r} \cdot \frac{1}{2} \leq \frac{1}{3}.$$

Hence $\widetilde{\mathcal{M}}_\alpha u$ is not continuous at the origin.

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