HARDY–SOBOLEV INEQUALITIES AND WEIGHTED CAPACITIES IN METRIC SPACES

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ABSTRACT. Let Ω be an open set in a metric measure space X. Our main result gives an equivalence between the validity of a weighted Hardy–Sobolev inequality in Ω and quasiadditivity of a weighted capacity with respect to Whitney covers of Ω. Important ingredients in the proof include the use of a discrete convolution as a capacity test function and a Maz'ya type characterization of weighted Hardy–Sobolev inequalities.

1. INTRODUCTION

Let $\Omega \subsetneq \mathbb{R}^n$ be an open set. We say that (q, p, β)-Hardy–Sobolev inequality holds in Ω, for $1 \le p, q < \infty$ and $\beta \in \mathbb{R}$, if there exists a constant $C > 0$ such that the inequality

$$(1) \qquad \left(\int_\Omega |u(x)|^q \, d(x, \Omega^c)^{\frac{q}{p}(n-p+\beta)-n} \, dx \right)^{\frac{1}{q}} \le C \left(\int_\Omega |\nabla u(x)|^p \, d(x, \Omega^c)^\beta \, dx \right)^{\frac{1}{p}}$$

is valid for all functions $u \in C_0^\infty(\Omega)$, or equivalently, by approximation, for all Sobolev functions $u \in W_0^{1,p}(\Omega)$ such that the support of u is a compact subset of Ω. Here $d(x, \Omega^c) := \operatorname{dist}(x, \Omega^c)$ denotes the distance from $x \in \Omega$ to the complement $\Omega^c := \mathbb{R}^n \setminus \Omega$. When $p = q$, the inequality in (1) is called the (weighted) (p, β)-Hardy inequality, and for $q = \frac{np}{n-p}$ one obtains a weighted Sobolev inequality, which in the unweighted case $\beta = 0$ reduces to the usual Sobolev inequality. Often it is natural (or even necessary) to restrict the parameters in (1) to the range $1 \le p \le q \le \frac{np}{n-p}$, giving a scale of inequalities interpolating between the (weighted) Hardy and Sobolev inequalities.

In this paper, we consider analogous inequalities for the so-called Newtonian (Sobolev) functions in more general metric spaces, under the standard assumptions that the space X is equipped with a doubling measure and supports Poincaré inequalities. The norm of the gradient on the right-hand side of (1) is then replaced by a (p-weak, p-integrable) upper gradient g_u of u. We are mainly interested in the interplay between Hardy–Sobolev inequalities and conditions given in terms of (weighted) capacities; see Section 3 for the definitions. We begin by proving a Maz'ya type characterization for the validity of Hardy–Sobolev inequalities in Theorem 3.1. This is a straight-forward generalization of many earlier results, but in particular it serves to illustrate that the weighted capacity is a natural tool in questions related to Hardy–Sobolev inequalities.

Our main results in Section 5 relate Hardy–Sobolev inequalities to quasiadditivity properties of weighted capacities. Recall that capacities are (typically) subadditive set functions, which, however, very seldom enjoy full additivity. Quasiadditivity is a weak converse of the subadditivity, involving a multiplicative constant and applicable only to certain types of sets, often given in terms of Whitney-type covers or decompositions. The formulations of the results in the general metric setting are slightly complicated due to the various parameters, see Section 5, but for an open set $\Omega \subsetneq \mathbb{R}^n$ our main result, Theorem 5.2, can be stated as follows.

Theorem 1.1. *Let $1 < p \leq q \leq \frac{np}{n-p}$ and $\beta \in \mathbb{R}$, and let $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ be a cover of an open set $\Omega \subsetneq \mathbb{R}^n$ by Whitney balls $B_i = B(x_i, cd(x_i, \Omega^c))$, with $0 < c < 1/53$. Then the following conditions are equivalent:*

  (i) *Hardy–Sobolev inequality* (1) *holds in Ω.*

  (ii) *There exist constants C_1 and C_2 such that the weighted relative capacity satisfies the quasiadditivity property*

$$\sum_{i=1}^{\infty} \operatorname{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \leq C_1 \operatorname{cap}_{p,\beta}(E, \Omega)^{\frac{q}{p}},$$

for all sets $E \Subset \Omega$, and the capacity lower bound

$$\operatorname{cap}_{p,\beta}(B_i, \Omega) \geq C_2 \, d(x_i, \Omega^c)^{n+\beta-p}$$

holds for every $i \in \mathbb{N}$.

Above, we write $E \Subset \Omega$ if the closure $\overline{E}$ is a compact subset of Ω. For such sets, the weighted relative capacity is defined by setting

$$\operatorname{cap}_{p,\beta}(E, \Omega) := \inf_u \int_\Omega |\nabla u(x)|^p \, d(x, \Omega^c)^\beta \, dx,$$

where the infimum is taken over all quasicontinuous representatives of $u \in W_0^{1,p}(\Omega)$ such that $u(x) \geq 1$ for all $x \in E$ and u has a compact support in Ω.

The case $\beta = 0$ of Theorem 1.1, but with respect to Whitney cube decompositions, was established in [14, Theorem 10.52]. The constant $1/53$ in the statement of Theorem 1.1 is not that essential and it has not been optimized. In fact, it could be replaced by a larger number (up to $1/3$) by using a further covering argument.

In general, both conditions in part (ii) of Theorem 1.1 are needed; this is illustrated by examples in Section 6. However, there are cases in which the capacity lower bound automatically holds, and therefore only the quasiadditivity property needs to be assumed in part (ii). This happens, for instance, in the unweighted ($\beta = 0$) case in $\mathbb{R}^n$ when $1 < p < n$; see Section 6 for further discussion.

In the metric space setting, the case $q = p$, $\beta = 0$ of our main result (Theorem 5.2) was established in [18]. The proofs in [18] were based on potential theoretic tools such as Harnack inequalities and the existence of capacitary potentials. In this paper, we use a different approach which combines and develops ideas from [9, 12, 14, 20] and is better suited to the weighted ($\beta \neq 0$) case and to exponents $q \geq p$. An important feature in the proof of Theorem 5.2 is the use of the so-called discrete convolution as a capacity test function. The definition of the discrete convolution is recalled in Section 4, where we also show that if u belongs to the Newtonian space $N_0^{1,p}(\Omega)$, then a local maximal function of g_u is a p-weak upper gradient of the discrete convolution u_t. This fact will be utilized in the proof of Theorem 5.2. Theorem 5.2 also contains a weaker variant of the quasiadditivity property, where instead of arbitrary sets $E \subset \Omega$ the quasiadditivity is tested using only unions of Whitney balls.

If the weight $w(x) = d(x, \Omega^c)^\beta$ is p-admissible in the metric space $X = (X, d, \mu)$, that is, the weighted measure $w \, d\mu$ is doubling and supports a $(1, p)$-Poincaré inequality, then the weighted cases of our main results could be obtained from the corresponding unweighted results applied in the weighted metric space $(X, d, w \, d\mu)$. In particular, then for $q = p$ the weighted case can be directly obtained from the results in [18]. Many examples of admissible weights are provided by weights from the Muckenhoupt classes, see [7, Theorem 4] and the remark after [7, Theorem 7]. Sufficient and necessary conditions for the weight $w(x) = d(x, \Omega^c)^\beta$ to be a Muckenhoupt weight have been given in [8] in terms of the (co)dimension of Ω^c. However, we emphasize that our more general approach applies

also in the cases where w is not admissible, and thus we obtain a unified theory for all weighted Hardy–Sobolev inequalities.

In addition to the results in [18, Theorem 1] and [14, Theorem 10.52], the quasiadditivity property has been considered in the Euclidean space $\mathbb{R}^n$ for instance in [2, 3] for Riesz capacities, and in [9, 20] ($q = p$) and [12] ($q \geq p$) for fractional capacities with the help of fractional Hardy(–Sobolev) inequalities. Sufficient conditions for different versions of (weighted) Hardy and Hardy–Sobolev inequalities, and hence also for the corresponding quasiadditivity, have been given for example in [8, 14, 17, 19, 21]; see also the references therein.

The outline for the rest of the paper is as follows. In Section 2 we recall the necessary background on analysis on metric spaces: doubling measures, p-weak upper gradients, Poincaré inequalities, and Whitney covers. Metric space variants of the Hardy–Sobolev inequality (1) and the weighted relative capacity are introduced in Section 3, where we also prove the Maz'ya type characterization of weighted Hardy–Sobolev inequalities and show how the capacity lower bound follows from the Hardy–Sobolev inequality. In Section 4 we prepare for our main results by proving the boundedness of a local maximal operator and showing how this maximal operator can be used to give an upper gradient for the discrete convolution. Section 5 contains our main results on quasiadditivity, and finally in Section 6 we consider the special case of Q-regular spaces and give examples illustrating the necessity of the conditions in our characterizations.

As usual, we let C denote positive constants whose exact value may change at each occurrence. By χ_E we denote the characteristic function of a set $E \subset X$; that is, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in X \setminus E$.

2. Preliminaries

2.1. Metric spaces with a doubling measure.
Let $X = (X, d, \mu)$ be a metric measure space. We assume for the rest of this paper that μ is a Borel measure on X, with $0 < \mu(B) < \infty$ whenever $B = B(x, r)$ is an open ball in X, and that μ is *doubling*, that is, there is a constant C_μ such that such that

$$\mu(2B) \leq C_\mu \, \mu(B) \tag{2}$$

for all balls B in X. Here we use the notation $tB = B(x, tr)$, when $0 < t < \infty$ and $B = B(x, r)$.

By iterating the doubling condition (2), we can find constants $Q > 0$ and $C > 0$ such that

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C\left(\frac{r}{R}\right)^Q \tag{3}$$

whenever $0 < r \leq R < \operatorname{diam} X$ and $y \in B(x, R)$; see [4, Lemma 3.3].

2.2. Upper gradients and Newtonian functions.
Let $1 \leq p < \infty$. We say that a μ-measurable function $g \colon X \to [0, \infty]$ is a *p-weak upper gradient* of $u \colon X \to [-\infty, \infty]$, if

$$|u(\gamma(0)) - u(\gamma(\ell_\gamma))| \leq \int_\gamma g \, ds \tag{4}$$

for p-almost every curve $\gamma \colon [0, \ell_\gamma] \to X$; that is, there exists a non-negative Borel function $\rho \in L^p_{\mathrm{loc}}(X; d\mu)$ such that $\int_\gamma \rho \, ds = \infty$ whenever (4) does not hold or is not defined. For a function $u \in L^p(X; d\mu)$, we denote by $\mathcal{D}^p(u)$ the set of all p-weak upper gradients $g \in L^p(X; d\mu)$ of u. We remark that $\mathcal{D}^p(u)$ can be empty. See [4] and [11] for detailed treatments of p-weak upper gradients.

Using p-weak upper gradients as a substitute for modulus of the weak derivative, one defines the norm

$$\|u\|_{N^{1,p}(X)} := \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where $1 \leq p < \infty$ and the infimum is taken over all $g \in \mathcal{D}^p(u)$. The *Newtonian space* $N^{1,p}(X)$ is the set

$$\left\{ u \colon X \to [-\infty, \infty] \,:\, \|u\|_{N^{1,p}(X)} < \infty \right\}$$

equipped with the norm $\|\cdot\|_{N^{1,p}(X)}$. We assume that functions in $N^{1,p}(X)$ are defined everywhere, and not just up to an equivalence class.

When $\Omega \subset X$ is an open set, we denote by $N_0^{1,p}(\Omega)$ the space of all Newtonian functions on X that vanish in the complement $\Omega^c = X \setminus \Omega$. Moreover, we let $N_c^{1,p}(\Omega)$ denote the space of all Newtonian functions on X whose support, i.e. the closure of the set where $u \neq 0$, is a bounded set having a strictly positive distance to the complement Ω^c. Then $N_c^{1,p}(\Omega) \subset N_0^{1,p}(\Omega)$ and $N_c^{1,p}(\Omega)$ is a natural class of test functions in weighted Hardy–Sobolev inequalities. If X is complete, it is equivalent to require that functions in $N_c^{1,p}(\Omega)$ have a compact support in Ω, but in non-complete spaces the above definition gives more flexibility.

We refer to [4, 11, 22] for more information on Newtonian spaces.

2.3. **Poincaré inequalities.** We say that the space $X = (X, d, \mu)$ supports a (q, p)-*Poincaré inequality* for $1 \leq q, p < \infty$ if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$(5) \qquad \left(\fint_B |u - u_B|^q \, d\mu \right)^{\frac{1}{q}} \leq C \, r \left(\fint_{\lambda B} g^p \, d\mu \right)^{\frac{1}{p}}$$

for all measurable functions u in X and all p-weak upper gradients g of u, where the left-hand side of (5) is defined to be ∞, if the mean value integral

$$u_B := \frac{1}{\mu(B)} \int_B u \, d\mu =: \fint_B u \, d\mu$$

is not defined.

The constant λ above is called the dilatation constant for the (q, p)-Poincaré inequality. The $(1, p)$-Poincaré inequality for $1 \leq p < Q$, where Q is as in (3), together with the doubling condition (2) implies a (q, p)-Poincaré inequality with $q = Qp/(Q-p)$; see [4, 10].

If X is complete and supports a $(1, p)$-Poincaré inequality for some $1 < p < \infty$, then there exists $1 < s < p$ such that X supports a $(1, s)$-Poincaré inequality; see [13]. Since we do not require the completeness from X in this paper, we can not refer to this self-improvement result and hence in some of our results we directly assume that X supports a $(1, s)$-Poincaré inequality for some $1 < s < p$.

2.4. **Whitney covers.** Let $\Omega \subsetneq X$ be an open set. When $x \in \Omega$, we denote by $d(x, \Omega^c) = \mathrm{dist}(x, \Omega^c)$ the distance from $x \in \Omega$ to the complement $\Omega^c := X \setminus \Omega$.

Let $0 < c < 1/3$. There exists a countable family $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ of balls $B_i = B(x_i, r_i)$, $r_i = c \, d(x_i, \Omega^c)$, such that $\mathcal{W}_c(\Omega)$ is a cover of Ω and the balls B_i have a uniformly bounded overlap, that is, there exists $1 \leq C < \infty$ such that

$$1 \leq \sum_{i=1}^{\infty} \chi_{B_i}(x) \leq C,$$

for every $x \in \Omega$. The collection $\mathcal{W}_c(\Omega)$ of *Whitney balls* B_i is called a *Whitney cover* of Ω.

The following lemma collects useful properties of Whitney covers. Proofs of properties (i), (ii) and (iii) easily follow from the definition of balls B_i and B_i^*, and property (iv) is proved, for instance, in [6].

Lemma 2.1. *Let $\Omega \subsetneq X$ be an open set and let $L > 1$. Suppose $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ is a family of Whitney balls as above, with $c \leq (3L)^{-1}$, and define $B_i^* := LB_i$. Then the following assertions hold:*

(i) $\Omega = \bigcup_{i \in \mathbb{N}} B_i$,

(ii) $B_i^* \subset \Omega$, *for every* $i \in \mathbb{N}$,

(iii) *if* $x \in B_i^*$, *then* $(1/c - L)r_i \leq d(x, \Omega^c) \leq (1/c + L)r_i$,

(iv) *there is* $M \in \mathbb{N}$ *such that* $\sum_{i \in \mathbb{N}} \chi_{B_i^*}(x) \leq M$ *for all* $x \in \Omega$.

3. Hardy–Sobolev inequality and relative capacity

In $\mathbb{R}^n$, the Hardy–Sobolev inequality (1) is written in terms of powers of the distance $d(x, \Omega^c)$. In the metric case, it turns out to be natural to interpret the part $d(x, \Omega^c)^{n(q-p)/p}$ as the measure of the ball $B(x, d(x, \Omega^c))$ to the power $(q-p)/p$. This leads to the following formulation, which in the case $X = \mathbb{R}^n$ is equivalent to (1). See also Section 6 for the case of Q-regular metric spaces.

Let $\Omega \subsetneq X$ be an open set. We say that a (q, p, β)-Hardy–Sobolev inequality holds in Ω, for $1 \leq p, q < \infty$ and $\beta \in \mathbb{R}$, if there exists a constant $C > 0$ such that the inequality

$$(6) \quad \left(\int_\Omega \frac{|u(x)|^q}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}} \, d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_\Omega g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{1}{p}}$$

is valid for all $u \in N_c^{1,p}(\Omega)$ and for all $g \in \mathcal{D}^p(u)$.

We write $E \Subset \Omega$ if the set E is bounded and $d(E, \Omega^c) > 0$, that is, E has a strictly positive distance to the complement Ω^c. The *relative* (p, β)-*capacity* of a set $E \Subset \Omega$, for $1 \leq p < \infty$ and $\beta \in \mathbb{R}$, is defined to be the number

$$\mathrm{cap}_{p,\beta}(E, \Omega) := \inf_u \inf_g \int_\Omega g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x),$$

where the infimum is taken over all $u \in N_c^{1,p}(\Omega)$ such that $u(x) \geq 1$ for all $x \in E$, and over all $g \in \mathcal{D}^p(u)$. A function u satisfying the above conditions is called a *capacity test function* for E. Observe that if $E \Subset \Omega$, then there exist capacity test functions for E and thus $\mathrm{cap}_{p,\beta}(E, \Omega) < \infty$. For instance, we can test the capacity with the Lipschitz function

$$\varphi(x) = \max \left\{ 0, 1 - \frac{2d(x, E)}{d(E, \Omega^c)} \right\}, \qquad x \in X.$$

The following Maz'ya-type characterization is one manifestation of the close connection between Hardy–Sobolev inequalities and relative capacities; see [21] for the origins of this kind of characterizations.

Theorem 3.1. *Let $1 \leq p \leq q < \infty$, $\beta \in \mathbb{R}$ and let $\Omega \subsetneq X$ be an open set. The (q, p, β)-Hardy–Sobolev inequality holds in Ω if and only if there is a constant C_1 such that*

$$(7) \quad \int_E \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x) \leq C_1 \, \mathrm{cap}_{p,\beta}(E, \Omega)^{\frac{q}{p}},$$

for all $E \Subset \Omega$.

Proof. The main lines of the proof follow the proof of [15, Theorem 4.1] where a similar characterization is obtained for the validity of the p-Hardy inequality, that is the case $p = q$ and $\beta = 0$.

First assume that (q, p, β)-Hardy–Sobolev inequality (6) holds in Ω, with a constant $C > 0$. Let $E \Subset \Omega$, and let $u \in N_c^{1,p}(\Omega)$ be such that $u(x) \geq 1$, for every $x \in E$. Then

$$
\text{(8)} \quad
\int_E \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x) \leq \int_\Omega \frac{|u(x)|^q}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}} \, d\mu(x)
$$

$$
\leq C^q \left(\int_\Omega g(x)^p d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{q}{p}},
$$

where $g \in \mathcal{D}^p(u)$. By taking infimum over all $g \in \mathcal{D}^p(u)$, and then over all functions u as above, we obtain (7) with $C_1 = C^q$.

Then assume that inequality (7) holds with a constant C_1 for all sets $E \Subset \Omega$. Let $u \in N_c^{1,p}(\Omega)$ and $g \in \mathcal{D}^p(u)$. For $j \in \mathbb{Z}$, define

$$
E_j = \{x \in \Omega : |u(x)| > 2^j\}.
$$

Since $E_j \Subset \Omega$ for every $j \in \mathbb{Z}$, by (7) we have

$$
\int_\Omega \frac{|u(x)|^q}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}} \, d\mu(x)
$$

$$
\leq \sum_{j=-\infty}^\infty 2^{(j+2)q} \int_{E_{j+1} \setminus E_{j+2}} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x)
$$

$$
\leq \sum_{j=-\infty}^\infty 2^{(j+2)q} \int_{E_{j+1}} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x)
$$

$$
\leq 4^q C_1 \sum_{j=-\infty}^\infty 2^{jq} \operatorname{cap}_{p,\beta}(E_{j+1}, \Omega)^{\frac{q}{p}}.
$$

For every $j \in \mathbb{Z}$, define $u_j \colon X \to [0, 1]$ by

$$
u_j(x) = \begin{cases} 1, & \text{if } |u(x)| \geq 2^{j+1}, \\ 2^{-j}|u(x)| - 1, & \text{if } 2^j < |u(x)| < 2^{j+1}, \\ 0, & \text{if } |u(x)| \leq 2^j. \end{cases}
$$

Then $u_j \in N_c^{1,p}(\Omega)$ and $u_j = 1$ in E_{j+1}. From the gluing lemma [4, Lemma 2.19] we obtain that $g_j = 2^{-j} g \chi_{E_j \setminus E_{j+1}} \in \mathcal{D}^p(u_j)$. Using u_j as a test function for the weighted capacity $\operatorname{cap}_{p,\beta}(E_{j+1}, \Omega)$, we obtain

$$
\operatorname{cap}_{p,\beta}(E_{j+1}, \Omega) \leq \int_\Omega g_j(x)^p d(x, \Omega^c)^\beta \, d\mu(x)
$$

$$
= \int_{E_j \setminus E_{j+1}} g_j(x)^p d(x, \Omega^c)^\beta \, d\mu(x) \leq 2^{-jp} \int_{E_j \setminus E_{j+1}} g(x)^p d(x, \Omega^c)^\beta \, d\mu(x).
$$

Since $q/p \geq 1$, it follows that

$$
\sum_{j=-\infty}^\infty 2^{jq} \operatorname{cap}_{p,\beta}(E_{j+1}, \Omega)^{\frac{q}{p}} \leq \sum_{j=-\infty}^\infty \left(\int_{E_j \setminus E_{j+1}} g(x)^p d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{q}{p}}
$$

$$
\leq \left(\sum_{j=-\infty}^\infty \int_{E_j \setminus E_{j+1}} g(x)^p d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{q}{p}}
$$

$$
= \left(\int_\Omega g(x)^p d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{q}{p}}.
$$

This shows that the (q,p,β)-Hardy–Sobolev inequality holds with the constant $C = 4C_1^{1/q}$ for every function $u \in N_c^{1,p}(\Omega)$ and for every $g \in \mathcal{D}^p(u)$. $\qquad\square$

Let $B = B(x,r) \in \mathcal{W}_c(\Omega)$, $0 < c < 1/3$, be a Whitney ball defined as above and set $B^* = LB$ with $L = 1/(3c)$. In this case $B^* \Subset \Omega$ and we can take the Lipschitz function

$$\varphi(x) = \max\left\{0, 1 - \frac{d(x,B)}{d(B, X \setminus B^*)}\right\}, \qquad x \in X,$$

as a test function for the capacity $\mathrm{cap}_{p,\beta}(B,\Omega)$. Then, by the properties of Whitney balls, see Lemma 2.1, and by the doubling condition for measure μ, it follows that there exists a constant C, depending on the parameter c and the doubling constant C_μ, such that

$$\mathrm{cap}_{p,\beta}(B,\Omega) \le C\mu(B)r^{\beta-p},$$

for every ball $B = B(x,r) \in \mathcal{W}_c(\Omega)$.

By the following lemma, if a (q,p,β)-Hardy–Sobolev inequality holds in Ω, then the corresponding lower bound for the capacity is valid as well, and, therefore, the value of $\mathrm{cap}_{p,\beta}(B,\Omega)$ is comparable to $\mu(B)r^{\beta-p}$ for every ball $B = B(x,r) \in \mathcal{W}_c(\Omega)$.

Lemma 3.2. *Let $1 \le p, q < \infty$ and $\beta \in \mathbb{R}$, and assume that a (q,p,β)-Hardy–Sobolev inequality (6) holds in an open set $\Omega \subsetneq X$. Then there exists a constant $C > 0$ such that*

$$(9) \qquad\qquad \mathrm{cap}_{p,\beta}(B,\Omega) \ge C\mu(B)r^{\beta-p}$$

for every ball $B = B(x,r) \in \mathcal{W}_c(\Omega)$.

Proof. By applying the estimate in (8) (which holds also if $q < p$) and the property (iii) of Whitney balls, we have

$$\mathrm{cap}_{p,\beta}(B,\Omega)^{\frac{q}{p}} \ge C \int_B \frac{\mu(B(y, d(y,\Omega^c)))^{\frac{q-p}{p}}}{d(y,\Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(y)$$

$$\ge C \int_B \mu(B)^{\frac{q-p}{p}} r^{\frac{q}{p}(\beta-p)} \, d\mu(y) = C\big(\mu(B)r^{\beta-p}\big)^{\frac{q}{p}}.$$

The claim follows by raising this to power $p/q > 0$. $\qquad\square$

4. Local maximal functions and discrete convolutions

Let $\Omega \subsetneq X$ be an open set and let $0 < \kappa < 1$. For a measurable function f on Ω, define the *local maximal function*

$$\mathcal{M}_{\Omega,\kappa}f(x) = \sup_{0 < r < \kappa d(x,\Omega^c)} \fint_{B(x,r)} |f| \, d\mu, \qquad x \in \Omega.$$

For convenience, we set $\mathcal{M}_{\Omega,\kappa}f(x) = 0$ if $x \in X \setminus \Omega$. Recall that the usual (centered) Hardy–Littlewood maximal operator $\mathcal{M}$ is defined for all $x \in X$ by using the integral averages as in the definition of $\mathcal{M}_{\Omega,\kappa}$ but omitting the upper bound $r < \kappa d(x,\Omega^c)$ for the radii. Hence $\mathcal{M}_{\Omega,\kappa}f(x) \le \mathcal{M}f(x)$ whenever f is measurable on X and $x \in X$.

It is well known that $\mathcal{M}$ is bounded on $L^s(X, d\mu)$ for all $1 < s < \infty$; see, for instance, [4, Section 3.2]. The next lemma gives a similar boundedness result for the local maximal operator $\mathcal{M}_{\Omega,\kappa}$ on the weighted space $L^s(\Omega; w_\beta \, d\mu)$, when $1 < s < \infty$, $0 < \kappa < 1/5$ and $w_\beta(x) = d(x,\Omega^c)^\beta$ is a distance weight, for $\beta \in \mathbb{R}$ and $x \in \Omega$.

Lemma 4.1. *Let $0 < \kappa < 1/5$ and $1 < s < \infty$, and let $\Omega \subsetneq X$ be an open set. Let $\beta \in \mathbb{R}$ and define $w_\beta(x) = d(x,\Omega^c)^\beta$, for $x \in \Omega$. Then $\mathcal{M}_{\Omega,\kappa}$ is bounded on $L^s(\Omega; w_\beta \, d\mu)$, that is, there is a constant C such that*

$$\int_\Omega \big(\mathcal{M}_{\Omega,\kappa}f\big)^s w_\beta \, d\mu \le C \int_\Omega |f|^s w_\beta \, d\mu,$$

for every $f \in L^s(\Omega; w_\beta \, d\mu)$.

Proof. Let $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ be a Whitney cover of Ω, with $c = 1/9$, and take $L = 3$. Suppose $x \in B_i = B(x_i, r_i)$. If $y \in B(x, \kappa d(x, \Omega^c))$, then

$$d(y, x_i) \leq r_i + \kappa d(x, \Omega^c) < r_i + \tfrac{1}{5}(r_i + 9r_i) = 3r_i,$$

which implies that $B(x, \kappa d(x, \Omega^c)) \subset B(x_i, 3r_i) = B_i^*$. Hence, by the definition of the local maximal function, for every $x \in B_i$,

$$\mathcal{M}_{\Omega,\kappa}(f)(x) = \mathcal{M}_{\Omega,\kappa}(f\chi_{B_i^*})(x) \leq \mathcal{M}(f\chi_{B_i^*})(x).$$

This observation, together with the properties of Whitney balls and the boundedness of $\mathcal{M}$ on $L^s(X, d\mu)$, gives

$$\int_\Omega (\mathcal{M}_{\Omega,\kappa}f(x))^s w_\beta(x) d\mu(x) \leq \sum_i \int_{B_i} (\mathcal{M}_{\Omega,\kappa}f(x))^s d(x, \Omega^c)^\beta d\mu(x)$$

$$\leq C \sum_i r_i^\beta \int_{B_i} (\mathcal{M}(f\chi_{B_i^*})(x))^s d\mu(x) \leq C \sum_i r_i^\beta \int_X |f(x)|^s \chi_{B_i^*}(x) d\mu(x)$$

$$\leq C \sum_i \int_{B_i^*} |f(x)|^s d(x, \Omega^c)^\beta d\mu(x) \leq C \int_\Omega |f(x)|^s w_\beta(x) d\mu(x).$$

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $0 < t < 1$ be a scaling parameter and let $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ be a Whitney cover of an open set $\Omega \subsetneq X$ with $c = t/18$. There is a sequence of Lipschitz functions $\{\varphi_i\}_{i\in\mathbb{N}}$ corresponding to the cover $\mathcal{W}_c(\Omega)$ such that φ_i satisfies the following properties for every $i \in \mathbb{N}$: $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ outside of the balls $6B_i$, $\varphi_i \geq \nu > 0$ on $3B_i$, and φ_i is Lipschitz with constant K/r_i, where ν and K only depend on the doubling constant of the measure μ. Moreover,

$$\sum_{i=1}^\infty \varphi_i(x) = 1$$

for every $x \in \Omega$. Then the *discrete convolution* of a locally integrable function u at the scale t is

$$(10) \qquad\qquad u_t(x) = \sum_{i=1}^\infty \varphi_i(x) u_{3B_i}, \qquad x \in X.$$

Notice that $u_t(x) \leq C\mathcal{M}u(x)$, for every $x \in X$, where $\mathcal{M}$ is the standard Hardy–Littlewood maximal operator on X. Hence, the boundedness of $\mathcal{M}$ implies that if $1 < s < \infty$ and $u \in L^s(X, d\mu)$, then $u_t \in L^s(X, d\mu)$ as well.

The proof of the following lemma is strongly inspired by the proof of [1, Lemma 5.1]. However, since Lemma 4.2 contains an additional parameter κ, we present the details for the convenience of the reader.

Lemma 4.2. *Let $1 < p < \infty$ and $0 < \kappa \leq 1$, and let $\Omega \subsetneq X$ be an open set. Assume that X supports a $(1, s)$-Poincaré inequality for some $1 < s < p$ with a dilation constant λ, and let $0 < t < \min\{1, 6\kappa/(3\lambda + 2\kappa)\}$. Then there is a constant C such that if $u \in N_0^{1,p}(\Omega)$ and $g \in \mathcal{D}^p(u)$, then $C(\mathcal{M}_{\Omega,\kappa}g^s)^{1/s} \in \mathcal{D}^p(u_t)$.*

Proof. Let $u \in N_0^{1,p}(\Omega)$ and $g \in \mathcal{D}^p(u)$. Since $\mathcal{M}_{\Omega,\kappa}g$ depends only on the values of g in Ω, by the gluing lemma [4, Lemma 2.19] we may assume that $g = 0$ in $X \setminus \Omega$.

Take a Whitney cover $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ with $c = t/18$, set $L = 6$, and write $u_t(x)$ for $x \in X$ as

$$u_t(x) = u(x) + \sum_{i=1}^\infty \varphi_i(x)\big(u_{B(x_i, 3r_i)} - u(x)\big).$$

By the properties of the Lipschitz functions φ_i, see above, and by the product rule, see for instance [4, Lemma 2.15], we have that

$$\left(\frac{K}{r_i}|u - u_{B(x_i,3r_i)}| + g\right)\chi_{B(x_i,6r_i)}$$

belongs to $\mathcal{D}^p(\varphi_i(u_{B(x_i,3r_i)} - u))$. This implies that

$$g + \sum_{i=1}^{\infty}\left(\frac{K}{r_i}|u - u_{B(x_i,3r_i)}| + g\right)\chi_{B(x_i,6r_i)} \in \mathcal{D}^p(u_t).$$

To estimate this p-weak upper gradient of u_t in terms of $\mathcal{M}_{\Omega,\kappa}g^s$, we fix a Lebesgue point $x \in B(x_i, 6r_i)$ of function u, notice that $B(x_i, 3r_i) \subset B(x, 9r_i)$, and write

(11) $$|u(x) - u_{B(x_i,3r_i)}| \le |u(x) - u_{B(x,9r_i)}| + |u_{B(x,9r_i)} - u_{B(x_i,3r_i)}|.$$

For the second term on the right-hand side of (11) we have

$$|u_{B(x,9r_i)} - u_{B(x_i,3r_i)}| \le \fint_{B(x_i,3r_i)} |u - u_{B(x,9r_i)}|\,d\mu \le C\fint_{B(x,9r_i)} |u - u_{B(x,9r_i)}|\,d\mu$$

$$\le Cr_i\left(\fint_{B(x,9\lambda r_i)} g(y)^s\,d\mu(y)\right)^{\frac{1}{s}} \le Cr_i(\mathcal{M}_{\Omega,\kappa}g^s(x))^{1/s}.$$

Here we use the fact that g is also an s-weak upper gradient of u, and the last estimate follows from the definition of maximal function $\mathcal{M}_{\Omega,\kappa}g^s$ and the inequality

$$9\lambda r_i \le 9\lambda\left(\frac{18}{t} - 6\right)^{-1} d(x, \Omega^c) < 9\lambda\left(\frac{9\lambda + 6\kappa}{\kappa} - 6\right)^{-1} d(x, \Omega^c) \le \kappa d(x, \Omega^c).$$

For the first term on the right-hand side of (11), by a standard telescoping argument we have for the Lebesgue point $x \in B(x_i, 6r_i)$ of u that

$$|u(x) - u_{B(x,9r_i)}| \le C\sum_{j=0}^{\infty}\fint_{B(x,3^{2-j}r_i)} |u - u_{B(x,3^{2-j}r_i)}|\,d\mu$$

$$\le C\sum_{j=0}^{\infty} 3^{2-j}r_i\left(\fint_{B(x,\lambda 3^{2-j}r_i)} g(y)^s\,d\mu(y)\right)^{\frac{1}{s}} \le Cr_i(\mathcal{M}_{\Omega,\kappa}g^s(x))^{1/s}.$$

The last inequality is true for the same reason as above since $B(x, 3^{2-j}\lambda r_i) \subset B(x, 9\lambda r_i)$ for every $j = 0, 1, \dots$.

Hence, we obtain for every Lebesgue point $x \in B(x_i, 6r_i)$ of u that

$$|u(x) - u_{B(x_i,3r_i)}| \le Cr_i(\mathcal{M}_{\Omega,\kappa}g^s(x))^{1/s}.$$

This, together with the fact that $g = 0$ in $X \setminus \Omega$, gives

$$g(x) + \sum_{i=1}^{\infty}\left(\frac{K}{r_i}|u(x) - u_{B(x_i,3r_i)}| + g\right)\chi_{B(x_i,6r_i)}(x) \le C(\mathcal{M}_{\Omega,\kappa}g^s(x))^{1/s},$$

for almost every $x \in X$. This, in turn, implies that $C(\mathcal{M}_{\Omega,\kappa}g^s)^{1/s}$ is a p-weak upper gradient of u_t; see e.g. [4, Corollary 1.44]. Notice that here the constant C does not depend on u. Finally, since $\mathcal{M}_{\Omega,\kappa}g^s \le \mathcal{M}g^s$ and $g \in \mathcal{D}^p(u)$, where $1 \le s < p$, the maximal function theorem (with exponent $p/s > 1$) implies that $(\mathcal{M}_{\Omega,\kappa}g^s)^{1/s} \in L^p(X; d\mu)$. Hence we conclude that $C(\mathcal{M}_{\Omega,\kappa}g^s)^{1/s} \in \mathcal{D}^p(u_t)$, as desired. $\qquad\square$

5. Quasiadditivity of capacity and Hardy–Sobolev inequalities

This section contains our main results, which relate weighted Hardy–Sobolev inequalities to the quasiadditivity of the weighted capacities. The general approach in the proofs is similar to those in [9, 12, 14, 20], but the present context requires different tools on the level of details. We first show that the validity of the (q, p, β)-Hardy–Sobolev inequality (6) in Ω, for $1 < p \leq q < \infty$, implies the quasiadditivity property for the weighted relative capacity with respect to Whitney balls.

Theorem 5.1. *Let* $1 < p \leq q < \infty$ *and let* $\Omega \subsetneq X$ *be an open set. Assume that* (q, p, β)-*Hardy–Sobolev inequality* (6) *holds in* Ω *with a constant* C_1, *and let* $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ *be a Whitney cover of* Ω *for some* $0 < c < 1/3$. *Then there exists a constant* C *such that*

$$\sum_{i \in \mathbb{N}} \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \leq C \, \mathrm{cap}_{p,\beta}(E, \Omega)^{\frac{q}{p}},$$

for every set $E \Subset \Omega$.

Proof. Let $E \Subset \Omega$. Let $u \in N_c^{1,p}(\Omega)$ be such that $u(x) \geq 1$, for every $x \in E$, and let $g \in \mathcal{D}^p(u)$.

Fix $L > 1$ satisfying $c < (3L)^{-1}$ and recall that $B_i^* = LB_i$, $i \in \mathbb{N}$. For every $i \in \mathbb{N}$ we choose a Lipschitz function φ_i satisfying the following properties: $\varphi_i(x) = 1$ for every $x \in B_i$, φ_i is K/r_i-Lipschitz, for some $K \geq 1$, and $0 \leq \varphi_i(x) \leq \chi_{B_i^*}(x)$, for every $x \in X$. For instance, we can take

$$\varphi_i(x) = \max\left\{0, 1 - \frac{d(x, B_i)}{d(B_i, X \setminus B_i^*)}\right\}, \qquad x \in X,$$

in which case we can choose $K = (L - 1)^{-1}$. Then, for every $i \in \mathbb{N}$, $u_i = u\varphi_i \in N_c^{1,p}(\Omega)$, and the product rule (see [4, Lemma 2.15]) implies that the function

$$g_i(x) = (g(x) + Kr_i^{-1}|u(x)|)\chi_{B_i^*}(x), \quad x \in X,$$

belongs to $\mathcal{D}^p(u_i)$. Hence u_i is a capacity test function for $E \cap B_i$ and

$$\mathrm{cap}_{p,\beta}(E \cap B_i, \Omega) \leq \int_\Omega g_i(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x).$$

Since $r_i^{-1} \leq Cd(x, \Omega^c)^{-1}$, for every $x \in B_i^*$,

$$\left(\int_\Omega g_i(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x)\right)^{\frac{q}{p}}$$

$$\leq C\left(\int_{B_i^*} g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x)\right)^{\frac{q}{p}} + C\left(\int_{B_i^*} |u(x)|^p d(x, \Omega^c)^{-p+\beta} \, d\mu(x)\right)^{\frac{q}{p}}.$$

Since $q \geq p$, we can write

$$\sum_{i=1}^\infty \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \leq \sum_{i=1}^\infty \left(\int_\Omega g_i(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x)\right)^{\frac{q}{p}}$$

$$\leq C\sum_{i=1}^\infty \left(\int_{B_i^*} g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x)\right)^{\frac{q}{p}} + C\sum_{i=1}^\infty \left(\int_{B_i^*} |u(x)|^p d(x, \Omega^c)^{-p+\beta} \, d\mu(x)\right)^{\frac{q}{p}}$$

$$\leq C\left(\sum_{i=1}^\infty \int_{B_i^*} g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x)\right)^{\frac{q}{p}} + C\sum_{i=1}^\infty \mu(B_i^*)^{\frac{q}{p}\frac{q-p}{q}} \int_{B_i^*} \frac{|u(x)|^q}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x).$$

For the first term on the right-hand side, due to the finite overlap of the balls B_i^*, we get

$$\left(\sum_{i=1}^{\infty} \int_{B_i^*} g(x)^p\, d(x,\Omega^c)^{\beta}\, d\mu(x)\right)^{\frac{q}{p}} \leq C\left(\int_{\Omega} g(x)^p\, d(x,\Omega^c)^{\beta}\, d\mu(x)\right)^{\frac{q}{p}}.$$

To estimate the second term, we notice from Lemma 2.1 that $B_i^* \subset B(x, d(x,\Omega^c))$, for every $x \in B_i^*$. This inclusion together with the finite overlap of the balls B_i^* and inequality (6) gives

$$\sum_{i=1}^{\infty} \mu(B_i^*)^{\frac{q-p}{p}} \int_{B_i^*} \frac{|u(x)|^q}{d(x,\Omega^c)^{\frac{q}{p}(p-\beta)}}\, d\mu(x)$$

$$\leq C \sum_{i=1}^{\infty} \int_{B_i^*} \frac{|u(x)|^q}{d(x,\Omega^c)^{\frac{q}{p}(p-\beta)}} \mu(B(x, d(x,\Omega^c)))^{\frac{q-p}{p}}\, d\mu(x)$$

$$\leq C \int_{\Omega} \frac{|u(x)|^q}{d(x,\Omega^c)^{\frac{q}{p}(p-\beta)}} \mu(B(x, d(x,\Omega^c)))^{\frac{q-p}{p}}\, d\mu(x)$$

$$\leq C\left(\int_{\Omega} g(x)^p\, d(x,\Omega^c)^{\beta}\, d\mu(x)\right)^{\frac{q}{p}}.$$

After collecting the estimates above, and taking the infimum first over all $g \in \mathcal{D}^p(u)$ and then over all functions u as above, the claim follows. $\qquad\square$

For $q \leq \frac{Qp}{Q-p}$ and sufficiently small parameters c, there is also a partial converse of Theorem 5.1. In addition to the quasiadditivity property in part (ii), the next result contains the *weak quasiadditivity* property in part (iii).

Theorem 5.2. *Let $1 < p \leq q \leq \frac{Qp}{Q-p}$ and assume that X supports a $(1,s)$-Poincaré inequality for some $1 < s < p$ with a dilation constant λ. Let $\Omega \subsetneq X$ be an open set and let $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ be a Whitney cover of Ω with $0 < c < \frac{1}{45\lambda+8}$. Then the following conditions are equivalent:*

(i) *The (q,p,β)-Hardy–Sobolev inequality (6) holds in Ω.*
(ii) *There exist constants C_1 and C_2 such that*

$$\sum_{i=1}^{\infty} \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \leq C_1 \mathrm{cap}_{p,\beta}(E,\Omega)^{\frac{q}{p}},$$

for every set $E \Subset \Omega$, and the capacity lower bound (9) holds with the constant C_2 for every ball $B_i \in \mathcal{W}_c(\Omega)$.
(iii) *There exist constants C_1 and C_2 such that*

$$\sum_{i\in I} \mathrm{cap}_{p,\beta}(B_i, \Omega)^{\frac{q}{p}} \leq C_1 \mathrm{cap}_{p,\beta}\left(\bigcup_{i\in I} B_i, \Omega\right)^{\frac{q}{p}},$$

whenever $I \subset \mathbb{N}$ is a finite set, and the capacity lower bound (9) holds with the constant C_2 for every ball $B_i \in \mathcal{W}_c(\Omega)$.

Proof. The implication from (i) to (ii) follows from Lemma 3.2 and Theorem 5.1. The implication from (ii) to (iii) follows by considering $E = \bigcup_{i\in I} B_i$.

Assume now that condition (iii) holds with a constant C_1. To show that (i) holds, by Theorem 3.1 it suffices to prove that there exists a constant C such that

$$(12) \qquad \int_E \frac{\mu(B(x, d(x,\Omega^c)))^{\frac{q-p}{p}}}{d(x,\Omega^c)^{\frac{q}{p}(p-\beta)}}\, d\mu(x) \leq C\, \mathrm{cap}_{p,\beta}(E,\Omega)^{\frac{q}{p}},$$

for every set $E \Subset \Omega$. To this end, fix $E \Subset \Omega$, let $u \in N_c^{1,p}(\Omega)$ be such that $u(x) \geq 1$ for every $x \in E$, and let $g \in \mathcal{D}^p(u)$. By considering $|u|$ instead of u, we may assume that $u \geq 0$ in X; observe that $|u| \in N_c^{1,p}(\Omega)$ and $g \in \mathcal{D}^p(|u|)$.

Partition $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ into two subfamilies: $\mathcal{W}^1 = \{B_i \in \mathcal{W}_c(\Omega) : u_{B_i} < \frac{1}{2}\}$ and $\mathcal{W}^2 = \mathcal{W}_c(\Omega) \setminus \mathcal{W}^1$. Observe that $\mathcal{W}^2$ is necessarily finite since $u \in N_c^{1,p}(\Omega)$. The left-hand side of (12) can be estimated from above by

$$(13) \qquad \left(\sum_{B \in \mathcal{W}^1} + \sum_{B \in \mathcal{W}^2} \right) \int_{E \cap B} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x).$$

To estimate the first sum, we observe that, for every $B_i \in \mathcal{W}^1$ and every $x \in E \cap B_i$,

$$\tfrac{1}{2} = 1 - \tfrac{1}{2} < u(x) - u_{B_i} = |u(x) - u_{B_i}|.$$

By the definition and the properties of Whitney balls, see Lemma 2.1, if $B_i \in \mathcal{W}_c(\Omega)$ and $x \in B_i$, then

$$B_i \subset B(x, d(x, \Omega^c)) \quad \text{and} \quad (1/c - 1)r_i \leq d(x, \Omega^c) \leq (1/c + 1)r_i.$$

Consequently,

$$(14) \qquad \mu(B_i) \leq \mu(B(x, d(x, \Omega^c))) \quad \text{and} \quad \mu(B(x, d(x, \Omega^c))) \leq C\mu(B_i)$$

for every $x \in B_i$.

Since X supports a $(1, s)$-Poincaré inequality for $1 < s < p$ with the dilation constant λ, then X supports a $(1, p)$-Poincaré inequality with the same dilation constant λ, and hence the (q, p)-Poincaré inequality with the dilation constant 2λ; see [4, Theorem 4.21].

The observations above lead to the estimates

$$\sum_{B \in \mathcal{W}^1} \int_{E \cap B} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}} \, d\mu(x) \leq C \sum_{B_i \in \mathcal{W}^1} \frac{\left(\mu(B_i) r_i^\beta\right)^{\frac{q}{p}}}{r_i^q} \int_{B_i} |u(x) - u_{B_i}|^q \, d\mu(x)$$

$$\leq C \sum_{B_i \in \mathcal{W}^1} \frac{\left(\mu(B_i) r_i^\beta\right)^{\frac{q}{p}}}{r_i^q} r_i^q \left(\fint_{2\lambda B_i} g(x)^p \, d\mu(x) \right)^{\frac{q}{p}}$$

$$\leq C \left(\sum_{B \in \mathcal{W}^1} \int_{2\lambda B} g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{q}{p}}$$

$$\leq C \left(\int_\Omega g(x)^p \, d(x, \Omega^c)^\beta \, d\mu(x) \right)^{\frac{q}{p}}.$$

The last inequality follows from the finite overlap of dilated Whitney balls $2\lambda B$ which is guaranteed by the relation between c and λ in the formulation of the theorem.

To estimate the second sum in (13), let $B_i = B(x_i, r_i) \in \mathcal{W}^2$ and $x \in B_i$. Recall that $r_i = cd(x_i, \Omega^c)$, where $c < \frac{1}{45\lambda + 8} < \frac{1}{20}$. Choose a number t such that $\frac{18c}{1 - 2c} < t < \frac{6}{15\lambda + 2}$; then, in particular, $0 < t < 1$. Let $\mathcal{W}_{t/18}(\Omega) = \{B(x_j', r_j') : j \in \mathbb{N}\}$ be a Whitney cover of Ω, $\{\varphi_j\}$ be a partition of unity related to $\mathcal{W}_{t/18}(\Omega)$, and u_t be the the discrete convolution of $u \geq 0$ at scale t as in (10). Then $x \in B(x_j', r_j')$ for some $j \in \mathbb{N}$ and the choice of t guarantees that $r_i \leq r_j'$, since by property (3) of Lemma 2.1 we have

$$(1/c - 1)r_i \leq d(x, \Omega^c) \leq (18/t + 1)r_j',$$

which implies

$$r_i \leq \frac{18/t + 1}{1/c - 1} r_j' \leq r_j'.$$

Similarly, we obtain that $r_j' \leq Cr_i$, for some $C > 0$, since

$$(18/t - 1)r_j' \leq d(x, \Omega^c) \leq (1/c + 1)r_i.$$

By the definition of the discrete convolution u_t, properties of functions φ_j, the inclusion $B(x_i, r_i) \subset B(x_j', 3r_j')$ (from $r_i \leq r_j'$), and the doubling condition for measure μ, we obtain

$$
\begin{aligned}
(15) \qquad u_t(x) &\geq \varphi_j(x) \fint_{B(x_j', 3r_j')} u(y)\, d\mu(y) \geq C \fint_{B(x_j', 3r_j')} u(y)\, d\mu(y) \\
&\geq C \fint_{B(x_i, r_i)} u(y)\, d\mu(y) \geq C u_{B_i} \geq \frac{C}{2},
\end{aligned}
$$

where C depends on λ, c, and the doubling constant of the measure μ.

Since $u \in N_c^{1,p}(\Omega)$, we have that $u \in L^p(X; d\mu)$, and hence $u_t \in L^p(X; d\mu)$; see the comment before Lemma 4.2. Furthermore, by Lemma 4.2 the set $\mathcal{D}^p(u_t)$ is nonempty, and as the sum in (10) has only finitely many non-zero terms for $u \in N_c^{1,p}(\Omega)$, the support of u_t is bounded and has a positive distance to Ω^c. Thus we conclude that $u_t \in N_c^{1,p}(\Omega)$.

By (15) there exists $C_2 > 0$ such that $C_2 u_t \geq 1$ in $\bigcup_{B \in \mathcal{W}^2} B$. Hence, $C_2 u_t$ is a capacity test function for $\mathrm{cap}_{p,\beta}\big(\bigcup_{B \in \mathcal{W}^2} B, \Omega\big)$. Since $t < 6/(15\lambda + 2)$, we can choose $0 < \kappa < 1/5$ such that $t < 6\kappa/(3\lambda + 2\kappa)$, and, by Lemma 4.2, there is a constant C_3 such that $C_3(\mathcal{M}_{\Omega,\kappa} g^s)^{1/s} \in \mathcal{D}^p(u_t)$.

Let $B_i = B(x_i, r_i) \in \mathcal{W}^2$. By inequalities (14) and (9), it follows that

$$
\begin{aligned}
\int_{E \cap B_i} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}}\, d\mu(x) &\leq C \mu(B_i)^{\frac{q}{p}-1} \int_{E \cap B_i} d(x, \Omega^c)^{\frac{q}{p}(\beta-p)}\, d\mu(x) \\
&\leq C\big(\mu(B_i) r_i^{\beta-p}\big)^{q/p} \leq C \, \mathrm{cap}_{p,\beta}(B_i, \Omega)^{q/p}.
\end{aligned}
$$

Since $\mathcal{W}^2$ is finite, we obtain from the assumed condition (iii) that

$$
\begin{aligned}
\sum_{B \in \mathcal{W}^2} \int_{E \cap B} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}}\, d\mu(x) &\leq C \sum_{B \in \mathcal{W}^2} \mathrm{cap}_{p,\beta}(B, \Omega)^{\frac{q}{p}} \\
&\leq C \, \mathrm{cap}_{p,\beta}\bigg(\bigcup_{B \in \mathcal{W}^2} B, \Omega \bigg)^{\frac{q}{p}} \\
&\leq C \bigg(\int_\Omega (\mathcal{M}_{\Omega,\kappa} g^s(x))^{p/s} d(x, \Omega^c)^\beta\, d\mu(x) \bigg)^{\frac{q}{p}} \\
&\leq C \bigg(\int_\Omega g(x)^p d(x, \Omega^c)^\beta\, d\mu \bigg)^{\frac{q}{p}},
\end{aligned}
$$

where the last inequality is valid since $s < p$, and thus the maximal operator $\mathcal{M}_{\Omega,\kappa}$ is bounded on $L^{p/s}(\Omega, w_\beta\, d\mu)$ by Lemma 4.1.

By combining the estimates for $\mathcal{W}^1$ and $\mathcal{W}^2$ we obtain

$$
\begin{aligned}
\int_E \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}}\, d\mu(x) &\leq \sum_{B \in \mathcal{W}} \int_{E \cap B} \frac{\mu(B(x, d(x, \Omega^c)))^{\frac{q-p}{p}}}{d(x, \Omega^c)^{\frac{q}{p}(p-\beta)}}\, d\mu(x) \\
&\leq C \bigg(\int_\Omega g(x)^p d(x, \Omega^c)^\beta\, d\mu \bigg)^{\frac{q}{p}}.
\end{aligned}
$$

The desired estimate (12) follows by taking infimum over all $g \in \mathcal{D}^p(u)$ and then over all functions u as above. $\qquad\square$

Theorem 5.2 gives the following implication for the validity of (q, p, β)-Hardy–Sobolev inequalities for different values of the parameter q. Observe that this is not completely obvious from the statement of the Hardy–Sobolev inequality.

Remark 5.3. Let $1 < p \leq q < p^* = \frac{Qp}{Q-p}$ and $\beta \in \mathbb{R}$, and assume that X supports a $(1,s)$-Poincaré inequality for some $1 < s < p$. If a (q, p, β)-Hardy–Sobolev inequality holds in an open set $\Omega \subsetneq X$, then also (q', p, β)-Hardy–Sobolev inequalities hold in Ω for every $q \leq q' \leq p^*$. This follows from Theorem 5.2 and the fact that if $q \leq q'$, then

$$\left(\sum_{i=1}^{\infty} \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q'}{p}} \right)^{\frac{p}{q'}} \leq \left(\sum_{i=1}^{\infty} \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \right)^{\frac{p}{q}}$$

whenever $E \subset \Omega$ and $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ is a Whitney cover of Ω.

6. Special cases and examples

The main results of the paper are obtained under the assumption that the measure μ is doubling. In particular, μ satisfies for some $Q > 0$ the inequality

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C \left(\frac{r}{R} \right)^Q,$$

whenever $0 < r \leq R < \mathrm{diam}\, X$ and $y \in B(x, R)$.

If X is connected (this is guaranteed in our setting by the Poincaré inequalities), then there exists also a constant Q_u satisfying $0 < Q_u \leq Q$ and such that

$$(16) \qquad \frac{\mu(B(y,r))}{\mu(B(x,R))} \leq C \left(\frac{r}{R} \right)^{Q_u},$$

whenever $0 < r \leq R < \mathrm{diam}\, X$ and $y \in B(x, R)$; see e.g. [4, Corollary 3.8].

If there are uniform upper and lower bounds for the measures of the balls, that is,

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q$$

for every $x \in X$ and all $0 < r < \mathrm{diam}(X)$, the measure μ is said to be *Ahlfors Q-regular*. The above exponent $Q > 0$ plays the same role as n does in $\mathbb{R}^n$. In the Q-regular case the (q, p, β)-Hardy–Sobolev inequality (6) can be written in a simpler form

$$(17) \qquad \left(\int_\Omega |u(x)|^q d(x, \Omega^c)^{\frac{q}{p}(Q-p+\beta)-Q} \, d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_\Omega g(x)^p d(x, \Omega^c)^\beta \, d\mu \right)^{\frac{1}{p}},$$

and Theorem 5.2 implies the following characterization. In $X = \mathbb{R}^n$ (equipped with the usual Euclidean distance and the Lebesgue measure), the corresponding result in terms of Whitney cubes has been considered in [14, Theorem 10.52] in the unweighted case $\beta = 0$, but the weighted result is new even in $\mathbb{R}^n$. Since $N^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ and $\mathbb{R}^n$ supports $(1,s)$-Poincaré inequalities for every $s \geq 1$ with $\lambda = 1$, Theorem 1.1 follows from Corollary 6.1.

Corollary 6.1. *Let X be a metric space equipped with an Ahlfors Q-regular measure μ and let $1 < p \leq q \leq \frac{Qp}{Q-p}$. Assume that X supports a $(1,s)$-Poincaré inequality for some $1 < s < p$ with a dilation constant λ. Let $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$, $B_i = (x_i, r_i)$, be a Whitney cover of an open set $\Omega \subsetneq X$, with $0 < c < \frac{1}{45\lambda+8}$. Then the following conditions are equivalent:*

(i) *Hardy–Sobolev inequality (17) holds in Ω.*
(ii) *There exist constants C_1 and C_2 such that*

$$\sum_{i=1}^{\infty} \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \leq C_1 \, \mathrm{cap}_{p,\beta}(E, \Omega)^{\frac{q}{p}},$$

for every set $E \Subset \Omega$, and $\mathrm{cap}_{p,\beta}(B_i, \Omega) \geq C_2 r_i^{Q+\beta-p}$ for every $i \in \mathbb{N}$.

(iii) *There exist constants C_1 and C_2 such that*

$$\sum_{i \in I} \mathrm{cap}_{p,\beta}(B_i, \Omega)^{\frac{q}{p}} \le C_1 \, \mathrm{cap}_{p,\beta}\left(\bigcup_{i \in I} B_i, \Omega\right)^{\frac{q}{p}},$$

whenever $I \subset \mathbb{N}$ is a finite set, and $\mathrm{cap}_{p,\beta}(B_i, \Omega) \ge C_2 r_i^{Q+\beta-p}$ for every $i \in \mathbb{N}$.

When $1 < p < Q$ and $\beta = 0$, the capacity lower bound $\mathrm{cap}_p(B, \Omega) \ge C r^{Q-p}$ holds for all Whitney balls $B = B(x, r) \in \mathcal{W}_c(\Omega)$, at least if X is unbounded or $\mathrm{diam}(\Omega) < \frac{1}{4}\mathrm{diam}(X)$; see [18, Lemma 2] and [5, Proposition 6.1] for details. Here $\mathrm{cap}_p(B, \Omega) = \mathrm{cap}_{p,0}(B, \Omega)$. (More generally, the capacity lower bound (9) holds in the unweighted case $\beta = 0$ if $1 < p < Q_u$, by the same references.)

Consequently, in certain cases, for instance if X is an unbounded Q-regular space and $1 < p < Q$, $\beta = 0$, there is no need to assume the capacity lower bound in conditions (ii) and (iii) in Corollary 6.1. This also shows that the weighted capacity lower bound (9), for every Whitney ball, is not alone sufficient for the (q, p, β)-Hardy–Sobolev inequality in Theorem 5.2. For instance, consider $X = \mathbb{R}^n$ equipped with the usual Euclidean distance and the Lebesgue measure. Then $Q = n$, and for every $1 < p < n$ and $p \le q < \frac{np}{n-p}$ there are open sets where the $(q, p, 0)$-Hardy–Sobolev inequality does not hold. When $1 < p = q < n$, one such an example is given by the complement of any (non-empty) Ahlfors $(n-p)$-regular set, by [16, Theorem 1.1]; see also [19, Example 4.7] for counterexamples in more general cases.

On the other hand, even if the capacity lower bound (9) is not always needed, it is not possible to remove this part from conditions (ii) and (iii) in Theorem 5.2. This is illustrated by the next example.

Example 6.2. Consider $X = \mathbb{R}^n$, equipped with the usual Euclidean distance and the Lebesgue measure. Then $N^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$, and $|\nabla u| \in \mathcal{D}_p(u)$ if $u \in W^{1,p}(\mathbb{R}^n)$; see e.g. [4, A.1 and Proposition A.13].

Let $1 < p < \infty$ and $\beta \ge 0$, and choose $\Omega = B(0, 1) \subset \mathbb{R}^n$. For every $j \in \mathbb{N}$, let u_j be a Lipschitz continuous function in $\mathbb{R}^n$, satisfying the following properties: $u_j = 1$ in $B(0, 1 - 2^{-j})$, $u_j = 0$ in $\mathbb{R}^n \setminus B(0, 1 - 2^{-j-1})$, and $|\nabla u| \le 2^{j+1}$ almost everywhere in $A_j = B(0, 1 - 2^{-j-1}) \setminus B(0, 1 - 2^{-j})$. Then we have for every $j \in \mathbb{N}$ that $u_j \in W_0^{1,p}(\Omega)$ has a compact support in Ω and

$$\int_\Omega |\nabla u_j(x)|^p \, d(x, \Omega^c)^\beta \, d\mu \le \mu(A_j) 2^{(j+1)p} 2^{-j\beta} \le C 2^{-j(1-p+\beta)}.$$

If $\beta > p - 1$, then $2^{-j(1-p+\beta)} \to 0$ as $j \to \infty$.

Let $\mathcal{W}_c(\Omega) = \{B_i : i \in \mathbb{N}\}$ be a Whitney cover of Ω. When $i \in \mathbb{N}$ is fixed, the functions u_j are test functions for the capacity $\mathrm{cap}_{p,\beta}(B_i, \Omega)$ for all sufficiently large j. Thus the above computation shows that if $\beta > p - 1$, then $\mathrm{cap}_{p,\beta}(B_i, \Omega) = 0$ for all Whitney balls $B_i \in \mathcal{W}_c(\Omega)$, and so the capacity lower bound (9) does not hold in this case. Lemma 3.2 (or a direct computation for the functions u_j) implies that neither does the (q, p, β)-Hardy–Sobolev inequality hold when $\beta > p - 1$. Nevertheless, the quasiadditivity property

$$\sum_{i \in \mathbb{N}} \mathrm{cap}_{p,\beta}(E \cap B_i, \Omega)^{\frac{q}{p}} \le C \, \mathrm{cap}_{p,\beta}(E, \Omega)^{\frac{q}{p}}$$

holds trivially for every set $E \Subset \Omega$, since all the terms on the left-hand side are zero. This shows that in addition to the quasiadditivity property, also the capacity lower bound is really needed in assertions (ii) and (iii) of Theorem 5.2 and in the corresponding assertions of Corollary 6.1.

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