THE EQUIVALENCE BETWEEN POINTWISE HARDY INEQUALITIES AND UNIFORM FATNESS

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Abstract. We prove an equivalence result between the validity of a pointwise Hardy inequality in a domain and uniform capacity density of the complement. This result is new even in Euclidean spaces, but our methods apply in general metric spaces as well. We also present a new transparent proof for the fact that uniform capacity density implies the classical integral version of the Hardy inequality in the setting of metric spaces. In addition, we consider the relations between the above concepts and certain Hausdorff content conditions.

1. Introduction

Let $\Omega \subsetneq \mathbb{R}^n$ be a domain and let $u \in C_0^\infty(\Omega)$. The inequality

$$|u(x)| \leq C\, \mathrm{d}(x,\partial\Omega)\big(\mathcal{M}_{2\,\mathrm{d}(x,\partial\Omega)}\,|\nabla u|^p(x)\big)^{1/p}, \quad x \in \Omega, \tag{1.1}$$

where $\mathcal{M}_R$ is the restricted Hardy–Littlewood maximal operator and $1 \leq p < \infty$, can be viewed as a pointwise variant of the classical p-Hardy inequality

$$\int_\Omega \frac{|u(x)|^p}{\mathrm{d}(x,\partial\Omega)^p}\,dx \leq C \int_\Omega |\nabla u(x)|^p\,dx. \tag{1.2}$$

We say that the domain Ω admits the pointwise p-Hardy inequality, if there exists a constant $C > 0$ such that inequality (1.1) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$. As our main result, we prove the following characterization for such domains. Recall that uniform p-fatness is a capacity density condition; the exact definition is given in Section 2.

Theorem 1.1. *Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality if and only if the complement of Ω is uniformly p-fat.*

The origins of Hardy inequalities lie in the one-dimensional considerations by Hardy, see [15, §330] and the references therein. In $\mathbb{R}^n$, for $n \geq 2$, Hardy-type inequalities first appeared in the paper of Nečas [28] in the context of Lipschitz domains. However, it has been well-known since the works of Ancona [3] ($p = 2$), Lewis [26], and Wannebo [31], that the regularity of the boundary is not essential for Hardy inequalities. Indeed, uniform p-fatness of the complement suffices for a domain to admit the integral p-Hardy inequality (1.2). Uniform n-fatness of the complement is also necessary for the n-Hardy inequality, see [3] and [26], but this is not true for $p < n$.

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Pointwise Hardy inequalities were introduced by Hajłasz [12] and Kinnunen and Martio [21]. In these works it was shown that uniform p-fatness of the complement guarantees that the domain admits even the pointwise p-Hardy inequality; this is the sufficiency part of Theorem 1.1.

Using the boundedness of the Hardy–Littlewood maximal operator it is easy to see that a pointwise q-Hardy inequality for some $q < p$ implies the p-Hardy inequality (1.2). This method does not work if we start with a pointwise p-Hardy inequality, as only weak type estimates are available when the exponent is not allowed to increase. Indeed, it has been an open question since the first appearance of pointwise Hardy inequalities whether the pointwise p-Hardy inequality implies the integral p-Hardy inequality with the same exponent.

Now, by a remarkable result of Lewis [26], uniform p-fatness has the following self-improvement property: If $1 < p < \infty$ and a set $E \subset \mathbb{R}^n$ is uniformly p-fat, then E is also uniformly q-fat for some $1 < q < p$. Thus Theorem 1.1 has the striking consequence that pointwise p-Hardy inequalities, for $1 < p < \infty$, enjoy this same property. In particular, we obtain a positive answer to the above question:

Corollary 1.2. *Let $1 < p < \infty$. If a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then Ω admits the integral p-Hardy inequality.*

In fact, by using the approach of Wannebo, we obtain for Corollary 1.2 another proof in which we avoid the use of the rather deep self-improvement of uniform fatness; see Section 5. In addition, we establish a further equivalence between the conditions of Theorem 1.1 and certain Poincaré type boundary conditions, see Theorem 2.2. Notice also the inclusion of the case $p = 1$ in Theorem 1.1. On the contrary, the usual 1-Hardy inequality does not hold even in smooth domains.

We remark that it was recently shown in [25] that a domain $\Omega \subset \mathbb{R}^n$ admits a pointwise q-Hardy inequality for some $1 < q < p$ if and only if the complement of Ω is uniformly p-fat (note here the difference between our terminology and that of [25]). This result is nevertheless significantly weaker than Theorem 1.1, as the crucial end-point $q = p$ is not reached.

The second purpose of this paper is to generalize parts of the existing theory of Euclidean Hardy inequalities to the setting of metric measure spaces. As a part of this scheme we also state and prove Theorem 1.1 in this more general setting. The relevant parts of the analysis in metric spaces, as well as the exact formulations of our main results, can be found in Section 2. In Section 3, we prove that uniform p-fatness of the complement implies the pointwise p-Hardy inequality also in metric spaces. The necessity part of Theorem 1.1 is then obtained in Section 4. Section 5 contains a transparent proof for the fact that uniform p-fatness of the complement (and thus also the pointwise p-Hardy inequality) is sufficient for Ω to admit the usual integral version of the p-Hardy inequality. Finally, in Section 6, we give further generalizations of the results from [25] to metric spaces by linking pointwise Hardy inequalities and uniform fatness to certain Hausdorff content density conditions. In the special case of Carnot–Carathéodory spaces similar generalizations were recently obtained in [10]. Different aspects of

Hardy inequalities in the metric setting have also been studied in [6], [19], [23], and [24].

2. Preliminaries and the main results

2.1. **Metric spaces.** We assume that $X = (X, \mathrm{d}, \mu)$ is a complete metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x,r) = \{y \in X : \mathrm{d}(y,x) < r\}$. For $0 < t < \infty$, we write $tB = B(x,tr)$, and $\overline{B}$ is the corresponding closed ball. We assume that μ is *doubling*, which means that there is a constant $c_D \geq 1$, called *the doubling constant of* μ, such that

$$\mu(2B) \leq c_D\, \mu(B)$$

for all balls B of X. Note that the doubling condition together with completeness implies that the space is proper, that is, closed balls of X are compact.

The doubling condition gives an upper bound for the dimension of X. By this we mean that there is a constant $C = C(c_D) > 0$ such that, for $s = \log_2 c_D$,

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\Big(\frac{r}{R}\Big)^s \tag{2.1}$$

whenever $0 < r \leq R < \operatorname{diam} X$ and $y \in B(x,R)$. Inequality (2.1) may hold for some smaller exponents than $\log_2 c_D$, too. In such cases we let s denote the infimum of the exponents for which (2.1) holds and say that s is *the doubling dimension* of X.

When $\Omega \subset \mathbb{R}^n$, we obtain, by the density of smooth functions in the Sobolev space $W_0^{1,p}(\Omega)$, that the Hardy inequality (1.2) holds for all $u \in W_0^{1,p}(\Omega)$ if it holds for all smooth functions $\varphi \in C_0^\infty(\Omega)$. General metric spaces lack the notion of smooth functions, but there exists a natural counterpart of Sobolev spaces, defined by Shanmugalingam [29] and based on the use of *upper gradients*. We say that a Borel function $g \geq 0$ is an upper gradient of a function u on an open set $\Omega \subset X$, if for all curves γ joining points x and y in Ω we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds, \tag{2.2}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. By a *curve* we mean a nonconstant, rectifiable, continuous mapping from a compact interval to X.

If $g \geq 0$ is a measurable function and (2.2) only fails for a curve family with zero p-modulus, then g is *a p-weak upper gradient* of u on Ω. For the p-modulus on metric measure spaces and the properties of upper gradients, see for example [11], [17], [29], and [30]. We use the notation g_u for a p-weak upper gradient of u. The Sobolev space $N^{1,p}(\Omega)$ consists of those functions $u \in L^p(\Omega)$ that have a p-weak upper gradient $g_u \in L^p(\Omega)$ in Ω. The space $N^{1,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{N^{1,p}(\Omega)} = \Big(\int_\Omega |u|^p\, d\mu + \inf_g \int_\Omega |g|^p\, d\mu\Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients $g \in L^p(\Omega)$ of u. In the Euclidean space with the Lebesgue measure, $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ for all domains $\Omega \subset \mathbb{R}^n$ and $g_u = |\nabla u|$ is a minimal upper gradient of u.

For a measurable set $E \subset X$, the *Sobolev space with zero boundary values* is

$$N_0^{1,p}(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ in } X \setminus E\}.$$

By [30, Theorem 4.4], also the space $N_0^{1,p}(E)$, equipped with the norm inherited from $N^{1,p}(X)$, is a Banach space. Note that often the definition of $N_0^{1,p}(\Omega)$ is given so that the functions are only required to vanish in $X \setminus E$ outside a set of zero p-capacity. However, our definition gives the same space because functions in $N^{1,p}(X)$ are p-quasicontinuous by [4].

In order to be able to develop the basic machinery of analysis in the metric space X, we need to assume, in addition to the doubling condition, that the geometry of X is rich enough. In practice, this means that there must exist sufficiently many rectifiable curves everywhere in X. This requirement is in a sense quantified by assuming that the space X supports a *(weak)* $(1,p)$-*Poincaré inequality.* That is, we assume that there exist constants $c_P > 0$ and $\tau \geq 1$ such that for all balls $B \subset X$, all locally integrable functions u and for all p-weak upper gradients g_u of u, we have

$$\fint_B |u - u_B| \, d\mu \leq c_P r \Big(\fint_{\tau B} g_u^p \, d\mu \Big)^{1/p},$$

where

$$u_B = \fint_B u \, d\mu = \mu(B)^{-1} \int_B u \, d\mu$$

is the integral average of u over B.

Standard examples of doubling metric spaces supporting Poincaré inequalities include (weighted) Euclidean spaces, compact Riemannian manifolds, metric graphs, and Carnot–Carathéodory spaces. See for instance [14] and [11], and the references therein, for more extensive lists of examples and applications.

2.2. **Capacity and fatness.** Let $\Omega \subset X$ be an open set and let $E \subset \Omega$. The *p-capacity* of E with respect to Ω is

$$\operatorname{cap}_p(E, \Omega) = \inf \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all functions $u \in N_0^{1,p}(\Omega)$ such that $u|_E = 1$. If there are no such functions u, then $\operatorname{cap}_p(E, \Omega) = \infty$. Since the norm of an upper gradient does not increase under truncation, we may assume that $0 \leq u \leq 1$. Note also that because functions in $N^{1,p}(X)$ are p-quasicontinuous by [4], our definition of p-capacity agrees with the classical definition where admissible functions are required to satisfy $u = 1$ in a neighborhood of E.

There exists a constant $C > 0$ such that the following comparison between the p-capacity and measure holds for each $1 \leq p < \infty$: For all balls $B = B(x, r)$ with $0 < r < (1/6) \operatorname{diam} X$ and for each $E \subset B$

$$\frac{\mu(E)}{Cr^p} \leq \operatorname{cap}_p(E, 2B) \leq \frac{C\mu(B)}{r^p}. \tag{2.3}$$

The lower bound can be obtained by considering $(1,p)$-Poincaré inequality for all admissible functions $0 \le u \le 1$ for the capacity $\mathrm{cap}_p(E, 2B)$ in the ball $3B$. For more details, see for example [5, Lemma 3.3].

We say that a set $E \subset X$ is *(uniformly) p-fat*, $1 \le p < \infty$, if there exists a constant $c_0 > 0$ such that

$$\mathrm{cap}_p(E \cap \overline{B}(x,r), B(x,2r)) \ge c_0 \, \mathrm{cap}_p(\overline{B}(x,r), B(x,2r)) \tag{2.4}$$

for all $x \in E$ and all $0 < r < (1/6)\,\mathrm{diam}\, X$. Notice that by the double inequality (2.3), $\mathrm{cap}_p(\overline{B}(x,r), B(x,2r))$ is always comparable to $\mu(B)r^{-p}$. There are many natural examples of uniformly p-fat sets. For instance, all nonempty subsets of X are uniformly p-fat for all $p > s$, where s is the doubling dimension of X. Also complements of simply connected subdomains of $\mathbb{R}^2$ and sets satisfying measure density condition

$$\mu(B(x,r) \cap E) \ge C\mu(B(x,r)) \quad \text{for all } x \in E, \quad r > 0,$$

are uniformly p-fat for all $1 \le p < \infty$. The p-fatness condition is stronger than the Wiener criterion and it is important for example in the study of boundary regularity of $\mathcal{A}$-harmonic functions, see [16].

As mentioned in the introduction, uniform fatness is closely related to pointwise Hardy inequalities.

Definition 2.1. Let $1 \le p < \infty$. An open set $\Omega \subsetneq X$ admits *the pointwise p-Hardy inequality* if there exist constants $c_H > 0$ and $L \ge 1$ such that, for all $u \in N_0^{1,p}(\Omega)$,

$$|u(x)| \le c_H \, \mathrm{d}_\Omega(x) \Big(\mathcal{M}_{L\, \mathrm{d}_\Omega(x)} \, g_u^p(x) \Big)^{1/p} \tag{2.5}$$

holds at almost every $x \in \Omega$.

Above

$$\mathcal{M}_R \, u(x) = \sup_{0<r\le R} \fint_{B(x,r)} |u| \, d\mu$$

is the restricted Hardy–Littlewood maximal function of a locally integrable function u. By the maximal theorem [14, Thm 14.13], $\mathcal{M}_R$ is bounded on $L^p(X)$ for each $1 < p \le \infty$. Contrary to the Euclidean case, here $\mathrm{d}_\Omega(x) = \mathrm{d}(x, \Omega^c)$ is the distance from $x \in \Omega$ to the complement $\Omega^c = X \setminus \Omega$. We use the this distance because in metric spaces $\mathrm{d}(x, \partial\Omega)$ may be larger than $\mathrm{d}(x, \Omega^c)$. See however the end of Section 6 for a related discussion.

2.3. **Main results.** We are now ready to give the general formulation of our main result, which shows, even in the metric setting, the equivalence between uniform p-fatness of the complement, validity of the pointwise p-Hardy inequality, and two Poincaré type inequalities. Here $\tau \ge 1$ is the dilatation constant from the $(1,p)$-Poincaré inequality.

Theorem 2.2. *Let $1 \le p < \infty$ and let X be a complete, doubling metric measure space supporting a $(1,p)$-Poincaré inequality. Then, for an open set $\Omega \subsetneq X$, the following assertions are quantitatively equivalent:*

(a) *The complement Ω^c is uniformly p-fat.*

(b) *For all $B = B(w, r)$, with $w \in \Omega^c$ and $r > 0$, and every $u \in N_0^{1,p}(\Omega)$*

$$\int_B |u|^p \, d\mu \leq C r^p \int_{5\tau B} g_u^p \, d\mu \,. \tag{2.6}$$

(c) *For all $x \in \Omega$ and every $u \in N_0^{1,p}(\Omega)$*

$$|u_{B_x}|^p \leq C \, \mathrm{d}_\Omega(x)^p \fint_{20\tau B_x} g_u^p \, d\mu, \tag{2.7}$$

where $B_x = B(x, \mathrm{d}_\Omega(x))$.

(d) *The open set Ω admits the pointwise p-Hardy inequality* (2.5)*, and we may choose the dilatation constant to be $L = 20\tau$.*

Remark 2.3. It can be seen from the proof of Theorem 2.2 that the conditions (a)–(d) are equivalent also in a local sense, if interpreted correctly. Indeed, if one of the conditions holds near a point $w_0 \in \Omega^c$, then the other conditions hold near w_0 as well if we only consider sufficiently small radii in the uniform fatness condition (2.4) and in the Poincaré type inequality (2.6).

As the self-improvement of uniform fatness was generalized to the metric space setting by Björn, MacManus and Shanmugalingam in [6], we obtain for $1 < p < \infty$ the following important corollary to Theorem 2.2.

Corollary 2.4. *For $1 < p < \infty$ each of the assertions in Theorem 2.2 possesses a self-improvement property. More precisely, if one of the assertions (a)–(d) holds for $1 < p < \infty$, then there exists some $1 < q < p$ so that the same assertion (and thus each of them) holds with the exponent q and constants depending only on p and the associated data.*

Notice that we only assume that X supports a $(1, p)$-Poincaré inequality, but in the above corollary we actually need that X supports a $(1, q)$-Poincaré inequality for some $q < p$ as well. By a result of Keith and Zhong [18], this is in fact always true if X is complete, doubling and supports a weak $(1, p)$-Poincaré inequality.

In the previous literature concerning pointwise Hardy inequalities (see e.g. [12] and [25]), a sort of a self-improvement has actually been an a priori assumption when the passage from pointwise inequalities to the usual Hardy inequality was considered. Now, by Corollary 2.4, such an extra assumption becomes unnecessary. Especially, using the maximal theorem for an exponent $1 < q < p$, for which Ω still admits the pointwise inequality, we obtain the following corollary just as in the Euclidean case.

Corollary 2.5. *If an open set $\Omega \subset X$ admits the pointwise p-Hardy inequality* (2.5) *for some $1 < p < \infty$, then Ω admits the p-Hardy inequality, that is, there exists $C > 0$ such that*

$$\int_\Omega \frac{u(x)^p}{\mathrm{d}_\Omega(x)^p} \, d\mu \leq C \int_\Omega g_u(x)^p \, d\mu$$

for every $u \in N_0^{1,p}(\Omega)$.

However, the result of Corollary 2.5, when viewed as a consequence of Theorem 2.2, depends on a heavy machinery of non-trivial results already in the Euclidean setting, let alone in general metric spaces, as the self-improvement of uniform fatness is involved. In particular, the theory of Cheeger derivatives is needed in the metric case. The ideas of Wannebo [31] lead to an alternative proof for Corollary 2.5, which is based on completely elementary tools and methods, and especially avoids the use of the self-improvement. Using this approach, we give in Theorem 5.1 a direct proof for the fact that uniform p-fatness of the complement of Ω implies that Ω admits the p-Hardy inequality. Note that this result was first generalized to metric spaces in [6], but there the proof was based on the self-improvement. As the pointwise p-Hardy inequality implies the uniform fatness of the complement by Theorem 2.2, Corollary 2.5 follows.

It would also be interesting to acquire an alternative proof for Corollary 2.4 by showing the self-improvement directly for one of the conditions (b)–(d) in Theorem 2.2. Let us remark here that self-improving properties of integral Hardy inequalities were considered in [24], but these results and methods do not seem apply for pointwise inequalities.

3. From fatness to pointwise Hardy

This section deals with the proofs of the implications (a)⇒(b)⇒(c)⇒(d) of Theorem 2.2. The implication (a)⇒(d), that uniform p-fatness of the complement implies the pointwise p-Hardy inequality, is a generalization of an Euclidean result of Kinnunen and Martio [21, Thm 3.9] and Hajłasz [12, Thm 2].

Our proof utilizes the following Sobolev type inequality, proved in the classical case by Maz'ya (c.f. [27, Ch. 10]) and in the metric setting by Björn [5, Proposition 3.2]. We recall the main ideas of the proof for the sake of completeness.

Lemma 3.1. *There is a constant $C > 0$ such that for each $u \in N^{1,p}(X)$ and for all balls $B \subset X$ we have*

$$\fint_B |u|^p \, d\mu \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u = 0\}, B)} \int_{5\tau B} g_u^p \, d\mu, \tag{3.1}$$

where τ is from the $(1,p)$-Poincaré inequality.

Proof. Let $B = B(x,r)$ be a ball and let φ be a $2/r$-Lipschitz function such that $0 \le \varphi \le 1$, $\varphi = 1$ on $1/2B$ and $\varphi = 0$ outside B. We may assume that $u \ge 0$ in B. The function

$$v = \varphi(1 - u/\bar{u}),$$

where $\bar{u} = (\fint_B u^p \, d\mu)^{1/p}$, is a test function for the capacity in (3.1). The claim follows by estimating the integral of g_v^p,

$$g_v = |1 - u/\bar{u}| 2r^{-1} + g_u/\bar{u}.$$

Here one needs a (p,p)-Poincaré inequality, which by [14, Theorem 5.1] follows from the $(1,p)$-Poincaré inequality with dilatation constant 5τ. □

We also need the following pointwise inequality for $N^{1,p}$-functions in terms of the maximal function of the p-weak upper gradient: There is a constant

$C > 0$, depending only on the doubling constant and the constants of the Poincaré inequality, such that

$$|u(x) - u_B| \le Cr\big(\mathcal{M}_{\tau r}\, g_u^p(x)\big)^{1/p} \tag{3.2}$$

whenever $B = B(x,r)$ is a ball and x is a Lebesgue point of u. Estimate (3.2) follows easily from a standard telescoping argument, see for example [13]. Note that u has Lebesgue points almost everywhere in the p-capacity sense, see [20], [22].

Proof of Theorem 2.2 (a) $\Longrightarrow$ *(b)* $\Longrightarrow$ *(c)* $\Longrightarrow$ *(d).*
(a) $\Longrightarrow$ (b): Let $u \in N_0^{1,p}(\Omega)$ and let $B = B(w,r)$, where $w \in \Omega^c$. Assume first that $0 < r < (1/6)\operatorname{diam} X$. Since u vanishes outside Ω, we have $\Omega^c \subset \{u = 0\}$. Using the p-fatness of Ω^c, estimate (2.3), and the doubling property of μ, we obtain

$$\begin{aligned}\operatorname{cap}_p(\tfrac{1}{2}B \cap \{u=0\}, B) &\ge \operatorname{cap}_p(\tfrac{1}{2}B \cap \Omega^c, B)\\ &\ge c_0 \operatorname{cap}_p(\tfrac{1}{2}B, B) \ge C\mu(B) r^{-p}.\end{aligned}$$

This, together with Lemma 3.1, gives

$$\int_B |u|^p \, d\mu \le \frac{C\mu(B)}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)} \int_{5\tau B} g_u^p \, d\mu \le Cr^p \int_{5\tau B} g_u^p \, d\mu.$$

If $(1/6)\operatorname{diam} X \le r \le \operatorname{diam} X$, we take $\widetilde{B} = B\big(w, (1/7)\operatorname{diam} X\big)$. From the triangle inequality it follows that

$$\int_B |u|^p \, d\mu \le C\Big(\int_B |u - u_B|^p \, d\mu + \mu(B)|u_{\widetilde{B}}|^p + \mu(B)|u_{\widetilde{B}} - u_B|^p\Big).$$

We can then use the $(1,p)$-Poincaré inequality, the above case for the ball $\widetilde{B}$, and the doubling property, and the claim for B follows with simple calculations.

Finally, if $r > \operatorname{diam} X$, the claim is clear by the previous cases.
(b) $\Longrightarrow$ (c): Let $u \in N_0^{1,p}(\Omega)$, $x \in \Omega$, and let $B_x = B(x, \mathrm{d}_\Omega(x))$. Choose a point $w \in \Omega^c$ so that

$$R = \mathrm{d}(x,w) \le 2\,\mathrm{d}_\Omega(x),$$

and let $B_0 = B(w,R)$. Now

$$|u_{B_x}| \le |u_{B_x} - u_{B_0}| + |u|_{B_0},$$

where, by the $(1,p)$-Poincaré inequality, the fact that $B_0 \subset 4B_x$ and $B_x \subset 2B_0$, and the doubling property,

$$|u_{B_x} - u_{B_0}| \le C\,\mathrm{d}_\Omega(x)\Big(⨍_{4\tau B_x} g_u^p \, d\mu\Big)^{1/p}.$$

Using the Hölder inequality, assumption (b), and the doubling property, we obtain

$$|u|_{B_0} \le \Big(⨍_{B_0} |u|^p \, d\mu\Big)^{1/p} \le CR\Big(⨍_{5\tau B_0} g_u^p \, d\mu\Big)^{1/p} \le C\,\mathrm{d}_\Omega(x)\Big(⨍_{20\tau B_x} g_u^p \, d\mu\Big)^{1/p}.$$

The claim follows by combining these two estimates.
(c) $\Longrightarrow$ (d): Let $u \in N_0^{1,p}(\Omega)$ and let $x \in \Omega$ be a Lebesgue point of u. Now

$$|u(x)| \le |u(x) - u_{B_x}| + |u_{B_x}|,$$

where, by (3.2)

$$|u(x) - u_{B_x}| \le C\,\mathrm{d}_\Omega(x)\big(\mathcal{M}_{\tau\,\mathrm{d}_\Omega(x)}\,g_u^p(x)\big)^{1/p},$$

and by (c)

$$|u_{B_x}| \le C\,\mathrm{d}_\Omega(x)\bigg(\fint_{20\tau B_x} g_u^p\,d\mu\bigg)^{1/p} \le C\,\mathrm{d}_\Omega(x)\big(\mathcal{M}_{20\tau\,\mathrm{d}_\Omega(x)}\,g_u^p(x)\big)^{1/p}.$$

The pointwise p-Hardy inequality follows from the above estimates. □

By slightly modifying the proof above or the proof in [21, Thm 3.9], we obtain a p-Hardy inequality containing a fractional maximal function of the upper gradient.

Corollary 3.2. *Let $1 \le p < \infty$ and let $\Omega \subset X$ be an open set whose complement is uniformly p-fat. Then there is a constant $C > 0$, independent of Ω, such that for all $0 \le \alpha < p$ and for all $u \in N_0^{1,p}(\Omega)$,*

$$|u(x)| \le C\,\mathrm{d}_\Omega(x)^{1-\alpha/p}\big(\mathcal{M}_{\alpha,20\tau\,\mathrm{d}_\Omega(x)}\,g_u^p(x)\big)^{1/p} \tag{3.3}$$

whenever $x \in \Omega$ is a Lebesgue point of u.

Here, for $\alpha \ge 0$, the restricted fractional maximal function of a locally integrable function u is

$$\mathcal{M}_{\alpha,R}\,u(x) = \sup_{0<r\le R} r^\alpha \fint_{B(x,r)} |u|\,d\mu.$$

4. From pointwise Hardy to fatness

In this section we prove the following lemma, from which the part (d)⇒(a) of Theorem 2.2 and the previously unknown necessity part of Theorem 1.1 follow.

Lemma 4.1. *Let $1 \le p < \infty$ and let $\Omega \subset X$ be an open set. If Ω admits the pointwise p-Hardy inequality* (2.5)*, then Ω^c is uniformly p-fat. The constant in the uniform fatness condition* (2.4) *depends only on p, c_H, and the constants related to X.*

Proof. Let $B = B(w,R)$, where $w \in \Omega^c$ and $0 < R < (1/6)\operatorname{diam} X$. By (2.3), it suffices to find a constant $C > 0$, independent of w and R, such that

$$\mu(B)R^{-p} \le C\int_{2B} g_v^p\,d\mu \tag{4.1}$$

whenever g_v is an upper gradient of a function $v \in N_0^{1,p}(2B)$ satisfying $0 \le v \le 1$ and $v = 1$ in $\Omega^c \cap \overline{B}$. By the quasicontinuity of $N^{1,p}$-functions, we may assume that $v = 1$ in an open neighborhood of $\Omega^c \cap \overline{B}$.

Let $l = \big[2(L+1)\big]^{-1}$, where L is from the pointwise p-Hardy inequality (2.5). The doubling condition implies that $\mu(lB) \ge l^s\mu(B)/c_D$. If now $v_B > l^s/2c_D$, we obtain from the Poincaré inequality for $v \in N_0^{1,p}(2B)$ (see for example [5, Proposition 3.1]) that

$$1 \le C\fint_B |v|\,d\mu \le CR\bigg(\fint_{2B} g_v^p\,d\mu\bigg)^{1/p},$$

and (4.1) follows by the doubling condition.

We may hence assume that $v_B \le l^s/2c_D$. Let $\psi \in N_0^{1,p}(B)$ be a cut-off function, defined as

$$\psi(x) = \max\Big\{0, 1 - \tfrac{4}{R}\,\mathrm{d}\big(x, \tfrac{1}{2}B\big)\Big\},$$

and take

$$u = \min\{\psi, 1 - v\}.$$

Since $1 - v = 0$ in an open set containing $\Omega^c \cap B$ and $N^{1,p}(X)$ is a lattice, we have that $u \in N_0^{1,p}(\Omega)$. Moreover, u has an upper gradient g_u such that $g_u = g_v$ in $1/2B$.

We define $C_1 = l^s/4c_D$ and

$$E = \big\{x \in lB : u(x) > C_1 \text{ and (2.5) holds for } u \text{ at } x\big\},$$

and claim that

$$\mu(E) \ge C_1 \mu(B). \tag{4.2}$$

To see this, first notice that $u = 1 - v$ in lB and that $\mu(lB) \ge 4C_1\mu(B)$. As $v_B \le l^s/2c_D = 2C_1$, we obtain

$$\begin{aligned}\int_{lB} u\,d\mu = \int_{lB} (1-v)\,d\mu &\ge \int_B (1-v)\,d\mu - \mu(B \setminus lB) \\ &\ge (1 - 2C_1)\mu(B) - \mu(B) + \mu(lB) \\ &\ge 2C_1\mu(B).\end{aligned} \tag{4.3}$$

Since the pointwise p-Hardy holds for almost every $x \in \Omega$, we have $u \le C_1$ almost everywhere in $lB \setminus E$. Thus a direct computation using estimate (4.3) yields (4.2):

$$\begin{aligned}\mu(E) \ge \int_E u\,d\mu &= \int_{lB} u\,d\mu - \int_{lB\setminus E} u\,d\mu \\ &\ge 2C_1\mu(B) - \int_{lB} C_1\,d\mu \\ &\ge 2C_1\mu(B) - C_1\mu(B) = C_1\mu(B).\end{aligned}$$

To continue the proof, we fix for each $x \in E$ a radius $0 < r_x \le L\,\mathrm{d}_\Omega(x)$ such that

$$\mathcal{M}_{L\,\mathrm{d}_\Omega(x)}\, g_u^p(x) \le 2 \fint_{B(x,r_x)} g_u^p\,d\mu.$$

By the standard $5r$-covering theorem (see e.g. [8]), there are pairwise disjoint balls $B_i = B(x_i, r_i)$, where $x_i \in E$ and $r_i = r_{x_i}$ are as above, so that $E \subset \bigcup_{i=1}^\infty 5B_i$. It follows immediately from (4.2) and the doubling condition that

$$\mu(B) \le C_1^{-1}\mu(E) \le C\sum_{i=1}^\infty \mu(B_i). \tag{4.4}$$

As $x_i \in lB$ and $w \notin \Omega$, we have $\mathrm{d}_\Omega(x_i) \le lR$. Hence, by the choice of l, we obtain for each $y \in B_i$ that

$$\mathrm{d}(w,y) \le \mathrm{d}(w,x_i) + \mathrm{d}(x_i,y) \le lR + L\,\mathrm{d}_\Omega(x_i) \le lR(1+L) = R/2,$$

and so $B_i \subset 1/2B$. This means, in particular, that $g_u = g_v$ in each B_i. Since $u(x_i) > C_1$ for each i, the pointwise p-Hardy inequality (2.5) and the choice of the radii r_i imply that

$$C_1^p \le |u(x_i)|^p \le C\,\mathrm{d}_\Omega(x_i)^p\,\mathcal{M}_{L\,\mathrm{d}_\Omega(x_i)}\,g_u^p(x) \le CR^p\mu(B_i)^{-1}\int_{B_i} g_u^p\,d\mu,$$

and so

$$\mu(B_i) \le CR^p\int_{B_i} g_v^p\,d\mu.$$

Inserting this into (4.4) leads us to

$$\mu(B) \le CR^p\sum_{i=1}^{\infty}\int_{B_i} g_v^p\,d\mu \le CR^p\int_{2B} g_v^p\,d\mu,$$

where we used the fact that the balls $B_i \subset 2B$ are pairwise disjoint. This proves estimate (4.1), and the lemma follows. □

5. From fatness to Hardy

The purpose of this section is to give a straight-forward proof for the fact that uniform p-fatness of the complement Ω^c suffices for Ω to admit the p-Hardy inequality. Our proof follows the ideas of Wannebo [31]. A similar method was also used in [7] in the context of Orlicz–Hardy inequalities. As mentioned earlier, the following result first appeared in the metric space setting in [6], where the proof was based on the self-improvement of uniform p-fatness.

Theorem 5.1. *Let $1 < p < \infty$ and let $\Omega \subset X$ be an open set. If Ω^c is uniformly p-fat then Ω admits the p-Hardy inequality, quantitatively.*

Proof. To make the proof as simple as possible, let us assume that the dilatation constant in the right-hand side of Theorem 2.2 (b) is 2. The general case follows by obvious modifications. Let

$$\Omega_n = \{x \in \Omega\,:\,2^{-n} \le \mathrm{d}_\Omega(x) < 2^{-n+1}\}$$

and

$$\widetilde{\Omega}_n = \bigcup_{k=n}^{\infty}\Omega_k.$$

Let $\mathcal{F}_n$ be a cover of Ω_n with balls of radius 2^{-n-2} such that their center points are not included in any other ball in $\mathcal{F}_n$. Associate to each ball $B \in \mathcal{F}_n$ a bigger ball $\widetilde{B} \supset B$, whose radius is 2^{-n+2} and whose center point is on $\partial\Omega$. Note that $2\widetilde{B} \cap \Omega \subset \widetilde{\Omega}_{n-2}$ and that

$$\sum_{B\in\mathcal{F}_n}\chi_B < C \quad\text{and}\quad \sum_{B\in\mathcal{F}_n}\chi_{2\widetilde{B}} < C,$$

where the constant $C > 0$ only depends on the doubling constant of μ.

Let $u \in N_0^{1,p}(\Omega)$. The condition (b) of Theorem 2.2 (which follows from the uniform p-fatness of the complement) implies that for every $B \in \mathcal{F}_n$ we have

$$\int_B |u|^p\,d\mu \le \int_{\widetilde{B}} |u|^p\,d\mu \le C2^{-np}\int_{2\widetilde{B}} g_u^p\,d\mu.$$

By summing up the inequalities above, we obtain

$$\begin{aligned}\int_{\Omega_n} |u|^p d\mu &\le \sum_{B\in\mathcal{F}_n} \int_B |u|^p d\mu \le C2^{-np} \sum_{B\in\mathcal{F}_n} \int_{2\widetilde{B}} g_u^p d\mu \\ &\le C2^{-np} \int_{\widetilde{\Omega}_{n-2}} g_u^p d\mu.\end{aligned} \tag{5.1}$$

Let $0 < \beta < 1$ be a small constant to be fixed later. We multiply (5.1) by $2^{n(p+\beta)}$ and sum the inequalities to obtain

$$\begin{aligned}\int_\Omega |u(x)|^p \,\mathrm{d}_\Omega(x)^{-p-\beta}\, d\mu &\le \sum_{n=-\infty}^{\infty} \int_{\Omega_n} |u(x)|^p 2^{n(p+\beta)}\, d\mu \\ &\le C \sum_{n=-\infty}^{\infty} 2^{n\beta} \int_{\widetilde{\Omega}_{n-2}} g_u(x)^p\, d\mu \\ &= C \sum_{k=-\infty}^{\infty} \left(\sum_{n=0}^{\infty} 2^{(k+2-n)\beta} \int_{\Omega_k} g_u(x)^p\, d\mu \right) \\ &\le C \sum_{k=-\infty}^{\infty} \frac{2^{k\beta}}{\beta} \int_{\Omega_k} g_u(x)^p\, d\mu \\ &\le \frac{C}{\beta} \int_\Omega g_u(x)^p \,\mathrm{d}_\Omega(x)^{-\beta}\, d\mu.\end{aligned} \tag{5.2}$$

In the calculations above, we used the fact that $2^{-k} \le \mathrm{d}_\Omega(x) \le 2 \cdot 2^{-k}$ for every $x \in \Omega_k$.

Now let $v \in N^{1,p}(\Omega)$ be a function with a compact support in Ω and let

$$u(x) = v(x)\,\mathrm{d}_\Omega(x)^{\beta/p}.$$

Then the function

$$g_u(x) = g_v(x)\,\mathrm{d}_\Omega(x)^{\beta/p} + \frac{\beta}{p} v(x)\,\mathrm{d}_\Omega(x)^{\beta/p-1}$$

is a p-weak upper gradient of u. Thus, by (5.2), we have

$$\begin{aligned}\int_\Omega \frac{v(x)^p}{\mathrm{d}_\Omega(x)^p}\, d\mu &= \int_\Omega \frac{u(x)^p}{\mathrm{d}_\Omega(x)^{p+\beta}}\, d\mu \le \frac{C}{\beta} \int_\Omega \frac{g_u(x)^p}{\mathrm{d}_\Omega(x)^\beta}\, d\mu \\ &\le \frac{C}{\beta} \int_\Omega g_v(x)^p\, d\mu + \frac{C}{\beta}\frac{\beta^p}{p^p} \int_\Omega \frac{v(x)^p}{\mathrm{d}_\Omega(x)^p}\, d\mu.\end{aligned}$$

If $\beta > 0$ is small enough, the last term on the right-hand side can be included on the left-hand side and we obtain

$$\int_\Omega \frac{v(x)^p}{\mathrm{d}_\Omega(x)^p}\, d\mu \le C \int_\Omega g_v(x)^p\, d\mu.$$

This completes the proof because functions with compact support are dense in $N_0^{1,p}(\Omega)$, see [30, Theorem 4.8]. □

Notice that the requirement $p > 1$ is essential in Theorem 5.1. For instance, smooth domains in $\mathbb{R}^n$ admit the pointwise 1-Hardy inequality but not the integral 1-Hardy.

6. Hausdorff contents

It is well-known that capacities and Hausdorff contents are closely related both in Euclidean spaces and general metric spaces, see e.g. [16] and [17]. In metric spaces we follow [1], [2], and [22], and use Hausdorff contents $\mathscr{H}^t_R$, defined by applying the Carathéodory construction to functions

$$h(B(x,r)) = \frac{\mu(B(x,r))}{r^t},$$

where $r \le R$. Thus the *Hausdorff content of codimension* t of a set $E \subset X$ is given by

$$\mathscr{H}^t_R(E) = \inf\Big\{\sum_{i\in I} h(B(x_i,r_i)) : E \subset \bigcup_{i\in I} B(x_i,r_i),\ r_i \le R\Big\}.$$

Here we may actually assume that $x_i \in E$, as this increases $\mathscr{H}^t_R(E)$ at most by a constant factor.

If the space X is Q-regular, then $\mathscr{H}^t_\infty(E)$ is comparable with the usual Hausdorff content $\mathcal{H}^{Q-t}_\infty(E)$, which is defined by using the gauge function $h(B(x,r)) = r^{Q-t}$. Recall that Q-regularity means that there are constants $c_1, c_2 > 0$ such that

$$c_1 r^Q \le \mu(B(x,r)) \le c_2 r^Q$$

for all balls $B(x,r)$ in X.

Now, by slightly modifying the argument in [17, Thm. 5.9] (see also [9] and [10]), one can show that if $E \subset X$ is a closed set and there exists some $1 \le q < p$ and a constant $C > 0$ so that for all $w \in E$ and every $R > 0$,

$$\mathscr{H}^q_{R/2}\big(E \cap \overline{B}(w,R)\big) \ge C\mu(\overline{B}(w,R))R^{-q}, \tag{6.1}$$

then $E \subset X$ is uniformly p-fat. Conversely, by rewriting the argument of [16, Thm. 2.27] (see also [9, Thm. 4.9]) for the content $\mathscr{H}^p_{R/2}$, it is not hard to see that uniform p-fatness of E leads to (6.1), but with q replaced by p. Using the self-improvement of uniform fatness, we then conclude that uniform p-fatness of E implies the existence of an exponent $q < p$ for which (6.1) holds. Hence (6.1), with an exponent $1 \le q < p$, is actually equivalent with the uniform p-fatness of E.

In this section, we investigate similar density conditions for the boundary of a domain $\Omega \subset X$. To this end, we consider a version of the pointwise Hardy inequality where the distance is taken to the boundary instead of the complement. We define

$$\delta_\Omega(x) = \mathrm{d}(x, \partial\Omega) \text{ for } x \in \Omega.$$

The following lemma is a metric space generalization of a result from [25] and [10].

Lemma 6.1. *Let $1 \le p < \infty$ and let $\Omega \subset X$ be an open set. Assume that Ω admits the pointwise p-Hardy inequality*

$$|u(x)| \le c_H \delta_\Omega(x)\Big(\mathcal{M}_{L\delta_\Omega(x)}\, g_u^p(x)\Big)^{1/p} \tag{6.2}$$

for all $u \in N^{1,p}_0(\Omega)$. Then

$$\mathscr{H}^p_{\delta_\Omega(x)}\big(\partial\Omega \cap \overline{B}(x, 2L\delta_\Omega(x))\big) \ge C\delta_\Omega(x)^{-p}\mu\big(\overline{B}(x,\delta_\Omega(x))\big). \tag{6.3}$$

for all $x \in \Omega$.

Proof. Let $x \in \Omega$. We define $R = \delta_\Omega(x)$, $B = \overline{B}(x,R)$, and

$$E = \partial\Omega \cap 2LB.$$

Let $\{B_i\}_{i=1}^N$, where $B_i = B(w_i, r_i)$ with $w_i \in E$ and $0 < r_i \le R$, be a covering of E; we may assume that the covering is finite by the compactness of E.

It is now enough to show that there exists a constant $C > 0$, independent of the particular covering, such that

$$\sum_{i=1}^N \mu(B_i){r_i}^{-p} \ge C\mu(B)R^{-p}. \tag{6.4}$$

If $r_i \ge R/4$ for some $1 \le i \le N$, then, by (2.1) and the fact that $r_i^{-p} \ge R^{-p}$, we have

$$\mu(B_i)r_i^{-p} \ge C\mu(B)\Big(\frac{r_i}{R}\Big)^s R^{-p} \ge C\mu(B)R^{-p},$$

from which (6.4) readily follows.

We may hence assume that $r_i < R/4$ for all $1 \le i \le N$. Now, define

$$\varphi(y) = \min_{1\le i\le N}\big\{1,\, r_i^{-1}\,\mathrm{d}(y, B_i)\big\}$$

and let $\psi \in N_0^{1,p}(2LB)$ be a cut-off function such that $0 \le \psi \le 1$ and $\psi(y) = 1$ for all $y \in LB$. Then the function

$$u = \min\{\psi, \varphi\}\chi_\Omega$$

belongs to $N_0^{1,p}(\Omega)$. As $r_i < R/4$ for all $1 \le i \le N$, it follows that $\mathrm{d}(x, 2B_i) \ge R/2$ for all $1 \le i \le N$, and so $u(x) = 1$.

In addition, u has an upper gradient g_u such that

$$g_u(y)^p \le \sum_{i=1}^N r_i^{-p}\chi_{2B_i}(y) \tag{6.5}$$

for a.e. $y \in LB$. Especially, we must have $r > R/2$ in order to obtain something positive when estimating $\mathcal{M}_{LR}\, {g_u}^p(x)$. As the pointwise inequality (6.2) holds for the continuous function $u \in N_0^{1,p}(\Omega)$ at every $x \in \Omega$, we have

$$\begin{aligned}1 = |u(x)|^p &\le CR^p\,\mathcal{M}_{LR}\, g_u^p(x) \le CR^p \sup_{R/2\le r\le LR} \fint_{B(x,r)} g_u^p\,d\mu\\ &\le CR^p\mu\big(\tfrac12 B\big)^{-1}\int_{LB} g_u^p\,d\mu \le CR^p\mu(B)^{-1}\sum_{i=1}^N \mu(2B_i)\, r_i^{-p},\end{aligned}$$

where the last inequality is a consequence of (6.5). Estimate (6.4) then easily follows with the help of the doubling property. □

Next we show that the inner boundary density condition (6.3) is actually almost equivalent to the pointwise p-Hardy inequality. The proof below uses an idea from [17], but is new in the context of Hardy inequalities.

Theorem 6.2. *Let $1 < p < \infty$ and let $\Omega \subset X$ be an open set. If estimate (6.3) holds with an exponent $1 \le q < p$ for all $x \in \Omega$, then Ω admits the pointwise p-Hardy inequality (6.2), but possibly with a different dilatation constant in the maximal function.*

Proof. Let us first assume that $u \in N_0^{1,p}(\Omega)$ has a compact support in Ω. Let $B = \overline{B}(x, R)$, where $x \in \Omega$ and $R = \delta_\Omega(x)$. We are going to show that

$$|u_B|^p \le C\delta_\Omega(x)^p \fint_{3\tau LB} g_u^p \, d\mu, \tag{6.6}$$

where $C > 0$ and $\lambda \ge 1$ are independent of x, whence the pointwise p-Hardy inequality follows for almost every $x \in \Omega$ by Theorem 2.2.

If $u_B = 0$, the claim (6.6) is true, and so we may assume that $|u_B| > 0$, and in fact, by homogeneity, that $|u_B| = 1$. Let $w \in \partial\Omega \cap 2LB$ and let $B_k = B(w, r_k)$, where $r_k = (5\tau 2^k)^{-1}R$, $k \in \mathbb{N}$. It then follows that

$$1 = |u(w) - u_B| \le |u_{B_0}| + |u_{B_0} - u_B|.$$

Now, if $|u_{B_0}| < 1/2$, we infer, using the $(1,p)$-Poincaré inequality, the facts $B_0 \subset 3LB$ and $B \subset 3LB_0$, and the doubling property, that

$$\tfrac{1}{2} \le |u_{B_0} - u_B| \le |u_{B_0} - u_{3B}| + |u_B - u_{3B}| \le CR\Big(\fint_{3\tau LB} g_u^p \, d\mu\Big)^{1/p}.$$

As $|u_B| = 1$, the claim follows.

Thus we may assume that $1/2 \le |u_{B_0}| = |u(w) - u_{B_0}|$ for every $w \in \partial\Omega \cap 2LB$. A standard chaining argument, using the $(1,p)$-Poincaré inequality (see for example [14]) and the assumption that the support of u is compact, leads us to estimate

$$1 \le C\sum_{k=0}^{\infty} r_k \left(\fint_{\tau B_k} g_u^p \, d\mu\right)^{1/p}. \tag{6.7}$$

From (6.7) it follows that there must be a constant $C_1 > 0$, independent of u and w, and at least one index $k_w \in \mathbb{N}$ such that

$$r_{k_w}\left(\fint_{\tau B_{k_w}} g_u^p \, d\mu\right)^{1/p} \ge C_1 2^{-k_w(1-q/p)} = C_1\left(\frac{r_{k_w}}{R}\right)^{1-q/p}.$$

In particular, we obtain for each $w \in \partial\Omega \cap 2LB$ a radius $r_w \le R/(5\tau)$ and a ball $B_w = B(w, r_w)$ such that

$$\mu(\tau B_w)\, r_w^{-q} \le CR^{p-q} \int_{\tau B_w} g_u^p \, d\mu. \tag{6.8}$$

Again, the $5r$-covering lemma implies the existence of points $w_1, w_2, \dots, w_N \in \partial\Omega \cap 2LB$ such that if we set $r_i = r_{w_i}$, then the balls $\tau B_i = B(w_i, \tau r_i)$ are pairwise disjoint, but still $\partial\Omega \cap 2LB \subset \bigcup_{i=1}^N 5\tau B_i$. Assumption (6.3), the doubling property, estimate (6.8), and the pairwise disjointness of the balls

$\tau B_i \subset 3\tau LB$ then yield

$$
\begin{aligned}
R^{-q}\mu(B) &\leq C\mathscr{H}^q_R(\partial\Omega \cap 2LB) \\
&\leq C\sum_{i=1}^{N} \mu(5\tau B_i)(5\tau r_i)^{-q} \leq C\sum_{i=1}^{N} \mu(\tau B_i) r_i{}^{-q} \\
&\leq C\sum_{i=1}^{N} R^{p-q}\int_{\tau B_i} g_u^p \, d\mu \leq CR^{p-q}\int_{3\tau LB} g_u^p \, d\mu.
\end{aligned}
\tag{6.9}
$$

As we assumed $|u_B| = 1$, estimate (6.6) now follows from (6.9) and the doubling condition.

For a general $u \in N_0^{1,p}(\Omega)$ estimate (6.6) follows by a suitable approximation with compactly supported functions. □

If there now exists a constant $C \geq 1$ such that

$$
\mathrm{d}_\Omega(x) \leq \delta_\Omega(x) \leq C\,\mathrm{d}_\Omega(x) \quad \text{for each } x \in \Omega, \tag{6.10}
$$

then it is clear that pointwise inequalities (2.5) and (6.2) are quantitatively equivalent. In particular, if the inner boundary density condition (6.3) with codimension q holds for all $x \in \Omega$, then Theorems 6.2 and 2.2 imply that Ω^c is uniformly p-fat for all $p > q$. On the other hand, easy examples show that Ω^c need not be uniformly q-fat, or equivalently, Ω need not admit the pointwise q-Hardy inequality, if $q > 1$. Hence some information is inevitably lost once we pass from the pointwise p-Hardy inequality or uniform p-fatness (for $1 < p < \infty$) to Hausdorff contents; in the case $p = 1$ there is indeed an equivalence, cf. [22]. However, by the self-improvement of the assertions of Theorem 2.2, we can still have the following equivalent characterization in terms of Hausdorff contents (see also [25] and [10]). Note that here we need to use again the fact that X supports a $(1, q)$-Poincaré inequality for some $q < p$.

Corollary 6.3. *Assume that $\Omega \subset X$ is such that (6.10) holds. Then all of the assertions in Theorem 2.2, with an exponent $1 < p < \infty$, are (quantitatively) equivalent to the following density condition: There exist some $1 < q < p$ and constants $C > 0$ and $L \geq 1$ such that*

$$
\mathscr{H}^q_{\delta_\Omega(x)}\big(\partial\Omega \cap \overline{B}(x, L\delta_\Omega(x))\big) \geq C\delta_\Omega(x)^{-q}\mu\big(\overline{B}(x, \delta_\Omega(x))\big)
$$

for all $x \in \Omega$.

It is worth a mention that uniform p-fatness of the boundary $\partial\Omega$ is of course sufficient for the uniform p-fatness of the complement and the pointwise p-Hardy inequality, but not necessary, as cusp-type domains in $\mathbb{R}^n$, $n \geq 3$, show (cf. [25]). Thus it really is essential that we consider above the density of the boundary only as seen from within the domain, in the sense of (6.3).

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