A NOTE ON THE DIMENSIONS OF ASSOUAD AND AIKAWA

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Abstract. We show that in Euclidean and other regular metric spaces the notions of dimensions defined by Assouad and Aikawa coincide. In addition, we study the relationship between these two dimensions and a related codimension in more general metric spaces and give an application of the Aikawa (co)dimension for the Hardy inequalities.

1. Introduction

In this note, we consider two notions of dimension which arise from quite different settings but still turn out to be intimately connected with each other. First, we have the Assouad dimension, a purely metrical concept that was defined by Patrice Assouad (see [3, 4]) while studying which metric spaces are bi-Lipschitz embeddable in some $\mathbb{R}^n$; a thorough discussion of this dimension can be found in [12]. The second dimension under consideration was used by Hiroaki Aikawa [1] (see also [2, Sect. 7]) in his results on the quasiadditivity properties of capacities. This notion, for $E \subset \mathbb{R}^n$, was defined in terms of the behavior of integrals of the powers of the distance function $\operatorname{dist}(y, E)$. Subsequently, the same concept has proven to be useful for instance in connection with the Hardy inequalities, cf. [10]. The exact definitions of the Assouad and the Aikawa dimension can be found in Section 3.

It seems that the possibility of a connection between these two dimensions has been overlooked in the past. We fix this gap by proving as our main result

Theorem 1.1. *Let $E \subset \mathbb{R}^n$. Then the Assouad dimension of E equals the Aikawa dimension of E.*

More generally, if X is a Q-regular metric space, then the above conclusion holds for each $E \subset X$ as well. If the space X is not Q-regular, the situation becomes more subtle: We propose an extension of the Aikawa dimension and a related codimension in this setting and show the usefulness of these concepts with a result concerning the Hardy inequalities. However, we also give an example which shows that this more general concept no longer agrees with the Assouad dimension.

The paper is organized as follows. In Section 2, we briefly recall the notation and the terminology related to general metric spaces. Section 3 contains discussion on different notions of dimension. Our main results are stated and proved in Section 4, which also contains an example of a set whose Aikawa dimension is strictly larger than its Assouad dimension. In Section 5 we give an alternative characterization for

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the Aikawa codimension, and finally, in Section 6, we study the relationship between the Aikawa (co)dimension and the Hardy inequalities.

2. Metric Spaces

We assume that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X : d(y, x) < r\}$. For $0 < t < \infty$, we write $tB = B(x, tr)$. When $E \subset X$, we let $\operatorname{diam}(E)$ denote the diameter of E, and for $x \in X$ we use $d(x, E)$ to denote the distance from x to E. When $\delta > 0$, we define *the δ-neighborhood* of E (in X) as $E_\delta = \{y \in X : d(y, E) < \delta\}$.

We assume that the measure μ is *doubling*, which means that there is a constant $c_D \geq 1$, called *the doubling constant of* μ, such that

$$\mu(2B) \leq c_D\, \mu(B)$$

for all balls $B \subset X$.

The doubling condition gives an upper bound for the dimension of X. By this we mean that there is a constant $C > 0$ and an exponent $s \geq 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\Big(\frac{r}{R}\Big)^s \tag{1}$$

whenever $0 < r \leq R < \operatorname{diam}(X)$, $x \in X$, and $y \in B(x, R)$. Inequality (1) holds certainly with $s = \log_2 c_D$ (and $C = C(c_D) > 0$), but it may hold for some smaller exponents as well. With a slight abuse of the notation the infimum of the exponents for which (1) holds is also denoted by s and called *the doubling dimension* of X.

The measure μ is Q-regular, for $Q \geq 1$, if there is a constant $c_Q \geq 1$ such that

$$c_Q^{-1} r^Q \leq \mu(B(x,r)) \leq c_Q r^Q$$

for all $x \in X$ and every $0 < r < \operatorname{diam}(X)$. It is immediate that the doubling dimension of a Q-regular space is Q.

In general, C and c will denote positive constants whose value is not necessarily the same at each occurrence. By writing $C = C(K, \lambda)$, we indicate that the constant depends only on K and λ. If there exist constants $c_1, c_2 > 0$ such that $c_1 F \leq G \leq c_2 F$, we write $F \approx G$ and say that F and G are comparable.

3. Concepts of dimension

3.1. **Hausdorff and Minkowski.** We define *λ-Hausdorff contents* of a set $E \subset X$, for $0 < r \leq \infty$, $\lambda > 0$, as

$$\mathcal{H}^\lambda_r(E) = \inf\Big\{\sum_k r_k^\lambda : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \leq r\Big\},$$

and the *λ-Hausdorff measure* of E is $\mathcal{H}^\lambda(E) = \lim_{r\to 0} \mathcal{H}^\lambda_r(E)$. The *Hausdorff dimension* of E is then the number

$$\dim_{\mathcal{H}}(E) = \inf\{\lambda > 0 : \mathcal{H}^\lambda(E) = 0\}.$$

When the balls covering the set $E \subset X$ are required to be of equal radii, we obtain *λ-Minkowski contents* of E:

$$\mathcal{M}^\lambda_r(E) = \inf\Big\{N r^\lambda : E \subset \bigcup_{k=1}^N B(x_k, r),\ x_i \in E\Big\}.$$

The *lower and upper Minkowski dimension* of E are then defined to be

$$\underline{\dim}_{\mathcal{M}}(E) = \inf\Big\{\lambda > 0 : \liminf_{r\to 0} \mathcal{M}_r^{\lambda}(E) = 0\Big\}$$

and

$$\overline{\dim}_{\mathcal{M}}(E) = \inf\Big\{\lambda > 0 : \limsup_{r\to 0} \mathcal{M}_r^{\lambda}(E) = 0\Big\},$$

respectively.

In general metric spaces it is often more convenient to use modified versions of $\mathcal{H}_r^{\lambda}$ and $\mathcal{M}_r^{\lambda}$, namely the *Hausdorff and Minkowski contents of codimension q*. The former is defined for a set $E \subset X$ by

$$\widetilde{\mathcal{H}}_r^q(E) = \inf\Big\{\sum_k \mu(B_k)\, r_k^{-q} : E \subset \bigcup_k B_k,\ x_k \in E,\ 0 < r_k \le r\Big\},$$

where we write $B_k = B(x_k, r_k)$, and the latter by

$$\widetilde{\mathcal{M}}_r^q(E) = \inf\Big\{r^{-q} \sum_k \mu(B(x_k, r)) : E \subset \bigcup_k B(x_k, r),\ x_k \in E\Big\}.$$

Then, naturally, the *Hausdorff measure of codimension* q is defined as

$$\widetilde{\mathcal{H}}^q(E) = \lim_{r\to 0} \widetilde{\mathcal{H}}_r^q(E).$$

Note that in a Q-regular space $\widetilde{\mathcal{H}}_r^q \approx \mathcal{H}_r^{Q-q}$ and $\widetilde{\mathcal{M}}_r^q \approx \mathcal{M}_r^{Q-q}$.

Finally, the *Hausdorff and the (upper and lower) Minkowski codimensions* are defined respectively as

$$\operatorname{codim}_{\mathcal{H}}(E) = \sup\{q > 0 : \widetilde{\mathcal{H}}^q(E) = 0\},$$

$$\underline{\operatorname{codim}}_{\mathcal{M}}(E) = \sup\Big\{q > 0 : \liminf_{r\to 0} \widetilde{\mathcal{M}}_r^q(E) = 0\Big\},$$

and

$$\overline{\operatorname{codim}}_{\mathcal{M}}(E) = \sup\Big\{q > 0 : \limsup_{r\to 0} \widetilde{\mathcal{M}}_r^q(E) = 0\Big\}.$$

3.2. **Assouad.** There are many known equivalent definitions for the Assouad dimension and the same concept has appeared on many occasions under different names (in fact, we add one more to this list in the present paper as far as Q-regular spaces are concerned). See Luukkainen [12], especially his Remark 3.6, for a historical account (and references) on alternative definitions and terminology. We use the following definition essentially from Heinonen [6].

Definition 3.1. When $E \subset X$, we let $\operatorname{Cov}(E)$ denote the set of all $\beta > 0$ for which the following covering property holds: There exists $c_\beta \ge 1$ such that, for all $0 < \varepsilon < 1/2$, each subset $F \subset E$ can be covered by at most $c_\beta \varepsilon^{-\beta}$ balls of radius $r = \varepsilon \operatorname{diam}(F)$. *The Assouad dimension* of $E \subset X$ is then defined to be

$$\dim_{\mathcal{AS}}(E) = \inf\{\beta \in \operatorname{Cov}(E)\}.$$

This is a purely 'intrinsic' and 'metrical' definition. It is worth a mention that, according to [12, Remark 3.6], the first appearance of a related concept was in Bouligand [5] (from 1928), and there the definition was 'external' and using also the measure near the set E — much in the spirit of the definition of the Aikawa dimension below. In fact, in non-regular spaces the 'Bouligand-type' approach leads to an alternative characterization of the Aikawa codimension; see Section 5.

Let us remark here that the Assouad dimension has recently attained considerable interest in analysis on metric spaces, for instance in questions related to quasisymmetric uniformization; see e.g. [6], [7], [13], and [15].

3.3. **Aikawa.** In the paper [1], Aikawa introduced an 'external' notion of dimension given in terms of integrals of the distance function. In a general metric space $X = (X, \mu, d)$ of doubling dimension s we can state the definition as follows:

Definition 3.2. Let $E \subset X$ and let $G(E)$ be the set of those $t > 0$ for which there exists a constant c_t such that

$$\int_{B(x,r)} d(y,E)^{t-s}\, d\mu \le c_t r^{t-s} \mu(B(x,r)) \tag{2}$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(E)$. Then *the Aikawa dimension* of E is defined to be $\dim_{\mathcal{AI}}(E) = \inf G(E)$.

We use above the convention that if the set $E \subset X$ has positive measure, then $\dim_{\mathcal{AI}}(E) = s$, and thus for each $E \subset X$ we have $0 \le \dim_{\mathcal{AI}}(E) \le s$.

It is clear that the above definition agrees with the definition from [1, 2] in $\mathbb{R}^n$ and also in Q-regular spaces. In particular, the requirement $r < \operatorname{diam}(E)$ is unnecessary in regular spaces. However, if the space X is not regular, then, in accordance with Section 3.1, it is more convenient to rewrite (2) without an explicit use of the doubling dimension s; this leads to the related codimension:

Definition 3.3. *The Aikawa co-dimension* of $E \subset X$ is $\operatorname{codim}_{\mathcal{AI}}(E) = \sup F(E)$, where $F(E)$ is the set of those $q > 0$ for which there exists a constant c_q such that

$$\int_{B(x,r)} d(y,E)^{-q}\, d\mu \le c_q r^{-q} \mu(B(x,r))$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(E)$.

If the space X has doubling dimension s, then $\operatorname{codim}_{\mathcal{AI}}(E) = s - \dim_{\mathcal{AI}}(E)$ for every $E \subset X$. To prove this we only need to observe that always $\operatorname{codim}_{\mathcal{AI}}(E) \le s$:

Lemma 3.4. *Let s be the doubling dimension of X and let $E \subset X$. Then*

$$0 \le \operatorname{codim}_{\mathcal{AI}}(E) \le s.$$

Proof. Let $q > s$ and $c > 0$. As $q > s$, we may actually assume that the estimate (1) holds for s. It suffices to show that there are $x \in E$ and $0 < R < \operatorname{diam}(E)$ such that

$$\mu(B(x,R)) < \frac{1}{c} R^q \int_{B(x,R)} d(y,E)^{-q}\, d\mu. \tag{3}$$

Assume first that $\operatorname{diam}(E) > 1$. Let $x \in E$, $R > 1$, and $0 < r < \min\{1, C/c\}$, where C is the constant from (1). Then, using (1), we have that

$$\begin{aligned} \mu(B(x,R)) &\le \frac{1}{C}\Big(\frac{R}{r}\Big)^s \mu(B(x,r)) \le \frac{1}{C}\frac{R^q}{r^s} r^q \int_{B(x,r)} d(y,E)^{-q}\, d\mu \\ &\le \frac{r}{C} R^q \int_{B(x,r)} d(y,E)^{-q}\, d\mu < \frac{1}{c}\int_{B(x,r)} d(y,E)^{-q}\, d\mu. \end{aligned}$$

Assume then that $\operatorname{diam}(E) \le 1$. Let $d/2 \le R < d$ and $0 < r < \min\{d/2, (dC)/(2c)\}$. Then $R^s \le (2/d)^{q-s} R^q$ and a similar estimate as above shows that (3) holds. Hence the claim follows. □

3.4. **Basic relations.** It is well-known that for each $E \subset X$ we have

$$\dim_{\mathcal{H}}(E) \le \dim_{\mathcal{AS}}(E) \le s\,; \tag{4}$$

the second inequality follows with a simple calculation using the estimate (1) and a basic covering theorem. Moreover, if $E \subset X$ is bounded, then

$$\dim_{\mathcal{H}}(E) \le \underline{\dim}_{\mathcal{M}}(E) \le \overline{\dim}_{\mathcal{M}}(E) \le \dim_{\mathcal{AS}}(E); \tag{5}$$

see for instance [12, Thm A.5].

All of the inequalities in (5) can be strict. For example, if $E = \{(j^{-1}, 0, \dots, 0) : j \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}^n$, then $\dim_{\mathcal{H}}(E) = 0$, $\dim_{\mathcal{M}}(E) = 1/2$, and $\dim_{\mathcal{AS}}(E) = 1$. However, for many sufficiently regular sets all of the dimensions considered above agree. In particular, if X is Q-regular and if a bounded subset $E \subset X$ is *(Ahlfors)* λ*-regular*, i.e., $\mathcal{H}^\lambda(E \cap B(x,r)) \approx r^\lambda$ whenever $x \in E$ and $0 < r < \operatorname{diam}(E)$, then it is not hard to show that $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{AS}}(E) = \lambda\big(= \dim_{\mathcal{AI}}(E)\big)$; (cf. e.g. [10, Lemma 2.1]).

For the codimensions we have the following order relations (compare to (4) and (5)):

Lemma 3.5. *Let* $E \subset X$. *Then*

$$\operatorname{codim}_{\mathcal{H}}(E) \ge \operatorname{codim}_{\mathcal{AI}}(E).$$

Moreover, if $E \subset X$ *is bounded, then*

$$\operatorname{codim}_{\mathcal{H}}(E) \ge \underline{\operatorname{codim}}_{\mathcal{M}}(E) \ge \overline{\operatorname{codim}}_{\mathcal{M}}(E) \ge \operatorname{codim}_{\mathcal{AI}}(E). \tag{6}$$

Proof. The first two inequalities in (6) are almost immediate from the definitions. The last one is closely related to [9, Lemma 2.6], and it follows from the fact that if $q < \operatorname{codim}_{\mathcal{AI}}(E)$, $0 < r < \operatorname{diam}(E)$, and $x_0 \in E$ is a fixed point, then

$$\widetilde{\mathcal{M}}^q_r(E) \le C \int_{B(x_0, 2\operatorname{diam}(E))} d(y,E)^{-q}\, d\mu(y) < \infty$$

(compare this to the calculation in (7) below).

The first claim of the Lemma, for an unbounded E, follows from (6) by dividing E into bounded subsets. □

Remark 3.6. The proof of Lemma 3.5 reveals one essential difference between the Minkowski and the Aikawa (co)dimensions: The Minkowski dimension of $E \subset X$ is related to the finiteness of the integrals $\int_{E_\delta} d(y,E)^{-q}\, d\mu$, whereas when estimating the Aikawa dimension we require locally uniform estimates for all the integrals $\int_{B(x,r)} d(y,E)^{-q}\, d\mu$, where $x \in E$ and $0 < r < \operatorname{diam}(E)$.

4. Main results

Let us now turn to our main results. We begin by proving that in a metric space X with a doubling measure the Aikawa dimension of a subset $E \subset X$ is at least the Assouad dimension of E, and then show that if the measure μ is Q-regular, then also the converse is true and hence the Aikawa dimension and the Assouad dimension coincide in X. In particular, this last claim is true in Euclidean spaces equipped with the corresponding Lebesgue measures. Without the Q-regularity, the Aikawa dimension can be strictly larger as our Example 4.3 shows.

Theorem 4.1. *Let* μ *be a doubling measure and let* $E \subset X$. *Then* $\dim_{\mathcal{AS}}(E) \le \dim_{\mathcal{AI}}(E)$.

Proof. If $\dim_{\mathcal{AI}}(E) = s$, then the claim follows from (4). Now, let

$$\dim_{\mathcal{AI}}(E) < t < t' < s < s'$$

be such that $t-s = t'-s'$. Let $F \subset E$ be a non-empty subset with $\operatorname{diam}(F) = d$, take any $w \in F$, and let $0 < \varepsilon < 1/2$. By the $5r$-covering theorem, there is a covering $\{5B_i\}$ of F, where $B_i = B(w_i, \varepsilon d)$, $w_i \in F$, such that the balls B_i are pairwise disjoint. Then $\bigcup_i B_i \subset F_{\varepsilon d}$.

Using the definition of the Aikawa dimension, and the fact that $d(y,E) \le \varepsilon d$ for each $y \in F_{\varepsilon d}$ (recall that here $F_{\varepsilon d}$ is the εd-neighborhood of F), we obtain

$$\begin{aligned}
\sum_i \mu(B_i) &\le \mu(F_{\varepsilon d}) \le (\varepsilon d)^{s-t} \int_{F_{\varepsilon d}} d(y,E)^{t-s}\, d\mu \\
&\le (\varepsilon d)^{s'-t'} \int_{B(w,2d)} d(y,E)^{t-s}\, d\mu \qquad (7)\\
&\le (\varepsilon d)^{s'-t'} c_t (2d)^{t-s} \mu(B(w,2d)) \\
&\le c\, \varepsilon^{s'-t'} \mu(B(w,2d)).
\end{aligned}$$

Since $B_i \subset B(w,2d)$ for each i, the lower density property (1) for $s' > s$ implies

$$\sum_i \mu(B_i) \ge cN\varepsilon^{s'} \mu(B(w,2d)),$$

where N is the number of balls B_i. Combining this with estimate (7) we obtain $N \le c\varepsilon^{-t'}$, and thus $\dim_{\mathcal{AS}}(E) \le t'$. As this holds for all $\dim_{\mathcal{AI}}(E) < t' < s$, we conclude that $\dim_{\mathcal{AS}}(E) \le \dim_{\mathcal{AI}}(E)$. □

Theorem 4.2. *Let μ be a Q-regular measure and let $E \subset X$. Then* $\dim_{\mathcal{AI}}(E) \le \dim_{\mathcal{AS}}(E)$.

Proof. Let $E \subset X$, let $t > \dim_{\mathcal{AS}}(E)$, and let $\beta = \frac{t+\dim_{\mathcal{AS}}(E)}{2} \in \operatorname{Cov}(E)$ (cf. Definition 3.1). Let $B = B(w,r)$, where $w \in E$ and $0 < r < \operatorname{diam}(E)$.

For each $j \in \{0,1,\dots\}$, let

$$E_j = \{x \in B(w,2r) : d(x,E) < 2^{-j}r\}$$

and $A_j = E_j \setminus E_{j+1}$. As $\beta \in \operatorname{Cov}(E)$, each E_j can be covered by $N_j = C_1 2^{j\beta}$ balls of radius $4r2^{-j}$. If B_i^j are such balls, then, by the Q-regularity,

$$\mu(E_j) \le \sum_i \mu(B_i^j) \le N_j c_Q (4r2^{-j})^Q \le C(2^{-j})^{Q-\beta}\mu(B). \qquad (8)$$

Using (8) and the fact that if $x \in A_j$ then $d(x,E)$ is comparable to $2^{-j}r$, we obtain

$$\begin{aligned}
\int_B d(x,E)^{t-Q}\, d\mu &\le \sum_{j=0}^{\infty} \int_{A_j} d(x,E)^{t-Q}\, d\mu \le C \sum_{j=0}^{\infty} \mu(E_j)(2^{-j}r)^{t-Q} \\
&\le Cr^{t-Q}\mu(B) \sum_{j=0}^{\infty} (2^{-j})^{t-\beta} \le Cr^{t-Q}\mu(B),
\end{aligned}$$

and thus $\dim_{\mathcal{AI}}(E) \le t$. Since this holds for all $t > \dim_{\mathcal{AS}}(E)$, we conclude that $\dim_{\mathcal{AI}}(E) \le \dim_{\mathcal{AS}}(E)$. □

Proof of Theorem 1.1. The claim follows from Theorems 4.1 and 4.2. □

Let us now show that the conclusion of the previous theorem need not hold if the space is not Q-regular:

Example 4.3. Let $X = \mathbb{R}^2$ with the Euclidean distance and the measure

$$d\mu = (1 + |x_2|^\alpha) d\mathcal{L}^2, \qquad -1 < \alpha < 0,$$

and let $E = \mathbb{R} = \{x \in \mathbb{R}^2 : x_2 = 0\}$ (we write $x = (x_1, x_2)$ for the points and $\mathcal{L}^2$ for the Lebesgue measure in $\mathbb{R}^2$). It is clear that $\dim_{\mathcal{AS}}(E) = 1$.

For a cube $Q = (-r, r) \times (-r, r)$, $r > 0$, we have that

$$\mu(Q) = \int_Q 1 + |x_2|^\alpha \, dx = \mathcal{L}^2(Q) + \frac{4}{\alpha + 1} r^{2+\alpha},$$

which is comparable with $r^2 + r^{2+\alpha}$. The same measure estimate holds for any cube $Q = (x - r, x + r) \times (-r, r)$ whose center is in E and for any ball with center in E and radius r, and thus in fact for all balls $B \subset \mathbb{R}^2$. It follows from this that the measure μ is doubling with doubling dimension $s = 2$. However, μ is not Q-regular for any $Q \geq 1$.

To calculate the Aikawa dimension of E, we have to check for which exponents $t > 0$ inequality (2) holds. It is enough to consider cubes $Q = (-r, r) \times (-r, r)$, $r > 0$. Since $d(x, E) = |x_2|$ for each $x \in Q$, we have that

$$\int_Q d(x, E)^{t-2} \, d\mu = \int_Q \frac{1 + |x_2|^\alpha}{d(x, E)^{t-2}} \, dx = \int_Q |x_2|^{t-2} \, dx + \int_Q |x_2|^{t-2+\alpha} \, dx.$$

Thus, if $0 < t \leq 1 - \alpha$, then $\int_Q d(x, E)^{t-2} \, d\mu = \infty$, and if $t > 1 - \alpha$, then

$$\int_Q d(x, E)^{t-2} \, d\mu = \frac{4}{t - 1} r^t + \frac{4}{t - 1 + \alpha} r^{t+\alpha}.$$

This, together with the above estimate $\mu(Q) \approx r^2 + r^{2+\alpha}$, shows that

$$\int_Q d(x, E)^{t-2} \, d\mu \leq C(t) r^{t-2} \mu(Q)$$

for $t > 1 - \alpha$. We conclude that $\dim_{\mathcal{AI}}(E) = 1 - \alpha$, which is strictly larger than $1 = \dim_{\mathcal{AS}}(E)$ for any $-1 < \alpha < 0$.

5. Bouligand-type characterization

It is possible to give an equivalent definition for the Aikawa codimension of $E \subset X$ by looking only the measures of the neighborhoods of E, as follows; for the Assouad dimension in $\mathbb{R}^n$ a similar characterization is discussed in [12, Thm A.12].

Theorem 5.1. *Let $E \subset X$ and let $\widetilde{F}(E)$ be the set of those $\alpha > 0$ for which there exists a constant c_α such that*

$$\mu\big(E_{tR} \cap B(x, R)\big) \leq c_\alpha t^\alpha \mu(B(x, R))$$

for every $x \in E$, $0 < R < \operatorname{diam}(E)$, and $0 < t < 1$. Then $\operatorname{codim}_{\mathcal{AI}}(E) = \sup \widetilde{F}(E)$.

Proof. Let us first show that $\operatorname{codim}_{\mathcal{AI}}(E) \geq \sup \widetilde{F}(E)$. Let $q < \sup \widetilde{F}(E)$, take $\alpha = \frac{q + \sup \widetilde{F}(E)}{2}$, and fix $B = B(w, R)$, where $w \in E$ and $0 < R < \operatorname{diam}(E)$. Denote, as in the proof of Theorem 4.2,

$$E_j = \{x \in B(w, R) : d(x, E) < 2^{-j} R\}$$

and $A_j = E_j \setminus E_{j+1}$ for $j \in \{0, 1, \ldots\}$. As $\alpha \in \widetilde{F}(E)$, we have

$$\mu(E_j) \le c_\alpha (2^{-j})^\alpha \mu(B),$$

and thus

$$\begin{aligned}\int_B d(x,E)^{-q}\,d\mu &\le \sum_{j=0}^{\infty} \int_{A_j} d(x,E)^{-q}\,d\mu \le C \sum_{j=0}^{\infty} \mu(E_j)(2^{-j}R)^{-q} \\ &\le CR^{-q}\mu(B) \sum_{j=0}^{\infty} (2^{-j})^{\alpha-q} \le CR^{-q}\mu(B).\end{aligned}$$

Hence $q \le \operatorname{codim}_{\mathcal{AI}}(E)$, and we conclude that $\sup \widetilde{F}(E) \le \operatorname{codim}_{\mathcal{AI}}(E)$.

To prove the converse inequality we let $\alpha < \operatorname{codim}_{\mathcal{AI}}(E)$. Then, for each $x \in E$, $0 < R < \operatorname{diam}(E)$, and $0 < t < 1$, we have

$$\mu\big(E_{tR} \cap B(x,R)\big) \le (tR)^\alpha \int_{B(x,R)} d(y,E)^{-\alpha}\,d\mu \le c_\alpha t^\alpha R^\alpha R^{-\alpha} \mu(B(x,R)).$$

Thus $\alpha \in \widetilde{F}(E)$, and the claim follows. □

6. Hardy inequalities

In this section we assume that $X = (X, \mu, d)$ is a metric measure space equipped with a doubling measure μ and *supporting a p-Poincaré inequality.* We refer to [6] (see also [8]) for the basic information on such spaces.

When $\Omega \subset X$ is a domain, we write $\Omega^c = X \setminus \Omega$ and $d_\Omega(x) = \operatorname{dist}(x, \Omega^c)$ for $x \in \Omega$. The pointwise Lipschitz constant of a function $u \colon \Omega \to \mathbb{R}$ at $x \in \Omega$ is

$$\operatorname{Lip}(u;x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x,y)}.$$

We say that a domain $\Omega \subset X$ admits *the p-Hardy inequality* if there exists a constant $C > 0$ such that

$$\int_\Omega \frac{u(x)^p}{d_\Omega(x)^p}\,d\mu(x) \le C \int_\Omega \operatorname{Lip}(u;x)^p\,d\mu(x) \tag{9}$$

whenever u is a Lipschitz function with a compact support in Ω. See [8] and [9] for more details on Hardy inequalities in metric spaces.

The connection between Hardy inequalities and the Aikawa dimension was recently studied in [10] (see also [9]), but a similar concept of dimension appeared in this context already in the unpublished works of Wannebo in the 80's, see [14, pp. 13–14]. In the metric setting we have the following dichotomy concerning the dimension of the complement a domain admitting the p-Hardy inequality:

Theorem 6.1. *Let $1 < p < \infty$ and assume that a domain $\Omega \subset X$ admits the p-Hardy inequality. Then there exists $\varepsilon > 0$, depending only on the given data, such that either (i) $\operatorname{codim}_{\mathcal{H}}(\Omega^c) \le p - \varepsilon$ or (ii) $\operatorname{codim}_{\mathcal{AI}}(\Omega^c) \ge p + \varepsilon$.*

In fact, such a dichotomy holds also locally, in the following sense.

Theorem 6.2. *Let $1 < p < \infty$ and assume that a domain $\Omega \subset X$ admits the p-Hardy inequality. Then there exists $\varepsilon > 0$, depending only on the given data, such that for each ball $B_0 \subset X$ either*

$$\operatorname{codim}_{\mathcal{H}}(2B_0 \cap \Omega^c) \le p - \varepsilon$$

or

$$\operatorname{codim}_{\mathcal{AI}}(B_0 \cap \Omega^c) \geq p + \varepsilon.$$

For Q-regular spaces, Theorems 6.1 and 6.2 were essentially proven in [9], although with a slightly weaker formulation involving Hausdorff and *Minkowski* dimensions; see also [10] for alternative proofs of Theorems 6.1 and 6.2 (in $\mathbb{R}^n$). Let us recall here the main points of the proof, including the minor modifications needed in the case of a non-regular space X.

Proof of Theorem 6.2. We proceed just as in [9, Corollary 2.7]: First, there exists $\varepsilon_0 > 0$ such that Ω admits q-Hardy inequalities for all $p - \varepsilon_0 < q \leq p$ with a uniform constant ([9, Thm. 2.2]). Let $0 < \varepsilon < \varepsilon_0/2$ to be fixed later, and take an arbitrary ball $B_0 = B(x_0, r_0)$. If $\operatorname{codim}_{\mathcal{H}}(2B_0 \cap \Omega^c) \leq p - \varepsilon$, we are done. Thus we may assume

$$\operatorname{codim}_{\mathcal{H}}(2B_0 \cap \Omega^c) > p - \varepsilon, \tag{10}$$

whence it follows that the $(p-\varepsilon)$-Hardy inequality actually holds for all Lipschitz functions with a compact support in $\Omega \cup 2B_0$; here one can use either (i) a capacitary argument, since from (10) it follows that the set $\Omega \cup 2B_0$ is of variational $(p-\varepsilon)$-capacity zero (see e.g. [11, Prop. 4.1]), or (ii) direct approximation as in [10]. Furthermore, there exists $\delta > 0$ (independent of the particular ε) such that

$$\int_{B(x,r)} d(y, \Omega^c)^{-p+\varepsilon-\delta}\, d\mu \leq C r^{-p+\varepsilon-\delta} \mu(B(x,r)) \tag{11}$$

whenever $x \in \Omega^c \cap B_0$ and $0 < r < r_0$ (this is [9, Lemma 2.4]; for an alternative approach see [10, Lemma 5.2]). If we now choose $\varepsilon = \min\{\varepsilon_0/2, \delta/2\}$, it follows from (11) that $\operatorname{codim}_{\mathcal{AI}}(B_0 \cap \Omega^c) \geq p + \varepsilon$.

Notice that in the above reasoning we actually need q-Poincaré inequalities for some $q < p$, but by the self-improvement property of Poincaré inequalities these are available. □

The proof of the global dichotomy of Theorem 6.1 follows along the same lines.

Remark 6.3. Similar results hold for the so-called weighted Hardy inequalities, where (e.g.) a weight of the type $d_\Omega(x)^\beta$, $\beta \in \mathbb{R}$, is included in the both integrals of the inequality (9); see [10] for a treatment of the Euclidean case.

Remark 6.4. In Euclidean spaces the condition $\operatorname{codim}_{\mathcal{AI}}(\Omega^c) > p$ (i.e. $\dim_{\mathcal{AI}}(\Omega^c) < n - p$) is also sufficient for Ω to admit the p-Hardy inequality; cf. the discussion after Corollary 2.7 in [9], but note that there the Minkowski dimension should actually be replaced by the Aikawa (or equivalently the Assouad) dimension. It is in our plans to pursue analogous results in the metric space setting.

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