Hardy and Hardy–Sobolev inequalities on general open sets

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1. Introduction to Hardy inequalities

The original p-Hardy inequality

G.H. Hardy published in 1925 the inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p\, dx,$$

where $1 < p < \infty$ and $f \ge 0$ is measurable.

Taking $u(x) = \int_0^x f(t)\, dt$, the above *p-Hardy inequality* can be written as

$$\int_0^\infty \frac{|u(x)|^p}{x^p}\, dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |u'(x)|^p\, dx,$$

where $1 < p < \infty$ and u is absolutely continuous with $u(0) = 0$.

Hardy inequalities in $\mathbb{R}^n$

The 1-dimensional p-Hardy inequality

$$\int_0^\infty |u(x)|^p x^{-p}\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\,dx$$

can be generalized to higher dimensions in many ways.
We consider the following p-Hardy inequality in $\mathbb{R}^n$:

$$\int_\Omega |u(x)|^p \delta_{\partial\Omega}(x)^{-p}\,dx \le C \int_\Omega |\nabla u(x)|^p\,dx.$$

Here $\Omega \subset \mathbb{R}^n$ is an open set, $u \in C_0^\infty(\Omega)$, and $\delta_{\partial\Omega}(x) = \operatorname{dist}(x, \partial\Omega)$.

In addition, we are interested in the weighted (p,β)-Hardy inequality

$$\int_\Omega |u(x)|^p\,\delta_{\partial\Omega}(x)^{\beta-p}\,dx \le C \int_\Omega |\nabla u(x)|^p \delta_{\partial\Omega}(x)^{\beta}\,dx. \tag{1}$$

If there is $C > 0$ such that inequality (1) holds for all $u \in C_0^\infty(\Omega)$, we say that Ω *admits* a (p,β)-Hardy inequality.

Sufficient conditions for Hardy inequalities

The following are well known for weighted Hardy inequalities:

Theorem (Nečas 1962)

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits a (p, β)-Hardy inequality for all $\beta < p - 1$ (sharp).

However, the "smoothness" of the boundary is not that relevant:

Theorem (Wannebo 1990)

Let $1 < p < \infty$ and assume that Ω^c is uniformly p-fat. Then there exists $\beta_0 > 0$ such that Ω admits a (p, β)-Hardy inequality for all $\beta < \beta_0$.

In the unweighted case $\beta = 0$, this result was first proven in [Ancona 1986, $p = 2$] and [Lewis 1988, $1 < p < \infty$]; the latter is independent of [Wannebo 1990]). Note: the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ of a Lipschitz domain Ω is uniformly p-fat for all $p \geq 1$.

Sufficient conditions for Hardy inequalities, pt. 2

On the other hand, an open set $\Omega \subset \mathbb{R}^n$ can admit a (p, β)-Hardy inequality also if the complement Ω^c is small enough (contrary to the p-fatness condition).

The following is a combination of results from [Aikawa 1991], [Koskela–Zhong 2003], and [L. 2008]. Here we say that $E \subset \mathbb{R}^n$ satisfies the *Aikawa condition* for $s \geq 0$, and write $s \in \mathcal{A}(E)$, if there is $C > 0$ such that

$$\int_{B(x,r)} \operatorname{dist}(y, E)^{s-n} \, dy \leq C r^s$$

holds for all $x \in E$ and $0 < r < \operatorname{diam}(E)$.

Theorem

Let $1 < p < \infty$ and assume that $n - p \in \mathcal{A}(\Omega^c)$. Then Ω admits a p-Hardy inequality (sharp).
Moreover, if $n - p + \beta \in \mathcal{A}(\Omega^c)$ and Ω satisfies an additional 'accessibility' condition (John-type), then Ω admits a (p, β)-Hardy inequality.

2. Density conditions and notions of dimension

Fatness and thickness

Uniform p-fatness is a capacitary condition, but this can be expressed equivalently using density conditions for Hausdorf contents.

Recall that the *Hausdorff (ϱ-)content* of dimension λ, for $E \subset \mathbb{R}^n$, is

$$\mathcal{H}^{\lambda}_{\varrho}(E) = \inf \left\{ \sum_k r_k^{\lambda} : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \leq \varrho \right\}.$$

The λ-*Hausdorff measure* of E is $\mathcal{H}^{\lambda}(E) = \lim_{\varrho \to 0} \mathcal{H}^{\lambda}_{\varrho}(E)$ and the *Hausdorff dimension* of E is $\dim_{\mathrm{H}}(A) = \inf\{\lambda \geq 0 : \mathcal{H}^{\lambda}_{(\infty)}(A) = 0\}$.

We say that a (closed) set $E \subset \mathbb{R}^n$ is λ-thick, if there exists $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap \overline{B}(w, r)\big) \geq C r^{\lambda} \quad \text{for all } r > 0,\ w \in E.$$

It is known that

Theorem

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, for $1 < p < \infty$, if and only if E is λ-thick for some $\lambda > n - p$.

Assouad dimensions

The above thickness and Aikawa conditions are closely related to the following Assouad dimensions:

Let $E \subset \mathbb{R}^n$. Consider all exponents $\lambda \geq 0$ for which there is $C \geq 1$ such that $E \cap B(w, R)$ can be covered by *at most* $C(r/R)^{-\lambda}$ balls of radius r for all $0 < r < R < \operatorname{diam}(E)$ and $w \in E$.

The infimum of such exponents λ is the *(upper) Assouad dimension* $\overline{\dim}_{\mathrm{A}}(E)$ (or often simply $\dim_{\mathrm{A}}(E)$)

Conversely: consider all $\lambda \geq 0$ for which there is $c > 0$ such that if $0 < r < R < \operatorname{diam}(E)$, then for every $w \in E$ *at least* $c(r/R)^{-\lambda}$ balls of radius r are needed to cover $E \cap B(w, R)$.

The supremum of all such λ is the *lower Assouad dimension* $\underline{\dim}_{\mathrm{A}}(E)$.

Some comments on Assouad dimensions

(Upper) Assouad dimension was introduced by P. Assouad around 1980 in connection to bi-Lipschitz embedding problem between metric and Euclidean spaces. However, equivalent (or closely related) concepts have appeared under different names, e.g. *(uniform) metric dimension*, some dating back (at least) to [Bouligand 1928]. See [Luukkainen 1998] for a nice account on the basic properties of (upper) Assouad dimension as well as some historical comments.

Lower Assouad dimension has (essentially) appeared under names *lower dimension*, *minimal dimensional number*, and *uniformity dimension*. Some basic properties of this have recently been discussed in [Fraser 2014] and [Käenmäki–L.–Vuorinen 2013].

Minkowski and Assouad

Once again:

$\overline{\dim}_{\mathrm{A}}(E)$ is the infimum of $\lambda \geq 0$ s.t. $E \cap B(w,R)$ can (always) be covered by at most $C(r/R)^{-\lambda}$ balls of radius $0 < r < R < \operatorname{diam}(E)$

$\underline{\dim}_{\mathrm{A}}(E)$ is the supremum of $\lambda \geq 0$ s.t. (always) at least $C(r/R)^{-\lambda}$ balls of radius $0 < r < R < \operatorname{diam}(E)$ are needed to cover $E \cap B(w,R)$

For comparison, recall the *upper and lower Minkowski dimensions* of a compact $E \subset \mathbb{R}^n$:

$\overline{\dim}_{\mathrm{M}}(E)$ is the infimum of $\lambda \geq 0$ s.t. E can be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$

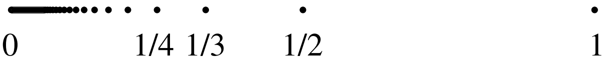
$\underline{\dim}_{\mathrm{M}}(E)$ is the supremum of $\lambda \geq 0$ s.t. at least $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$ are needed to cover E.

Thus $\quad \underline{\dim}_{\mathrm{A}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E).$

Examples (1)

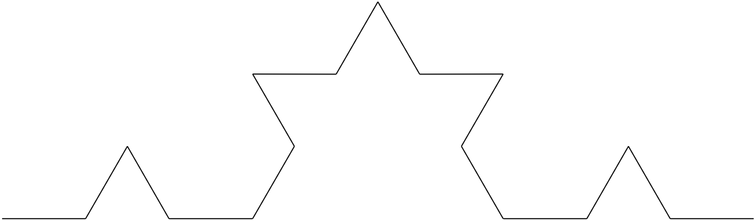
General idea: Assouad dimensions reflect the 'extreme' behavior of sets and take into account all scales $0 < r < d(E)$.

- If $E = \{0\} \cup [1,2] \subset \mathbb{R}$, then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ ($\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = 1$).
- $\underline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 0$ and $\overline{\dim}_{\mathrm{A}}(\mathbb{Z}) = 1$.
- If $E = \left\{\frac{1}{j} : j \in \mathbb{N}\right\} \cup \{0\} \subset \mathbb{R} \subset \mathbb{R}^n$, then then $\underline{\dim}_{\mathrm{A}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = 1$ ($\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = \frac{1}{2}$).

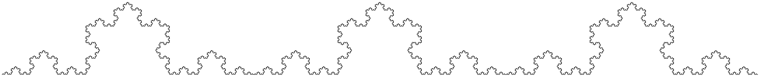


Examples (2)

- If $S \subset \mathbb{R}^2$ is an unbounded, locally rectifiable von Koch snowflake -type curve consisting of unit intervals, then $\underline{\dim}_{\mathrm{A}}(S) = 1$ and $\overline{\dim}_{\mathrm{A}}(E) = \log 4/\log 3$ (flat on small scales, fractal on large scales)



- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\underline{\dim}_{\mathrm{A}}(S) = 1$ and $\overline{\dim}_{\mathrm{A}}(E) = \log 4/\log 3$ (fractal on small scales, flat on large scales).



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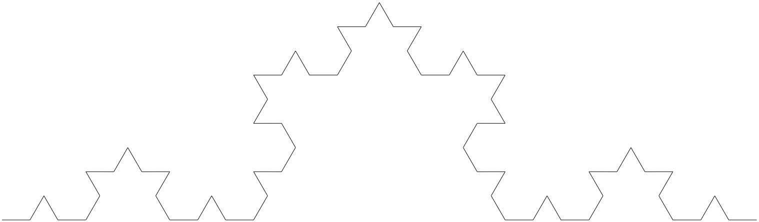

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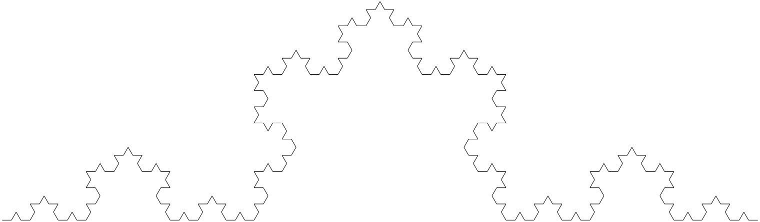

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Hausdorff, lower Assouad, and thickness

It can be shown that if $E \subset \mathbb{R}^n$ is closed, then $\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}(E \cap B)$ for all balls B centered at E. (However, e.g. $\underline{\dim}_{\mathrm{A}}(\mathbb{Q}) = 1$ but $\dim_{\mathrm{H}}(\mathbb{Q}) = 0$.)

In particular $\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}(E)$ for all closed sets $E \subset \mathbb{R}^n$.

The proof of $\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}(E \cap B)$ is actually based on the fact that for each $0 < \lambda < \underline{\dim}_{\mathrm{A}}(E)$

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap B(w,r)\big) \geq C r^{\lambda} \quad \text{for all } w \in E,\ 0 < r < \operatorname{diam}(E). \tag{2}$$

In fact, for closed $E \subset \mathbb{R}^n$ we have $\underline{\dim}_{\mathrm{A}}(E) = \sup\{\lambda \geq 0 : (2) \text{ holds}\}$, and thus for unbounded $E \subset \mathbb{R}^n$ it holds that

$$\underline{\dim}_{\mathrm{A}}(E) = \sup\{\lambda \geq 0 : E \text{ is } \lambda\text{-thick}\}.$$

Aikawa and upper Assouad

Recall the Aikawa condition $s \in \mathcal{A}(E)$ for $s \geq 0$: There is $C > 0$ such that

$$\int_{B(x,r)} \operatorname{dist}(y, E)^{s-n} \, dy \leq C r^s$$

for all $x \in E$ and $0 < r < \operatorname{diam}(E)$.

In [L.–Tuominen 2013] it was shown that the (upper) Assouad dimension $\overline{\dim}_{\mathrm{A}}(E)$ of $E \subset \mathbb{R}^n$ can be characterized as

$$\overline{\dim}_{\mathrm{A}}(E) = \inf\{s \geq 0 : s \in \mathcal{A}(E)\}.$$

3. Results for Hardy inequalities

Sufficient conditions for Hardy inequalities

Using Assouad dimensions we can formulate the following sufficient condition for Hardy inequalities. (Also metric space versions of this exist.)

Theorem (L. Jd'AM (to appear))

Let $1 < p < \infty$ *and* $\beta < p - 1$, *and let* $\Omega \subset \mathbb{R}^n$ *be an open set. If*

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_A(\Omega^c) > n - p + \beta,$$

then Ω *admits a* (p, β)*-Hardy inequality;*
in the latter case, if Ω *is unbounded, then also* Ω^c *has to be unbounded.*

For $\beta = 0$, the first condition is a reformulation of the *Aikawa condition* and the second is a reformulation of the *uniform p-fatness* condition!

The requirement $\beta < p - 1$ is optimal for this generality (but it can be removed under additional *accessibility* conditions). However, it is not 'natural' for the first condition if $\overline{\dim}_A(\Omega^c) < n - 1$.

A precise statement under uniform fatness

Condition $\underline{\dim}_A(\Omega^c) > n - p + \beta$ (or equivalently λ-thickness for $\lambda > n - p + \beta$) yields the following corollary in terms of uniform fatness:

Corollary (L. PAMS (2014))

Assume that Ω^c is uniformly q-fat for all $q > s \geq 1$. Then Ω admits a (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < \beta_0 = p - s$ (sharp).

For instance, if $\Omega \subset \mathbb{R}^2$ is simply connected, then Ω admits a (p, β)-Hardy inequality whenever $\beta < p - 1$ (Nečas had this for Lipschitz domains).

The idea of the proof of this part is quite simple if $\beta \geq 0$: by the assumption, Ω^c is uniformly $(p - \beta)$-fat, and so Ω admits a $(p - \beta)$-Hardy inequality. Then, given $u \in C_0^\infty(\Omega)$, we can use the $(p - \beta)$-Hardy inequality for the test function $v = |u|^{\beta/(p-\beta)}$, and the (p, β)-inequality for u follows with a simple calculation using Hölder's inequality.

Necessary conditions for Hardy inequalities

On the other hand, the following necessary condition, complementing the sufficient conditions, is (essentially) due to [Koskela–Zhong 2003, $\beta = 0$] and [L. 2008, $\beta \neq 0$].

Theorem

Assume that $1 < p < \infty$ and $\beta \neq p$, and that $\Omega \subset \mathbb{R}^n$ admits a (p, β)-Hardy inequality. Then

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \textit{or} \quad \dim_H(\Omega^c) > n - p + \beta\,.$$

Recall that always $\underline{\dim}_A(\Omega^c) \leq \dim_H(\Omega^c)$, and $\underline{\dim}_A(\Omega^c) > n - p + \beta$ is sufficient for (p, β)-Hardy. (Can not change $\underline{\dim}_A(\Omega^c) \leftrightarrow \dim_H(\Omega^c)$.)

The boundary dichotomy holds also locally: if $\Omega \subset \mathbb{R}^n$ admits a (p, β)-Hardy inequality, then for all balls $B \subset \mathbb{R}^n$ either

$$\overline{\dim}_A(B \cap \Omega^c) < n - p + \beta \quad \text{or} \quad \dim_H(2B \cap \Omega^c) > n - p + \beta.$$

Combining thick and thin parts

In accordance with the above local necessary conditions, it is possible to give also sufficient conditions with a mixture of 'thick' and 'thin' parts, for instance as follows:

Theorem (L. Jd'AM (to appear))

Let $1 < p < \infty$ and $\beta < p - 1$. Assume that $\Omega_0 \subset \mathbb{R}^n$ is an open set satisfying

$$\underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p + \beta,$$

and that $F \subset \overline{\Omega}_0$ is a closed set with

$$\overline{\dim}_{\mathrm{A}}(F) < n - p + \beta.$$

Then a (p, β)-Hardy inequality holds in $\Omega = \Omega_0 \setminus F$ for all $u \in C_0^\infty(\Omega_0)$. In particular, Ω admits a (p, β)-Hardy inequality.

4. Hardy–Sobolev inequalities

Hardy and Sobolev inequalities

Let $\Omega \subset \mathbb{R}^n$ be an open set, let $1 \leq p < n$, and denote $p^* = np/(n-p)$. Then there is $C > 0$ such that the Sobolev inequality

$$\left(\int_\Omega |u|^{p^*}\,dx\right)^{1/p^*} \leq C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$.

On the other hand, we have the p-Hardy inequality

$$\int_\Omega |u|^p\,\delta_{\partial\Omega}^{-p}\,dx \leq C\int_\Omega |\nabla u|^p\,dx$$

and the weighted (p,β)-Hardy inequality

$$\int_\Omega |u|^p\,\delta_{\partial\Omega}^{\beta-p}\,dx \leq C\int_\Omega |\nabla u|^p\,\delta_{\partial\Omega}^{\beta}\,dx,$$

which are not valid in all open sets $\Omega \subset \mathbb{R}^n$.

What is the connection between these?

The following Hardy–Sobolev inequalities form a natural interpolating scale in between the (weighted) Sobolev inequalities and the (weighted) Hardy inequalities.

We say that an open set $\Omega \subsetneq \mathbb{R}^n$ *admits a* (q, p, β)*-Hardy–Sobolev inequality* if there is $C > 0$ such that

$$\left(\int_\Omega |u|^q \, \delta_{\partial\Omega}^{(q/p)(n-p+\beta)-n} \, dx \right)^{1/q} \leq C \left(\int_\Omega |\nabla u|^p \, \delta_{\partial\Omega}^{\beta} \, dx \right)^{1/p} \tag{3}$$

for all $u \in C_0^\infty(\Omega)$.

Notice that the Sobolev inequality is the case $q = p^* = np/(n-p)$, $\beta = 0$ in (3) and the weighted (p, β)-Hardy inequality is the case $q = p$ in (3).

Some history of HS-inequalities

When $E \subset \mathbb{R}^n$ is an m-dimensional subspace, $1 \le m \le n-1$, $\Omega = \mathbb{R}^n \setminus E$, and $m < \frac{q}{p}(n-p+\beta)$, the global version of the (q,p,β)-Hardy–Sobolev inequality (for all $u \in C_0^\infty(\mathbb{R}^n)$) is due to [Maz'ya 1985].

[Badiale–Tarantello 2002] (essentially) rediscovered Maz'ya's result for $\beta = 0$, and applied this (case $m = 1$) to study the properties of the solutions of certain elliptic PDE's "related to the dynamics of galaxies". (More precisely,

$$-\Delta u(x) = \phi(r)|u|^{p-2}u,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r = \|(x_1, x_2)\|$, $u = u(r, x_3) > 0$, and $\phi \ge 0$ vanishes at 0 and at ∞).

For $m = 0$, i.e. $E = \{0\}$, the corresponding Hardy–Sobolev inequality is known as Caffarelli–Kohn–Nirenberg inequality, since this case first appeared in [CKN 1984].

"Interpolation"

In [L.–Vähäkangas 2015 (preprint)] we show that Hardy–Sobolev inequalities can be obtained from the (weighted) Hardy inequality with the help of the (unweighted) Sobolev inequality:

Theorem (LV. 2015)

Assume that $1 \leq p < n$ and $\beta \in \mathbb{R}$. If Ω admits a (p, p, β)-Hardy–Sobolev inequality (i.e., a (p, β)-Hardy inequality), then Ω admits (q, p, β)-Hardy–Sobolev inequalities for all exponents $p \leq q \leq p^*$.

Proof of the interpolation theorem

Step 1: (p,p,β)-HS $\implies$ (p^*,p,β)-HS (weighted Sobolev).
Let $u \in C_0^\infty(\Omega)$ and denote $g = |u|\delta_{\partial\Omega}^{\beta/p} \in \mathrm{Lip}_0(\Omega)$. Then using the Sobolev inequality for g and the (p,p,β)-HS inequality for u we obtain

$$\left(\int_\Omega |u|^{p^*}\delta_{\partial\Omega}^{\frac{n\beta}{n-p}}\right)^{1/p^*} = \left(\int_\Omega |g|^{p^*}\right)^{1/p^*}$$
$$\lesssim \left(\int_\Omega |\nabla g|^p\right)^{1/p} \lesssim \left(\int_\Omega |\nabla u|^p \delta_{\partial\Omega}^{\beta}\right)^{1/p} + \left(\int_\Omega |u|^p \delta_{\partial\Omega}^{\beta-p}\right)^{1/p}$$
$$\lesssim \left(\int_\Omega |\nabla u|^p \delta_{\partial\Omega}^{\beta}\right)^{1/p}$$

Step 2: The (p,p,β)- and (p^*,p,β)-HS inequalities and Hölder's inequality yield (q,p,β)-HS inequalities for all $p \le q \le p^*$:

$$\left(\int_\Omega |u|^q\, \delta_{\partial\Omega}^{(q/p)(n-p+\beta)-n}\right)^{1/q} \le \left(\int_\Omega |u|^p\, \delta_{\partial\Omega}^{\beta-p}\right)^{\frac{1}{q\alpha}} \left(\int_\Omega |u|^{p^*}\, \delta_{\partial\Omega}^{\frac{n\beta}{n-p}}\right)^{\frac{1}{q\alpha'}}.$$

5. Results for Hardy–Sobolev inequalities

Sufficient conditions for HS-inequalities

From the interpolation theorem we obtain corresponding results for Hardy–Sobolev inequalities for all $p \le q \le p^*$.

Theorem (LV. 2015)

Let $1 < p < \infty$ and $\beta < p - 1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. If

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_A(\Omega^c) > n - p + \beta,$$

then Ω admits a (q, p, β)-Hardy–Sobolev inequality for all $p \le q \le p^*$;
(p, β)-Hardy inequality;
in the latter case, if Ω is unbounded, then also Ω^c has to be unbounded.

Here the second bound $\underline{\dim}_A(\Omega^c) > n - p + \beta$ is rather sharp, but $\overline{\dim}_A(\Omega^c) < n - p + \beta$ can be weakened when $p < q < p^*$. Also the upper bound $\beta < p - 1$ can be changed to the weaker assumption that $\overline{\dim}_A(\Omega^c) < n - 1$ (thus improving the Hardy-case as well):

Sufficient conditions revisited

Theorem (LV. 2015)

Let $1 \le p \le q \le np/(n-p) < \infty$ and $\beta \in \mathbb{R}$. If $\Omega \subset \mathbb{R}^n$ is an open set and

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \min\left\{ \tfrac{q}{p}(n-p+\beta)\,,\ n-1 \right\},$$

then Ω admits a (q,p,β)-Hardy–Sobolev inequality.

The requirement $\overline{\dim}_{\mathrm{A}}(\Omega^c) < n-1$ can not be omitted (but it can be replaced with and upper bound for β).

An example is given by $\Omega = \mathbb{R}^n \setminus \partial B(0,1)$: for suitable functions $u_k \in C_0^\infty(B(0,1))$ the LHS of the (q,p,β)-HS has a positive lower bound, while the RHS tends to zero if $\beta > p-1 = p-n+\overline{\dim}_{\mathrm{A}}(\Omega^c)$.

Horiuchi and $P(s)$-condition

The proof of the previous theorem relies heavily on the work [Horiuchi, 1989], which studied embeddings between weighted Sobolev spaces and hence the non-homogeneous versions of Hardy–Sobolev inequalities.

In this connection Horiuchi defined that a closed set $E \subset \mathbb{R}^n$ of zero measure satisfies condition $P(s)$, for $0 \le s \le n$, if there is $C > 0$ such that for all balls B and all numbers η_1, η_2 satisfying $0 \le \eta_1 < \eta_2 \le \operatorname{diam}(B)$,

$$|B \cap (E_{\eta_2} \setminus E_{\eta_1})| \le \begin{cases} C\eta_2^{s-1}(\eta_2 - \eta_1)\operatorname{diam}(B)^{n-s} & \text{if } 1 \le s \le n \\ C(\eta_2 - \eta_1)^s \operatorname{diam}(B)^{n-s} & \text{if } 0 \le s < 1\,. \end{cases}$$

Here $E_\eta = \{x \in \mathbb{R}^n : \delta_E(x) < \eta\}$.

Horiuchi and Assouad

Horiuchi's $P(s)$-condition is clearly related to the dimension of E, but perhaps the following characterization is not completely obvious:

Theorem (LV. 2015)

Let $E \subset \mathbb{R}^n$ be a closed set with $|E| = 0$. Then

$$\overline{\dim}_A(E) = n - \sup\left\{0 \le s \le n \,:\, E \text{ satisfies } P(s)\right\}.$$

In particular, the $P(s)$-property holds for all $0 \le s < n - \overline{\dim}_A(E)$.

Knowing this, we can follow Horiuchi's original ideas to prove our sufficient condition for Hardy–Sobolev inequalities.

Necessary conditions for HS-inequalities

As with Hardy inequalities, there are corresponding necessary conditions for HS-inequalities as well.

Theorem (LV. 2015)

Assume that $1 \leq p \leq q < np/(n-p) < \infty$ *and that* $\Omega \subset \mathbb{R}^n$ *admits a* (q, p, β)*-HS inequality. If* $\beta \geq 0$ *and* $\frac{q}{p}(n - p + \beta) \neq n$*, then*

$$\overline{\dim}_A(\Omega^c) < \frac{q}{p}(n-p+\beta) \quad \text{or} \quad \dim_H(\Omega^c) \geq n - p + \beta\,.$$

If $\beta < 0$ *and* Ω^c *is compact and porous* ($\overline{\dim}_A(\Omega^c) < n$)*, then*

$$\overline{\dim}_A(\Omega^c) < \frac{q}{p}(n-p+\beta) \quad \text{or} \quad \underline{\dim}_M(\Omega^c) \geq n - p + \beta\,.$$

In particular, this shows that the numbers $\frac{q}{p}(n-p+\beta)$ and $n - p + \beta$ in the sufficient conditions are again sharp (although different dimensions in the lower bounds).

Necessary conditions for HS-inequalities

There are also local versions of the necessary conditions:

Theorem (LV. 2015)

Assume that $1 \leq p \leq q < np/(n-p) < \infty$ *and that* $\Omega \subset \mathbb{R}^n$ *admits a* (q,p,β)*-HS inequality. If* $\beta \geq 0$ *and* $\frac{q}{p}(n-p+\beta) \neq n$*, then for each ball* $B \subset \mathbb{R}^n$ *either*

$$\overline{\dim}_{\mathrm{A}}(\Omega^c \cap B) < \tfrac{q}{p}(n-p+\beta) \quad \text{or} \quad \dim_{\mathrm{H}}(\Omega^c \cap 2B) \geq n-p+\beta\,.$$

If $\beta < 0$ *and* Ω^c *is compact and porous (* $\overline{\dim}_{\mathrm{A}}(\Omega^c) < n$ *), then for each ball* $B \subset \mathbb{R}^n$ *either*

$$\overline{\dim}_{\mathrm{A}}(\Omega^c \cap B) < \tfrac{q}{p}(n-p+\beta) \quad \text{or} \quad \underline{\dim}_{\mathrm{M}}(\Omega^c \cap \ell B) \geq n-p+\beta\,,$$

where $\ell = 8\sqrt{n}$.

Unweighted characterization

In the results involving a 'thin' complement (corresponding to an upper bound for $\overline{\dim}_{\mathrm{A}}(\Omega^c)$), the HS-inequalities actually hold for all $u \in C_0^\infty(\mathbb{R}^n)$, not only for $u \in C_0^\infty(\Omega)$ as in the 'thick' case. Such inequalities are called *global* Hardy–Sobolev inequalities. In particular, we have the following characterization in the unweighted case $\beta = 0$.

Corollary (LV. 2015)

Let $E \neq \emptyset$ be a closed set in $\mathbb{R}^n$ and let $1 \leq p \leq q < np/(n-p) < \infty$. Then the global $(q,p,0)$-Hardy–Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^q \, \delta_E^{(q/p)(n-p)-n} \, dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{1/p}$$

holds for every $u \in C_0^\infty(\mathbb{R}^n)$ if and only if $\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n-p)$.

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