SELF-IMPROVEMENT OF UNIFORM FATNESS REVISITED

JUHA LEHRBÄCK, HELI TUOMINEN, AND ANTTI V. VÄHÄKANGAS

Abstract. We give a new proof for the self-improvement of uniform p-fatness in the setting of general metric spaces. Our proof is based on rather standard methods of geometric analysis, and in particular the proof avoids the use of deep results from potential theory and analysis on metric spaces that have been indispensable in the previous proofs of the self-improvement. A key ingredient in the proof is a self-improvement property for local Hardy inequalities.

1. Introduction

Self-improvement is among the most profound and beautiful phenomena in mathematical analysis, and a source of important tools in the proofs of several deep and perhaps even surprising results. Important examples of concepts enjoying self-improvement include reverse Hölder inequalities, Muckenhoupt's A_p classes of weights, Poincaré inequalities, and the main topics of this paper: Hardy inequalities and uniform p-fatness related to the variational p-capacity.

That a uniformly p-fat set E, for $1 < p < \infty$, is actually uniformly q-fat for some $1 \le q < p$ as well, was first proven by Lewis [14] in the Euclidean case $E \subset \mathbb{R}^n$. In fact, Lewis studied more general (α, p)-fatness conditions related to Riesz capacities, but when $\alpha = 1$ his setting is equivalent to that of the variational p-capacity. Another proof for the self-improvement of uniform p-fatness in (weighted) $\mathbb{R}^n$ was given by Mikkonen [18], and in [2] Björn, MacManus and Shanmugalingam generalized the self-improvement to more general metric spaces, essentially proving the following theorem (although in [2] the assumptions on the space X were slightly stronger).

Theorem 1.1. *Let $1 < p < \infty$ and let X be a complete metric measure space equipped with a doubling measure μ and supporting a $(1, p)$-Poincaré inequality. Assume that $E \subset X$ is a uniformly p-fat closed set. Then there exists $1 < q < p$ such that E is also uniformly q-fat (quantitatively).*

The proofs of the versions of Theorem 1.1 in [2, 14, 18] utilize deep results from linear and non-linear potential theory, and moreover the proof in [2] is based on the impressive theory of differential structures on metric spaces, established by Cheeger in [4].

In this paper, we use a different approach and establish a new proof for Theorem 1.1 with the help of local Hardy inequalities and their self-improvement properties. Our proof is completely new also in $\mathbb{R}^n$, where all previously known proofs have been based on the ideas either in [14] or in [18]. In addition, it turns out that with our approach it is possible to obtain the following generalization of Theorem 1.1 to a non-complete space X, where Cheeger's theory is not available.

Theorem 1.2. *Let $1 < p_0 < p < \infty$ and let X be a metric measure space equipped with a doubling measure μ and supporting a (p, p_0)-Poincaré inequality. Assume that $E \subset X$ is a uniformly p-fat closed set and that $E \cap \overline{B(w, r)}$ is compact for all $w \in E$ and all $r > 0$. Then there exists $p_0 < q < p$ such that E is also uniformly q-fat (quantitatively).*

2010 *Mathematics Subject Classification.* Primary 31C15, Secondary 31E05, 35A23.

Key words and phrases. Self-improvement, uniform fatness, local Hardy inequality, metric space.

We will also formulate a slightly stronger version of Theorem 1.2 later in Theorem 4.1. Recall that if X is as in Theorem 1.1 (i.e., complete, equipped with a doubling measure and supporting a $(1,p)$-Poincaré inequality), then a (p,p_0)-Poincaré inequality as in Theorem 1.2 follows from the well-known self-improvement properties of Poincaré inequalities; see Section 2.3 for more discussion.

It should perhaps be noted here that we define the variational p-capacity using Lipschitz test functions (the precise definition is given in Section 2.4). If X is complete, this definition agrees with the definition using Newtonian (or Sobolev) test functions, but in a non-complete space the resulting capacities can be different.

Let us turn to an outline of the ideas behind the proofs of Theorems 1.1 and 1.2. In [11] (see also [10, Theorem 3.3]) it was shown (essentially) that if X is as in Theorem 1.1, then a closed set $E \subset X$ is uniformly p-fat if and only if there is $C > 0$ such that the *boundary p-Poincaré inequality*

$$\int_{B(w,r)} |u|^p \, d\mu \leq C r^p \int_{B(w,\tau r)} g^p \, d\mu \tag{1}$$

holds for all $w \in E$ and all $r > 0$, whenever u is a Lipschitz function in X such that $u = 0$ in E and g is a (p-weak) upper gradient of u (in $\mathbb{R}^n$ one can always take $g = |\nabla u|$). Hence to obtain the self-improvement of uniform p-fatness, it would suffice to prove the self-improvement directly to inequality (1); this was actually mentioned in [11, p. 718] as a possible and interesting approach to self-improvement.

We will not give a direct proof for the self-improvement of (1), but we show in Theorem 3.1 that if X is as in Theorem 1.2 (in particular not necessarily complete) and $E \subset X$ is uniformly p-fat, then there exist $\varepsilon > 0$ and $C > 0$ such that the following $(p-\varepsilon)$-version of (1) holds for all $w \in E$ and all $r > 0$, whenever u is a Lipschitz function in X such that $u = 0$ in E:

$$\int_{B(w,r)} |u|^{p-\varepsilon} \, d\mu \leq C r^{p-\varepsilon} \int_{B(w,\tau r)} \operatorname{Lip}(u,\cdot)^{p-\varepsilon} \, d\mu \,. \tag{2}$$

Here $\operatorname{Lip}(u,x)$ is the upper pointwise Lipschitz constant of u at $x \in X$. We remark that in $\mathbb{R}^n$ inequality (2) can be obtained directly with $|\nabla u|$ instead of $\operatorname{Lip}(u,\cdot)$ on the right-hand side. More generally, if the space X is complete, then $\operatorname{Lip}(u,\cdot)$ is actually known to be a minimal weak upper gradient of u by the results of Cheeger [4]. Hence we can connect from inequality (2) back to uniform fatness, and now indeed to the better $(p-\varepsilon)$-uniform fatness, thus proving Theorem 1.1. In [11] this connection was established with the help of the so-called pointwise Hardy inequalities, but, for the sake of completeness, we show in Section 3 how Theorem 1.1 follows directly from the validity of (2). In particular, this way we avoid the use of pointwise Hardy inequalities in our proofs of Theorems 1.1 and 1.2, although it should be noted that our general approach has been partially suggested and motivated by these pointwise inequalities.

In a non-complete space X the validity of (2) does not immediately yield the uniform $(p-\varepsilon)$-fatness of E. Nevertheless, using as an additional tool the connection between uniform fatness and density conditions for suitable Hausdorff contents, we show in Section 4 how the improved boundary Poincaré inequality (2) can be used to prove also Theorem 1.2.

It is the proof of (2) (assuming uniform p-fatness) that constitutes the main challenge in our proofs of Theorems 1.1 and 1.2. In fact, we will establish (2) via a self-improvement property of suitable local Hardy inequalities. Recall that one of the consequences of the self-improvement of uniform fatness, noted in each of [2, 14, 18], is the validity of a p-Hardy inequality in the complement of a uniformly p-fat set $E \subset X$. However, using a method originating from Wannebo [19] (see also [11, Section 5]), it is also possible to prove such a p-Hardy inequality without using the self-improvement of uniform fatness. We use an adaptation of this latter method together with a novel 'local absorbtion argument'

(Lemma 5.3), and prove in the end of Section 5.2 the following local p-Hardy inequality when $E \subset X$ is uniformly p-fat.

Theorem 1.3. *Let $1 < p < \infty$ and let X be a metric measure space equipped with a doubling measure μ and supporting a $(1,p)$-Poincaré inequality. Assume that $E \subset X$ is a uniformly p-fat closed set. Then there exists a constant $C > 0$ such that the local p-Hardy inequality*

$$\int_{B\setminus E} \left(\frac{|u|}{d_E}\right)^p d\mu \le C \int_{32\tau^2 B} g^p\, d\mu \tag{3}$$

holds whenever u is a Lipschitz function in X such that $u = 0$ in E, g is a p-weak upper gradient of u, and $B = B(w,r)$ is a ball with $w \in E$ and $0 < r < (1/32)\operatorname{diam}(X)$.

Above we have abbreviated $d_E(x) = \operatorname{dist}(x,E)$. Notice in particular that we do not need to assume in Theorem 1.3 that the space X is complete.

The next step towards (2) is a self-improvement property for local p-Hardy inequalities (3). Here we need the assumption that X supports a (p,p_0)-Poincaré inequality for some $1 < p_0 < p$ (actually, it suffices to assume that X supports (q,q)-Poincaré inequalities for all $p_0 \le q \le p$, with uniform constants). In this case there exists $\varepsilon > 0$ such that a version of inequality (3) holds with the exponent $p-\varepsilon$, but now with the p-weak upper gradient g on the right-hand side of (3) replaced with the upper pointwise Lipschitz constant $\operatorname{Lip}(u,\cdot)$; see Proposition 5.7. The proof of this self-improvement for local Hardy inequalities is based on ideas used by Koskela and Zhong [13] in connection with the self-improvement of usual p-Hardy inequalities; the ideas in [13] were, in turn, inspired by the work of Lewis [15]. Again the absorbtion Lemma 5.3 is needed to obtain the local inequalities. The $(p-\varepsilon)$-version of the local Hardy inequality now easily yields the $(p-\varepsilon)$-version of the boundary Poincaré inequality (2), see Section 3, concluding the proofs of Theorems 1.1 and 1.2.

Admittedly, the proofs that were outlined above are somewhat lengthy and in many places still quite technical and delicate in the level of details, but one could argue that our general approach is nevertheless based on rather 'elementary' (or 'standard') tools. In particular, we do not need any sophisticated prerequisites concerning potential theory and we can also avoid completely the use of Cheeger's deep theory—or, if this theory is used to give a more direct proof to Theorem 1.1, the use is very explicit and localized; cf. the proof of Theorem 1.1 at the end of Section 3. In this sense we believe that our proof of Theorem 1.1 is more transparent (also in $\mathbb{R}^n$) than its predecessors and thus hopefully easier to adapt to further problems, for instance in connection with weighted capacities or capacities of fractional order smoothness.

2. Preliminaries

2.1. **Metric spaces.** We assume throughout the paper that $X = (X,d,\mu)$ is a metric measure space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B(x,r)) < \infty$ for all balls $B = B(x,r) = \{y \in X : d(y,x) < r\}$. As in [1, p. 2], we extend μ as a Borel regular (outer) measure on X. In particular, the space X is separable. Let us emphasize that we do not, in general, require X to be complete. If completeness is needed somewhere in the paper, we will mention this explicitly.

We also assume that the measure μ is *doubling*, meaning that there is a constant $C_D \ge 1$, called the *doubling constant of* μ, such that

$$\mu(2B) \le C_D\, \mu(B) \tag{4}$$

for all balls $B = B(x,r)$ of X. Here we use for $0 < t < \infty$ the notation $tB = B(x,tr)$. When $A \subset X$, we let $\overline{A}$ denote the closure of A, and hence $\overline{B}$ always refers to the closure of the ball B, not to the corresponding closed ball.

Let $A \subset X$. A function $u\colon A \to \mathbb{R}$ is said to be *(L-)Lipschitz*, for $0 \le L < \infty$, if

$$|u(x) - u(y)| \le L d(x,y) \qquad \text{for all } x,y \in A\,.$$

If $u : A \to \mathbb{R}$ is an L-Lipschitz function, then the classical McShane extension

$$\tilde{u}(x) = \inf_{y\in A}\{u(y) + Ld(x, y)\}\,, \qquad x \in X\,, \tag{5}$$

defines an L-Lipschitz function $\tilde{u} : X \to \mathbb{R}$ which satisfies $\tilde{u}|_A = u$. The set of all Lipschitz functions $u\colon A \to \mathbb{R}$ is denoted by $\mathrm{Lip}(A)$, and

$$\mathrm{Lip}_0(A) = \{u \in \mathrm{Lip}(X) \,:\, u = 0 \text{ in } X \setminus A\}\,.$$

2.2. **(Weak) upper gradients.** By a *curve* we mean a nonconstant, rectifiable, continuous mapping from a compact interval to X. We say that a Borel function $g \geq 0$ on X is an *upper gradient* of an extended real-valued function u on X, if for all curves γ joining arbitrary points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds\,, \tag{6}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. In addition, when $1 \leq p < \infty$, a measurable function $g \geq 0$ on X is a *p-weak upper gradient* of an extended real-valued function u on X if inequality (6) holds for p-almost every curve γ joining arbitrary points x and y in X; that is, there exists a non-negative Borel function $\rho \in L^p(X)$ such that $\int_\gamma \rho\, ds = \infty$ whenever (6) does not hold for the curve γ. We refer to [1] for more information on p-weak upper gradients.

When u is a (locally) Lipschitz function on X, the *upper pointwise Lipschitz constant* of u at $x \in X$ is defined as

$$\mathrm{Lip}(u, x) = \limsup_{r\to 0}\, \sup_{y\in B(x,r)} \frac{|u(y) - u(x)|}{r}\,. \tag{7}$$

The Borel function $\mathrm{Lip}(u, \cdot)$ is an upper gradient of u; cf. [1, Proposition 1.14]. Moreover, if X is complete and $1 < p < \infty$, then $\mathrm{Lip}(u, \cdot)$ is actually a so-called minimal p-weak upper gradient of u (in particular, this implies that $\mathrm{Lip}(u, \cdot) \leq g$ a.e. whenever $g \in L^p(X)$ is a p-weak upper gradient of u). This is a deep result of Cheeger, we refer to [4, Theorem 6.1] and [1, p. 342].

2.3. **Poincaré inequalities.** We say that the space X supports a (q, p)-*Poincaré inequality*, for $1 \leq q, p < \infty$, if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions u on X, and for all p-weak upper gradients g of u,

$$\left(\fint_B |u - u_B|^q\, d\mu\right)^{p/q} \leq C\, \mathrm{diam}(B)^p \fint_{\lambda B} g^p\, d\mu\,. \tag{8}$$

Here

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u\, d\mu$$

is the integral average of u over the ball B, and the left-hand side of (8) is interpreted as ∞ whenever u_B is not defined. We remark that X supports a (q, p)-Poincaré inequality with constants $C > 0$ and $\lambda \geq 1$ if, and only if, inequality (8) holds for all balls $B \subset X$, all functions $u \in L^1(X)$, and all upper gradients g of u; see [4, Proposition 4.13].

If $1 < p < \infty$ and X supports a $(1, p)$-Poincaré inequality (and the measure μ is doubling, as we assume throughout the paper), then X supports also a (p, p)-Poincaré inequality; see [1, Corollary 4.24]. If in addition X is complete, then there is an exponent $1 < p_0 < p$ such that X supports a (p, p_0)-Poincaré inequality and, consequently, also (q, q)-Poincaré inequalities with uniform constants whenever $p_0 \leq q \leq p$; for details we refer to [8] (see also [1, Theorem 4.30]) and to [1, Theorem 4.21]. Therefore the following (PI) condition, for a complete space X supporting a $(1, p)$-Poincaré inequality, is valid with the above exponents $1 < p_0 < p$.

However, since we do not in general assume that X is complete, we use in many of our results the following *a priori* assumption concerning the validity of (improved) Poincaré inequalities with uniform constants:

(PI) Let $1 < p < \infty$ be given. We assume that there are $1 < p_0 < p$, $C_P > 0$ and $\tau \geq 1$ such that X supports the (q,q)-Poincaré inequality

$$\fint_B |u - u_B|^q \, d\mu \leq C_P \operatorname{diam}(B)^q \fint_{\tau B} g^q \, d\mu \tag{9}$$

for every $p_0 \leq q \leq p$.

For simplicity, we will in the sequel use Poincaré inequalities with the constants $C_P > 0$ and $\tau \geq 1$. Indeed, if only a $(1,p)$-Poincaré inequality is assumed, this is just a matter of notation (in this case we may use both $(1,p)$-Poincaré and (p,p)-Poincaré inequality with the above constants). And if (PI) is assumed, the $(1,q)$-Poincaré inequalities (with $C_P > 0$ and $\tau \geq 1$) for $p_0 \leq q \leq p$ are all trivial consequences of (9) and Hölder's inequality.

2.4. **Capacity and fatness.** Let $\Omega \subset X$ be a bounded open set and let $K \subset \Omega$ be a closed set. We define the *(Lipschitz) variational p-capacity* of K with respect to Ω to be

$$\operatorname{cap}_p(K,\Omega) = \inf \int_\Omega g^p \, d\mu \,, \tag{10}$$

where the infimum is taken over all functions $u \in \operatorname{Lip}_0(\Omega)$, such that $u \geq 1$ in K, and all p-weak upper gradients g of u. If there are no such functions u, we set $\operatorname{cap}_p(K,\Omega) = \infty$.

Remark 2.1. If $\operatorname{cap}_p(K,\Omega) < \infty$, then the infimum in (10) can be restricted to $u \in \operatorname{Lip}_0(\Omega)$ satisfying $\chi_K \leq u \leq 1$ and to p-weak upper gradients g of u such that $g = g\chi_\Omega \in L^p(X)$. Indeed, if u is an admissible test function for $\operatorname{cap}_p(K,\Omega)$ and g is a p-weak upper gradient of u such that $g \in L^p(\Omega)$, then $\tilde{u} = \max\{0, \min\{1,u\}\}$ belongs to $\operatorname{Lip}_0(\Omega)$ and $\chi_K \leq \tilde{u} \leq 1$ on X. Moreover, the function g is clearly a p-weak upper gradient of $\tilde{u}$. By the glueing lemma [1, Lemma 2.19], we may further assume that $g = 0$ outside Ω. (Actually, since g need not belong to $L^p(X)$ but this is needed in the glueing lemma, we first define a function

$$\tilde{g} = g\chi_\Omega + \operatorname{Lip}(\tilde{u}, \cdot)\chi_{X\setminus\Omega} \in L^p(X)$$

that is a p-weak upper gradient of $\tilde{u}$, cf. the proof of [1, Theorem 2.6]. Now the glueing lemma applies, with $\tilde{g}$, showing that $g\chi_\Omega$ is a p-weak upper gradient of $\tilde{u}$.)

Let us remark here that if the metric space X is complete and supports a $(1,p)$-Poincaré inequality, then the above definition of $\operatorname{cap}_p(K,\Omega)$ is equivalent to the definition where the function u is assumed to belong to the *Newtonian space* $N_0^{1,p}(\Omega)$. However, we will not use the theory of Newtonian spaces in this paper, but rather refer to [1] for an introduction and basic properties of Newtonian functions. In particular, see [1, Theorem 6.19(x)] for the above-mentioned equivalence of capacities in the complete case.

On the other hand, if $X = \mathbb{R}^n$, equipped with the Euclidean metric and the Lebesgue measure (or more generally a p-admissible weight, see [6, 18]), then by standard approximation

$$\operatorname{cap}_p(K,\Omega) = \inf\biggl\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ in } K \biggr\} \tag{11}$$

for all closed (compact) $K \subset \Omega$, and therefore $\operatorname{cap}_p(K,\Omega)$ is the usual variational p-capacity of K. In this case all our results (and computations) concerning Lipschitz functions and their p-weak upper gradients (or upper pointwise Lipschitz constants) can be restated using functions in $C_0^\infty(\Omega)$ and the norms of their gradients. We recall that our approach is new even in this special case.

We say that a closed set $E \subset X$ is *uniformly p-fat*, for $1 \le p < \infty$, if there exists a constant $0 < c_0 \le 1$ such that

$$\operatorname{cap}_p(E \cap \overline{B(x,r)}, B(x,2r)) \ge c_0 \operatorname{cap}_p(\overline{B(x,r)}, B(x,2r)) \tag{12}$$

for all $x \in E$ and all $0 < r < (1/8)\operatorname{diam}(X)$. If there exists a constant $r_0 > 0$ such that condition (12) holds for all $x \in E$ and all $0 < r < r_0$, the closed set E is said to be *locally uniformly p-fat*.

Remark 2.2. Both Theorem 1.1 and Theorem 1.2 are formulated in terms of uniform fatness. However, the corresponding results are valid also when 'uniform fatness' is replaced by 'local uniform fatness' (in the assumptions with exponent p and in the conclusions with exponent q). In the sequel, we will exclusively focus on the case of uniformly fat sets. The minor modifications (required throughout the paper) in the local case are straightforward.

The self-improvement of uniform p-fatness (that is formulated, e.g., in Theorem 1.1) is critical in various applications; examples beyond the scope of Hardy inequalities include global higher integrability of both the gradients of solutions to PDE's [9, 18] and the upper gradients of certain quasiminimizers in metric measure spaces [16]. In [12] a quite simple proof for the self-improvement of uniform Q-fatness is provided in the setting of Ahlfors Q-regular metric measure spaces.

In the Euclidean space $\mathbb{R}^n$, the self-improvement property is known to hold also for more general (α, p)-fatness conditions related to Riesz capacities by the results of Lewis [14]. For $\alpha = 1$ these conditions are equivalent to the uniform p-fatness; cf. [9, p. 902].

In the rest of this paper (and hence in particular in our proof of the self-improvement of uniform fatness), we only need the following two basic facts concerning the variational p-capacity, which hold under the assumption that the space X supports a $(1,p)$-Poincaré inequality (and hence also a (p,p)-Poincaré inequality). First, there is a constant $C > 0$ such that, for each Lipschitz function u on X, all p-weak upper gradients g of u, and for all balls $B \subset X$, we have

$$\fint_B |u|^p \, d\mu \le \frac{C}{\operatorname{cap}_p(\overline{2^{-1}B} \cap \{u = 0\}, B)} \int_{\tau B} g^p \, d\mu \,. \tag{13}$$

Here $\{u = 0\} = \{x \in X : u(x) = 0\}$ and τ is the dilatation from the (p,p)-Poincaré inequality (9). This 'capacitary Poincaré inequality' is in the classical Euclidean case due to Maz'ya [17, Ch. 10]. For the metric space version, cf. [1, Proposition 6.21].

The second fact is a comparison between p-capacity and measure. Namely, there is a constant $C > 0$ such that for all balls $B = B(x,r)$ with $0 < r < (1/8)\operatorname{diam}(X)$ and for each closed set $E \subset \overline{B}$,

$$\frac{\mu(E)}{C\, r^p} \le \operatorname{cap}_p(E, 2B) \le \frac{C_D\, \mu(B)}{r^p}\,; \tag{14}$$

see, for instance [1, Proposition 6.16]. The $(1,p)$-Poincaré inequality is needed to ensure the validity of the lower bound in inequality (14).

2.5. **Tracking constants.** Our results are based on quantitative estimates and absorption arguments, where it is often crucial to track the dependencies of constants quantitatively. For this purpose, we will use the following notational convention:

- $C_{X,*,\cdots,*}$ denotes a positive constant which quantitatively depends on the quantities indicated by the $*$'s and (possibly) on: the doubling constant C_D of the measure μ in (4), the constants C_P and τ appearing in the (q,q)-Poincaré inequalities (9) and the constants appearing in the capacitary Poincaré inequality (13) and the comparison inequality (14).

Observe that $C_{X,*,\cdots,*}$ can implicitly depend on p via the estimates in inequalities (9), (13) and (14). However, any further dependencies on the exponent p will be explicitly indicated.

3. Improved boundary Poincaré inequalities

Recall, for the rest of the paper, that we assume X to be a metric space (not necessarily complete) equipped with a doubling measure μ. Further assumptions, concerning e.g. the validity of Poincaré inequalities, will be stated separately in each of the following results.

Our proof of the self-improvement of uniform fatness is based on the following improved boundary Poincaré inequalities.

Theorem 3.1. *Let $1 < p < \infty$ and suppose that X supports the improved (q,q)-Poincaré inequalities (PI) for $p_0 \le q \le p$. Assume that $E \subset X$ is a uniformly p-fat closed set. Then there exists constants $0 < \varepsilon < p - p_0$ and $C > 0$, quantitatively, such that inequality*

$$\int_{B(w,\rho)} |u|^{p-\varepsilon}\, d\mu \le C\rho^{p-\varepsilon} \int_{B(w,\tau\rho)} \operatorname{Lip}(u,\cdot)^{p-\varepsilon}\, d\mu$$

holds whenever $w \in E$, $\rho > 0$, and $u \in \operatorname{Lip}_0(X \setminus E)$.

Proof. Fix $w \in E$, a radius $\rho > 0$, and a function $u \in \operatorname{Lip}_0(X \setminus E)$. Clearly, we may assume that $\rho < (3/2)\operatorname{diam}(X)$. It is convenient to write $r = \rho/(12\tau^2)$ and $B = B(w,r)$. Let us assume, for the time being, that $0 < \varepsilon < p - p_0$ is given and $E_B \subset E \cap \overline{B}$ is *any* closed set such that $w \in E_B$. Since

$$\int_{B(w,\rho)} \frac{|u(x)|^{p-\varepsilon}}{\rho^{p-\varepsilon}}\, d\mu(x) \le \int_{B(w,\rho)\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x)\,,$$

it suffices to find quantitative constants $0 < \varepsilon < p - p_0$ and $C > 0$ (and a closed set $E_B \subset E$ as above) such that

$$\int_{B(w,\rho)\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) \le C \int_{B(w,\tau\rho)} \operatorname{Lip}(u,x)^{p-\varepsilon}\, d\mu(x)\,. \tag{15}$$

We establish this improved local Hardy inequality below in Proposition 5.7, and this proves the theorem. Let us remark here that the proof of Proposition 5.7 is rather involved and divided in Section 5 to the following three stages: 'Truncation' in §5.1, 'Local Hardy' in §5.2, and 'Improvement' in §5.3. □

From Theorem 3.1 (that is based on postponed Proposition 5.7) we obtain the following estimate for the capacity test-functions related to $\operatorname{cap}_p(E \cap \overline{B}, 2B)$. This estimate will be used in various settings to prove the self-improvement of uniform fatness.

Proposition 3.2. *Let $1 < p < \infty$ and suppose that X supports the improved (q,q)-Poincaré inequalities (PI) for $p_0 \le q \le p$. Assume that $E \subset X$ is a uniformly p-fat closed set. Then there exist constants $C > 0$ and $0 < \varepsilon < p - p_0$, quantitatively, such that for all balls $B = B(w,R)$, with $w \in E$ and $0 < R < (1/8)\operatorname{diam}(X)$, and for all functions $v \in \operatorname{Lip}_0(2B)$, with $0 \le v \le 1$ and $v = 1$ in $E \cap \overline{B}$, it holds that*

$$\mu(B)R^{-(p-\varepsilon)} \le C \int_{2B} \operatorname{Lip}(v,\cdot)^{p-\varepsilon}\, d\mu\,. \tag{16}$$

Proof. This proof is based on a similar idea as the proof of Lemma 2 in [11]. Let $0 < \varepsilon < p - p_0$ be given by Theorem 3.1, and write $q = p - \varepsilon$ and $\ell = (2\tau)^{-1} \le 1/2$, where τ is the dilatation constant from the (q,q)-Poincaré inequality (9). Fix w, R, and v as in the statement of the proposition. The doubling inequality (4) implies that there is a constant $C_1 = C_{C_D,\tau} > 0$ such that $\mu(\ell B) \ge C_1\mu(B)$. If $v_B > C_1/4$, we obtain from condition (PI) and the Sobolev inequality [1, Theorem 5.51] for $v \in \operatorname{Lip}_0(2B)$ that

$$C_1/4 \le \fint_B |v|\, d\mu \le C_D \fint_{2B} |v|\, d\mu \le CR\biggl(\fint_{2B} \operatorname{Lip}(v,\cdot)^q\, d\mu\biggr)^{1/q},$$

and from this (16) follows easily.

We may hence assume that $v_B \le C_1/4$. Let $\psi \in \operatorname{Lip}_0(B)$ be a cut-off function, defined as

$$\psi(x) = \max\Big\{0, 1 - \tfrac{2}{R}\operatorname{dist}\big(x, \tfrac12 B\big)\Big\},$$

and take

$$u = \min\{\psi, 1 - v\}.$$

Since $1 - v = 0$ in $E \cap \overline{B}$ and $\psi = 0$ in $X \setminus B$, we have that $u \in \operatorname{Lip}_0(X \setminus E)$. Observe that u coincides with $1 - v$ on $(1/2)B$, and therefore $\operatorname{Lip}(u, \cdot)|_{(1/2)B} = \operatorname{Lip}(v, \cdot)|_{(1/2)B}$.

Let $F = \{x \in \ell B : u(x) > 1/2\}$. We claim that $\mu(F) \ge (C_1/2)\mu(B)$. To prove this claim we assume the contrary, namely, that $\mu(F) < (C_1/2)\mu(B)$. Since $v \ge 0$ and $v = 1-u \ge 1/2$ in $\ell B \setminus F$, we obtain from the assumptions $\mu(\ell B) \ge C_1\mu(B)$ and $\mu(F) < (C_1/2)\mu(B)$ that

$$\begin{aligned}\int_B v\,d\mu \ge \int_{\ell B\setminus F} v\,d\mu &\ge \tfrac12\big(\mu(\ell B) - \mu(F)\big)\\ &> \tfrac12\big(C_1\mu(B) - (C_1/2)\mu(B)\big) = \tfrac14 C_1\mu(B)\,.\end{aligned}$$

This contradicts the assumption $v_B \le C_1/4$, and thus indeed $\mu(F) \ge (C_1/2)\mu(B)$.

Theorem 3.1, with $\rho = \ell R$, now implies that

$$(C_1/2)\mu(B) \le \mu(F) \le 2^q \int_{\ell B} |u|^q\,d\mu \le CR^q \int_{\tau\ell B} \operatorname{Lip}(u,\cdot)^q\,d\mu \le CR^q \int_{2B} \operatorname{Lip}(v,\cdot)^q\,d\mu\,.$$

This proves estimate (16) and concludes the proof. □

In a Euclidean space $\mathbb{R}^n$, which supports the $(1,p)$-Poincaré inequalities for all $1 \le p < \infty$, Proposition 3.2 yields immediately the self-improvement of uniform p-fatness. Indeed, we can replace in our argument the Lipschitz function $v \in \operatorname{Lip}_0(2B)$ with a function $\tilde v \in C_0^\infty(2B)$ and the pointwise Lipschitz constant $\operatorname{Lip}(v,\cdot)$ with $|\nabla \tilde v|$, whence the uniform $(p-\varepsilon)$-fatness of E follows from estimates (14) and (16).

More generally, in a complete metric space X supporting a $(1,p)$-Poincaré inequality, we can deduce the self-improvement of uniform fatness from Proposition 3.2 with the help of some deep facts concerning analysis on metric spaces (see the proof below). Nevertheless, with an additional argument using the interplay between uniform fatness and density conditions for suitable Hausdorff contents, it is possible to obtain a version of the self-improvement in a non-complete setting as well (Theorem 1.2), and hence in particular without the use of Cheeger's differentiation theory, but then the (p,p_0)-Poincaré inequality, or at least the validity of improved Poincaré inequalities (PI) for $p_0 \le q \le p$, has to be explicitly assumed for some exponent $1 < p_0 < p$; see Section 4 for details.

Proof of Theorem 1.1. Since the space X is assumed to be complete, the validity of the improved Poincaré inequalities (PI) follows from the $(1,p)$-Poincaré inequality, as discussed in Section 2.3. Hence we can apply Proposition 3.2. Moreover, by the deep result of Cheeger, [4, Theorem 6.1] (see also [1, Theorem A.7]), the upper pointwise Lipschitz constant $\operatorname{Lip}(v,\cdot)$ is a minimal $(p-\varepsilon)$-weak gradient of the Lipschitz function v, and so we obtain from estimates (14) and (16) (and Remark 2.1) that the set E is indeed uniformly $(p-\varepsilon)$-fat. □

4. Self-improvement of uniform fatness in non-complete spaces

In this section we provide the additional argument that is needed for the proof of the self-improvement result in the setting of non-complete metric spaces, Theorem 1.2. In fact, we prove the following slightly stronger result (by Hölder's inequality, the (p,p_0)-Poincaré inequality that was assumed in Theorem 1.2 implies the improved Poincaré inequalities (PI) for $p_0 \le q \le p$.)

Theorem 4.1. *Let $1 < p < \infty$ and suppose that X supports the improved (q,q)-Poincaré inequalities (PI) for $p_0 \le q \le p$. Assume that $E \subset X$ is a uniformly p-fat closed set and that $E \cap \overline{B(w,r)}$ is compact for all $w \in E$ and all $r > 0$. Then there exists $0 < \varepsilon < p - p_0$*

such that E is uniformly $(p-\varepsilon)$-fat; here both ε and the constant of uniform $(p-\varepsilon)$-fatness are quantitative.

The proof of Theorem 4.1 is based on Proposition 3.2, but we also need some auxiliary results related to Hausdorff contents. Note that these auxiliary results are essentially established in [11], but there the space X is assumed to be complete.

The *Hausdorff content of codimension* q of a set $K \subset X$ is defined by

$$\widetilde{\mathcal{H}}^q_\rho(K) = \inf\Biggl\{\sum_k \mu(B(x_k,r_k))\,r_k^{-q} : K \subset \bigcup_k B(x_k,r_k),\ x_k \in K,\ 0 < r_k \le \rho\Biggr\}.$$

Density conditions for these Hausdorff contents are known to be closely related to uniform fatness. Indeed, from Proposition 3.2 we obtain the following result.

Lemma 4.2. *Let $1 < p < \infty$ and suppose that X supports the improved (q,q)-Poincaré inequalities (PI) for $p_0 \le q \le p$. Assume that $E \subset X$ is a uniformly p-fat closed set. Then there exist constants $C > 0$ and $p_0 < q < p$, quantitatively, such that*

$$\widetilde{\mathcal{H}}^q_{R/2}\bigl(E \cap \overline{B(w,R)}\bigr) \ge C\mu\bigl(B(w,R)\bigr)R^{-q} \tag{17}$$

whenever $w \in E$ and $0 < R < (1/8)\operatorname{diam}(X)$ are such that $E \cap \overline{B(w,R)}$ is compact.

Proof. Fix $w \in E$ and $0 < R < (1/8)\operatorname{diam}(X)$, write $B = B(w,R)$, and assume that $E \cap \overline{B}$ is compact. Let $\{B_k\}$, where $B_k = B(x_k,r_k)$ with $x_k \in E \cap \overline{B}$ and $0 < r_k \le R/2$, be a cover of $E \cap \overline{B}$. Since $E \cap \overline{B}$ is compact, we may assume that this cover is finite, i.e. $E \cap \overline{B} \subset \bigcup_{k=1}^N B_k$. Also let $q = p - \varepsilon$, where ε is as in Proposition 3.2.

Define

$$v(x) = \max_{1\le k\le N} \bigl\{0, 1 - {r_k}^{-1}\operatorname{dist}(x,B_k)\bigr\}.$$

Then v is a Lipschitz function, $v = 1$ in $E \cap \overline{B}$, $v = 0$ outside $2B$, and $0 \le v \le 1$. Moreover, the upper pointwise Lipschitz constant of v satisfies $\operatorname{Lip}(v,x) \le \max_{1\le k\le N} {r_k}^{-1}\chi_{\overline{2B_k}}(x)$ for all $x \in X$, and hence

$$\operatorname{Lip}(v,x)^q \le \sum_{k=1}^N {r_k}^{-q}\chi_{\overline{2B_k}}(x)$$

for all $x \in 2B$. Thus we obtain from Proposition 3.2 (and the doubling condition) that

$$\mu(B)R^{-q} \le C\int_{2B} \operatorname{Lip}(v,x)^q\,d\mu(x) \le C\sum_{k=1}^N \mu\bigl(\overline{2B_k}\bigr){r_k}^{-q} \le C\sum_{k=1}^N \mu(B_k){r_k}^{-q}\,.$$

Taking the infimum over all such covers of $E \cap \overline{B}$ yields the claim. □

On the other hand, from (17) we get back to t-uniform fatness, for any $t > q$.

Lemma 4.3. *Let $1 < q < \infty$ and suppose that X supports a $(1,t)$-Poincaré inequality for all $t > q$. Let $E \subset X$ be a closed set. If there exists $C > 0$ such that the density condition*

$$\widetilde{\mathcal{H}}^q_{R/2}\bigl(E \cap \overline{B(w,R)}\bigr) \ge C\mu\bigl(B(w,R)\bigr)R^{-q} \tag{18}$$

holds for all $w \in E$ and all $0 < R < (1/8)\operatorname{diam}(X)$, then E is uniformly t-fat for all $t > q$.

Proof. Fix $t > q$. Let $w \in E$ and $0 < R < (1/8)\operatorname{diam}(X)$, write $B = B(w,R)$, and let $u \in \operatorname{Lip}_0(2B)$ be such that $0 \le u \le 1$ and $u = 1$ in $E \cap \overline{B}$. By the capacity comparison estimate (14) and Remark 2.1, it suffices to show that there exists a constant $C > 0$, independent of w, R and u, such that

$$\mu(B)R^{-t} \le C\int_{2B} g^t\,d\mu \tag{19}$$

for all t-weak upper gradients g of u such that $g = g\chi_{2B} \in L^t(X)$.

If $u_{2B} \geq 1/2$, then it follows from the Sobolev inequality [1, Theorem 5.51] that

$$1/2 \leq \fint_{2B} u \, d\mu \leq C_{X,p} R \biggl(\fint_{2B} g^t \, d\mu \biggr)^{1/t},$$

and from this (19) follows easily.

On the other hand, if $u_{2B} < 1/2$, we can use similar reasoning as in [11, p. 729] (which is based on the proof of [7, Theorem 5.9]), but let us recall the main steps for convenience. Since $t > q$ and $1/2 < u(x) - u_{2B} = |u(x) - u_{2B}|$ for each $x \in E \cap \overline{B}$, we can apply a well-known chaining argument (using also the continuity of u and the $(1,t)$-Poincaré inequality) to find for each $x \in E \cap \overline{B}$ a ball $B_x = B(x, r_x)$ with $0 < r_x \leq 3R$ such that

$$\mu(B_x) r_x^{-q} \leq C_{X,t,q} R^{t-q} \int_{\tau B_x} g^t \, d\mu \,. \tag{20}$$

The $5r$-covering lemma then yields us a countable collection of points $x_1, x_2, \ldots \in E \cap \overline{B}$ such that the corresponding balls $B_k = \tau B_{x_k}$ are pairwise disjoint, but the balls $5B_k$ cover $E \cap \overline{B}$. Using the assumption (18) for this particular cover and the doubling property of μ, we find that

$$\mu(B) R^{-q} \leq C \sum_k \mu(B_{x_k}) r_{x_k}^{-q} \,, \tag{21}$$

whence estimate (20) and the pairwise disjointness of the balls B_k yield the claim (19). (In particular, here we may assume that the radii of the balls $5B_k$ are all less than $R/2$, since otherwise the claim readily follows from the doubling property of μ and inequality (20) applied to a ball B_{x_k} with $5\tau r_{x_k} > R/2$). □

The proof of Theorem 4.1. Since we assumed that E is uniformly p-fat and $E \cap \overline{B(w,R)}$ is compact for all $w \in E$ and all $R > 0$, we have by Lemma 4.2 that

$$\widetilde{\mathcal{H}}^q_{R/2}\bigl(E \cap \overline{B(w,R)}\bigr) \geq C \mu\bigl(B(w,R)\bigr) R^{-q}$$

for all $w \in E$ and all $0 < R < (1/8)\operatorname{diam}(X)$, where $p_0 < q < p$. But now we can fix $q < t < p$, and Lemma 4.3 yields that E is uniformly t-fat. Notice, in particular, that the $(1,t)$-Poincaré inequality that is needed in the proof of Lemma 4.3 is valid by the assumed improved Poincaré inequalities (PI) since $p_0 < t < p$. □

5. Improved local Hardy inequalities

This section is devoted to the proof of the improved local Hardy inequality (15) that is reformulated as Proposition 5.7. The proof of this proposition is divided in the following three parts. In §5.1 we prepare for the localization of Hardy inequalities by truncating the set E and proving a local absorption lemma. In §5.2 we then obtain localized Hardy inequalities with exponent p, and in §5.3 we finally establish their self-improvement.

5.1. Truncation.

We begin with some technical tools that will be needed in the proofs of the local Hardy inequalities. The following truncation procedure provides us with the closed set $E_B \subset \overline{B}$ that was required in the proof of Theorem 3.1. A similar procedure was used in [14, p. 180] when proving the self-improvement of uniform (α,p)-fatness conditions in $\mathbb{R}^n$, and later also in [18], for weighted $\mathbb{R}^n$, and in [2], for general metric spaces.

We write $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 5.1. *Assume that $E \subset X$ is a closed set and that $B = B(w,r)$ for $w \in E$ and $r > 0$. Let $E_B^0 = E \cap \overline{\tfrac{1}{2}B}$, define inductively, for every $j \in \mathbb{N}$, that*

$$E_B^j = \bigcup_{x \in E_B^{j-1}} E \cap \overline{B(x, 2^{-j-1}r)}\,, \qquad \text{and set} \quad E_B = \overline{\bigcup_{j \in \mathbb{N}_0} E_B^j}.$$

Then the following statements hold:

(a) $w \in E_B$
(b) $E_B \subset E$
(c) $E_B \subset \overline{B}$
(d) $E_B^{j-1} \subset E_B^j \subset E_B$ *for every* $j \in \mathbb{N}$.

Proof. Part (a) is is true since $w \in E_B^0$. Part (b) follows from the facts that E is closed and $\cup_j E_B^j \subset E$ by definition. To verify (c), we fix $x \in E_B^j$. If $j = 0$, then $x \in \overline{B}$. If $j > 0$, then by induction we find a sequence $x_j, \ldots, x_0$ such $x_j = x$ and, for each $k = 0, \ldots, j$, $x_k \in E_B^k$ and $x_k \in E \cap \overline{B(x_{k-1}, 2^{-k-1}r)}$ if $k > 0$. It follows that

$$d(x, w) \le \sum_{k=1}^{j} d(x_k, x_{k-1}) + d(x_0, w) \le \sum_{k=1}^{j} 2^{-k-1}r + 2^{-1}r < r\,.$$

Hence, $x \in B(w, r) \subset \overline{B}$. We have shown that $E_B^j \subset \overline{B}$ whenever $j \in \mathbb{N}_0$, whence it follows that also $E_B \subset \overline{B}$. To prove (d) we fix $j \in \mathbb{N}$ and $x \in E_B^{j-1}$. By definition we have $x \in E$ and, hence, $x \in E \cap B(x, 2^{-j-1}r) \subset E_B^j$. □

Next we show that Lemma 5.1, in fact, truncates the set E to B in such a way that there are always certain balls $\widehat{B}$ whose intersection with the truncated set E_B contain big pieces of the original set E (by these balls we later employ the uniform fatness of E).

Lemma 5.2. *Let* E*,* B*, and* E_B *be as in Lemma 5.1. Suppose that* $m \in \mathbb{N}_0$ *and* $x \in X$ *is such that* $d_{E_B}(x) < 2^{-m+1}r$*. Then there exists a ball* $\widehat{B} = B(y_{x,m}, 2^{-m-1}r)$ *such that* $y_{x,m} \in E$*,*

$$\overline{2^{-1}\widehat{B}} \cap E = \overline{2^{-1}\widehat{B}} \cap E_B\,, \tag{22}$$

and $\sigma\widehat{B} \subset B(x, \sigma 2^{-m+2}r)$ *for every* $\sigma \ge 1$.

Proof. In this proof we will apply Lemma 5.1 several times without further notice. Since $d_{E_B}(x) < 2^{-m+1}r$ there exists $y \in \cup_{j\in\mathbb{N}_0} E_B^j \subset E$ such that $d(y, x) < 2^{-m+1}r$. Let us fix $j \in \mathbb{N}_0$ such that $y \in E_B^j$. There are two cases to be treated.

First, let us consider the case when $j > m \ge 0$. By induction, there are points $y_k \in E_B^k$ with $k = m, \ldots, j$ such that $y_j = y$ and $y_k \in E \cap \overline{B(y_{k-1}, 2^{-k-1}r)}$ for every $k = m+1, \ldots, j$. It follows that

$$d(y_m, y) = d(y_j, y_m) \le \sum_{k=m+1}^{j} d(y_k, y_{k-1}) \le \sum_{k=m+1}^{j} 2^{-k-1}r < 2^{-m-1}r\,.$$

Take $y_{x,m} = y_m \in E_B^m \subset E$ and $\widehat{B} = B(y_m, 2^{-m-1}r)$. If $\sigma \ge 1$ and $z \in \sigma\widehat{B}$, then

$$\begin{aligned} d(z, x) &\le d(z, y_m) + d(y_m, y) + d(y, x) \\ &\le \sigma 2^{-m-1}r + 2^{-m-1}r + 2^{-m+1}r < \sigma 2^{-m+2}r\,, \end{aligned}$$

and thus $\sigma\widehat{B} \subset B(x, \sigma 2^{-m+2}r)$. Moreover, since $y_m \in E_B^m$, we have

$$\overline{2^{-1}\widehat{B}} \cap E = E \cap \overline{B(y_m, 2^{-m-2}r)} \subset \bigcup_{z \in E_B^m} E \cap \overline{B(z, 2^{-m-2}r)} = E_B^{m+1} \subset E_B\,.$$

On the other hand $E_B \subset E$, and thus $\overline{2^{-1}\widehat{B}} \cap E = \overline{2^{-1}\widehat{B}} \cap E_B$.

Let us then consider the case $m \ge j \ge 0$. We take $y_{x,m} = y \in E$ and $\widehat{B} = B(y, 2^{-m-1}r)$. Then, for every $\sigma \ge 1$ and each $z \in \sigma\widehat{B}$,

$$d(z, x) \le d(z, y) + d(y, x) < \sigma 2^{-m-1}r + 2^{-m+1}r < \sigma 2^{-m+2}r\,,$$

and so $\sigma\widehat{B} \subset B(x, \sigma 2^{-m+2}r)$. Since $y \in E_B^j \subset E_B^m \subset E_B$ we can repeat the argument above, with y_m replaced by y, and it follows as above that $\overline{2^{-1}\widehat{B}} \cap E = \overline{2^{-1}\widehat{B}} \cap E_B$. □

One of the reasons for truncating the set E, *in the first place*, is to obtain the absorption Lemma 5.3. This lemma is needed twice during the rest of the paper, with slightly different contexts, and hence there are two different assumptions concerning the validity of Poincaré inequalities. The dependencies of the constants below are rather delicate, and it is important to track them carefully; to this end, recall our notational convention from §2.5.

Lemma 5.3. *Suppose that either*

(i) $1 \le q = p < \infty$ *and* X *supports a* $(1,p)$*-Poincaré inequality; or*
(ii) $1 < p_0 < p < \infty$ *and* X *supports the improved Poincaré inequalities (PI) for exponents* $p_0 \le q \le p$.

In addition, let E, B, *and* E_B *be as in Lemma 5.1, let* $\sigma \ge 1$ *and* $\varsigma \ge 2$, *and write* $B^* = \varsigma B$. *Assume that* $u \in \operatorname{Lip}(X)$ *is such that* $u = 0$ *on* E_B, *and that* g *is a* q*-weak upper gradient of* u *such that inequality*

$$\int_{B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) \le C_1 \int_{\sigma B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) + C_2 \int_{\sigma B^*} g(x)^q\,d\mu(x)$$

holds with some constants $C_1, C_2 > 0$. *Then*

$$C_3 \int_{\sigma B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) \le C_4 \int_{\tau\sigma B^*} g(x)^q\,d\mu(x)\,,$$

where $C_3 = 1 - C_1(1 + C_{C_D,\sigma,\varsigma,p})$ *and* $C_4 = (1 + C_2)C_{X,\sigma,\varsigma,q}$.

Proof. Since $E_B \subset \overline{B}$ and $\varsigma \ge 2$, we obtain the estimate

$$\begin{aligned}\int_{(\sigma B^*\setminus E_B)\setminus B^*} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) &\le r^{-q}\int_{\sigma B^*} |u(x)|^q\,d\mu(x)\\ &\le 3^q r^{-q}\biggl(\int_{\sigma B^*} |u(x) - u_{\sigma B^*}|^q\,d\mu(x) + \mu(\sigma B^*)|u_{\sigma B^*} - u_{B^*}|^q + \mu(\sigma B^*)|u_{B^*}|^q\biggr).\end{aligned}$$

By the doubling inequality (4) and the (q,q)-Poincaré inequality (9) (recall that in case (i), i.e. for $q = p$, this is a consequence of the $(1,p)$-Poincaré inequality, cf. [1, Corollary 4.24]) we obtain

$$\begin{aligned}&3^q r^{-q}\biggl(\int_{\sigma B^*} |u(x) - u_{\sigma B^*}|^q\,d\mu(x) + \mu(\sigma B^*)|u_{\sigma B^*} - u_{B^*}|^q\biggr)\\ &\qquad\le C_{\sigma,q,C_D} r^{-q}\int_{\sigma B^*} |u(x) - u_{\sigma B^*}|^q\,d\mu(x) \le C_{X,\sigma,\varsigma,q}\int_{\tau\sigma B^*} g(x)^q\,d\mu(x)\,.\end{aligned}$$

On the other hand, by the assumption,

$$\begin{aligned}3^q r^{-q}\mu(\sigma B^*)|u_{B^*}|^q &\le 3^q C_{\sigma,C_D} r^{-q}\int_{B^*\setminus E_B} |u(x)|^q\,d\mu(x)\\ &\le 3^q \varsigma^q C_{\sigma,C_D}\int_{B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x)\\ &\le 3^p \varsigma^p C_{\sigma,C_D} C_1 \int_{\sigma B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) + 3^q\varsigma^q C_{\sigma,C_D} C_2 \int_{\sigma B^*} g(x)^q\,d\mu(x)\,.\end{aligned}$$

Combining the estimates above, we find that

$$\begin{aligned}\int_{\sigma B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) &= \int_{B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) + \int_{(\sigma B^*\setminus E_B)\setminus B^*} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x)\\ &\le C_1(1 + C_{C_D,\sigma,\varsigma,p})\int_{\sigma B^*\setminus E_B} \frac{|u(x)|^q}{d_{E_B}(x)^q}\,d\mu(x) + C_4\int_{\tau\sigma B^*} g(x)^q\,d\mu(x)\,,\end{aligned}$$

where $C_4 = (1 + C_2)C_{X,\sigma,\varsigma,q}$. This concludes the proof. □

5.2. **Local Hardy.** In this section we prove Proposition 5.4 that gives a local p-Hardy inequality with respect to the truncated set E_B (Theorem 1.3 is also proved at the end of this section). This is done by adapting the proof of [11, Theorem 3] to the present setting. The proof in [11] is, in turn, based on the ideas of Wannebo [19]; see also [3].

Throughout this section, we will assume that X supports a $(1,p)$-Poincaré inequality, whence X supports also a (p,p)-Poincaré inequality. Both of these inequalities are assumed to be valid with constants $C_P > 0$ and $\tau \geq 1$.

Proposition 5.4. *Let $1 < p < \infty$ and suppose that X supports a $(1,p)$-Poincaré inequality. Assume that $E \subset X$ is a uniformly p-fat closed set, let $w \in E$ and $0 < r < (1/8)\operatorname{diam}(X)$, and let E_B be as in Lemma 5.1 for $B = B(w,r)$. Let $u \in \operatorname{Lip}(X)$ and let g be a p-weak upper gradient of u such that $u = 0 = g$ in an open set $U \subset X$ satisfying the condition $\operatorname{dist}(E_B, X \setminus U) > 0$ (or the condition $X = U$). Then*

$$\int_{8\tau B\setminus E_B} \frac{|u(x)|^p}{d_{E_B}(x)^p}\, d\mu(x) \leq C_H \int_{8\tau^2 B} g(x)^p\, d\mu(x)\,. \tag{23}$$

Here $C_H = C_{X,p,c_0}$ and the number $0 < c_0 \leq 1$ is the constant from the uniform p-fatness condition (12) for E.

Recall that we do not assume X to be complete, and hence it is not necessarily true that $\operatorname{dist}(K, X \setminus U) > 0$ whenever K is a closed subset of a bounded open set U. For this reason we make in Proposition 5.4 the explicit assumption that $\operatorname{dist}(E_B, X \setminus U) > 0$.

The proof of Proposition 5.4 is based upon covering and absorption arguments, and it will be completed at the end of this section. We begin with Lemmata 5.5 and 5.6 that provide information concerning the individual balls in the following covering families. For the rest of this section we assume that p, X, E, $B = B(w,r)$, and E_B are as in Proposition 5.4 (these are considered arbitrary but fixed).

For each $m \in \mathbb{Z}$, let us write

$$G_m = \{x \in 8\tau B \,:\, 2^{-m} r \leq d_{E_B}(x) < 2^{-m+1} r\}$$

and

$$\widetilde{G}_m = \bigcup_{k=m}^{\infty} G_k = \{x \in 8\tau B \,:\, 0 < d_{E_B}(x) < 2^{-m+1} r\}\,.$$

For every $m \in \mathbb{N}_0$ we let $\mathcal{G}_m$ be a (countable) cover of G_m with open balls $\widetilde{B}$ that are centered at G_m and of radius $2^{-m+2}r$. Moreover, we require that $\{2^{-1}\widetilde{B} \,:\, \widetilde{B} \in \mathcal{G}_m\}$ is a disjoint family, whence there exits $C = C_{C_D,\tau} > 0$ such that

$$\sum_{\widetilde{B}\in\mathcal{G}_m} \chi_{\tau\widetilde{B}} \leq C\,. \tag{24}$$

The existence of such a cover follows using a maximal packing argument and the doubling property of μ.

Lemma 5.5. *Let us define $\ell = \lceil \log_2(\tau)\rceil + 2$. Then, for each $m \in \mathbb{N}_0$ and every ball $\widetilde{B} \in \mathcal{G}_m$, we have*

$$\tau\widetilde{B} \setminus E_B \subset \widetilde{G}_{m-\ell}\,. \tag{25}$$

Proof. Fix $m \in \mathbb{N}_0$ and $\widetilde{B} \in \mathcal{G}_m$. By definition, we have $\widetilde{B} = B(x_{\widetilde{B}}, 2^{-m+2}r)$ with $x_{\widetilde{B}} \in G_m$. Let $x \in \tau\widetilde{B} \setminus E_B$. Then $d_{E_B}(x) > 0$. Moreover,

$$\begin{aligned} d_{E_B}(x) = \operatorname{dist}(x, E_B) &\leq d(x, x_{\widetilde{B}}) + \operatorname{dist}(x_{\widetilde{B}}, E_B) \\ &< \tau 2^{-m+2} r + 2^{-m+1} r < \tau 2^{-m+3} r \leq 2^{-(m-\ell)+1} r\,. \end{aligned}$$

Since $m \geq 0$, a modification of the previous estimate also yields

$$d(x, w) \leq \operatorname{dist}(x, \overline{B}) + r \leq \operatorname{dist}(x, E_B) + r < 4\tau r + 2r + r < 8\tau r\,,$$

and it follows that $x \in B(w, 8\tau r) = 8\tau B$. We can now conclude that $x \in \widetilde{G}_{m-\ell}$. □

The uniform p-fatness of E is exclusively used in the following lemma.

Lemma 5.6. *Let v be a Lipschitz function on X such that $v = 0$ on E_B and let g be a p-weak upper gradient of v. Then, for every $m \in \mathbb{N}_0$ and each $\widetilde{B} \in \mathcal{G}_m$,*

$$\int_{\widetilde{B}} |v(x)|^p \, d\mu(x) \le \frac{C_{X,p}}{c_0} 2^{-mp} r^p \int_{\tau \widetilde{B}} g(x)^p \, d\mu(x) \,. \tag{26}$$

Proof. Fix $m \in \mathbb{N}_0$ and $\widetilde{B} = B(x_{\widetilde{B}}, 2^{-m+2}r) \in \mathcal{G}_m$. Then, by definition, $x_{\widetilde{B}} \in G_m$. We apply Lemma 5.2 and thereby associate to $\widetilde{B}$ a smaller open ball $\widehat{B} \subset \widetilde{B}$, centered at E and of radius $2^{-m-1}r < (1/8)\operatorname{diam}(X)$. Note first that

$$\fint_{\widetilde{B}} |v(x)|^p \, d\mu(x) \le C_p \biggl(\fint_{\widetilde{B}} |v(x) - v_{\widetilde{B}}|^p \, d\mu(x) + |v_{\widetilde{B}} - v_{\widehat{B}}|^p + |v_{\widehat{B}}|^p \biggr) \,.$$

Here, by Hölder's inequality and the doubling condition (4),

$$|v_{\widetilde{B}} - v_{\widehat{B}}|^p \le \biggl(\fint_{\widehat{B}} |v(x) - v_{\widetilde{B}}| \, d\mu(x) \biggr)^p \le C_{C_D} \fint_{\widetilde{B}} |v(x) - v_{\widetilde{B}}|^p \, d\mu(x) \,,$$

and therefore, by the (p,p)-Poincaré inequality, we have that

$$C_p \biggl(\fint_{\widetilde{B}} |v(x) - v_{\widetilde{B}}|^p \, d\mu(x) + |v_{\widetilde{B}} - v_{\widehat{B}}|^p \biggr) \le C_{X,p} 2^{-mp} r^p \fint_{\tau \widetilde{B}} g(x)^p \, d\mu(x) \,.$$

On the other hand, by the capacitary Poincaré inequality (13),

$$|v_{\widehat{B}}|^p \le \fint_{\widehat{B}} |v(x)|^p \, d\mu(x) \le \frac{C_X}{\operatorname{cap}_p(\overline{2^{-1}\widehat{B}} \cap \{v = 0\}, \widehat{B})} \int_{\tau \widehat{B}} g(x)^p \, d\mu(x) \,.$$

Recall that $v(x) = 0$ whenever $x \in E_B$ (by assumption) and $\overline{2^{-1}\widehat{B}} \cap E_B = \overline{2^{-1}\widehat{B}} \cap E$ by Lemma 5.2. Therefore, using monotonicity, the uniform p-fatness condition (12), and the comparison inequality (14), we obtain

$$\begin{aligned} \operatorname{cap}_p(\overline{2^{-1}\widehat{B}} \cap \{v = 0\}, \widehat{B}) &\ge \operatorname{cap}_p(\overline{2^{-1}\widehat{B}} \cap E_B, \widehat{B}) = \operatorname{cap}_p(\overline{2^{-1}\widehat{B}} \cap E, \widehat{B}) \\ &\ge c_0 \operatorname{cap}_p(\overline{2^{-1}\widehat{B}}, \widehat{B}) \ge \frac{c_0\, \mu(2^{-1}\widehat{B})}{C_{X,p} 2^{-mp} r^p} \,. \end{aligned}$$

Finally, since $\tau \widehat{B} \subset \tau \widetilde{B}$, it follows that

$$C_p |v_{\widehat{B}}|^p \le C_p \fint_{\widehat{B}} |v(x)|^p \, d\mu(x) \le \frac{C_{X,p}}{c_0} \frac{2^{-mp} r^p}{\mu(2^{-1}\widehat{B})} \int_{\tau \widetilde{B}} g(x)^p \, d\mu(x) \,.$$

Inequality (26) follows from the above estimates and the doubling condition (4). □

Proof of Proposition 5.4. Let us first assume that $v \in \operatorname{Lip}_0(X \setminus E_B)$ and that g_v is a p-weak upper gradient of v that also vanishes in the set E_B. Then, by summing the inequalities (26) and using (24) and (25) we obtain, for every $m \in \mathbb{N}_0$,

$$\begin{aligned} \int_{G_m} |v(x)|^p \, d\mu(x) &\le \sum_{\widetilde{B} \in \mathcal{G}_m} \int_{\widetilde{B}} |v(x)|^p \, d\mu(x) \\ &\le \frac{C_{X,p}}{c_0} 2^{-mp} r^p \sum_{\widetilde{B} \in \mathcal{G}_m} \int_{\tau \widetilde{B} \setminus E_B} g_v(x)^p \, d\mu(x) \qquad (27) \\ &\le \frac{C_{X,p}}{c_0} 2^{-mp} r^p \int_{\widetilde{G}_{m-\ell}} g_v(x)^p \, d\mu(x) \,. \end{aligned}$$

Let $0 < \beta < 1$ be a small number, that will be fixed later. We multiply (27) by $2^{m(p+\beta)}r^{-p-\beta}$ and sum the inequalities to obtain the estimate

$$\begin{aligned}\int_{2B\setminus E_B} \frac{|v(x)|^p}{d_{E_B}(x)^{p+\beta}}\,d\mu(x) &\le \int_{\widetilde{G}_0} \frac{|v(x)|^p}{d_{E_B}(x)^{p+\beta}}\,d\mu(x) = \sum_{m=0}^{\infty} \int_{G_m} \frac{|v(x)|^p}{d_{E_B}(x)^{p+\beta}}\,d\mu(x) \\ &\le \sum_{m=0}^{\infty} 2^{m(p+\beta)} r^{-p-\beta} \int_{G_m} |v(x)|^p\,d\mu(x) \\ &\le \frac{C_{X,p}}{c_0} r^{-\beta} \sum_{m=0}^{\infty} 2^{m\beta} \int_{\widetilde{G}_{m-\ell}} g_v(x)^p\,d\mu(x) \\ &= \frac{C_{X,p}}{c_0} r^{-\beta} \sum_{k=-\ell}^{\infty} \sum_{m=0}^{k+\ell} 2^{m\beta} \int_{G_k} g_v(x)^p\,d\mu(x) \\ &\le \frac{C_{X,p}}{c_0\beta} \sum_{k=-\ell}^{\infty} 2^{k\beta} r^{-\beta} \int_{G_k} g_v(x)^p\,d\mu(x) \le \frac{C_{X,p}}{c_0\beta} \int_{8\tau B\setminus E_B} \frac{g_v(x)^p}{d_{E_B}(x)^\beta}\,d\mu(x)\,.\end{aligned} \tag{28}$$

Now we come to the main line of the argument. Let u be a Lipschitz function on X and let g be a p-weak upper gradient of u, both of which vanish in an open set $U \subset X$ satisfying the condition $\operatorname{dist}(E_B, X\setminus U) > 0$. We aim to show that inequality (23) holds, and so we may assume that $g \in L^p(8\tau B)$ (recall that $\tau \ge 1$).

Consider the Lipschitz function on $A = 8\tau B \cup (X \setminus 10\tau B)$ that coincides with u in $8\tau B$ and vanishes outside $10\tau B$, and let $\tilde{u}$ be the McShane extension (5) of this function to all of X. Then the function

$$\tilde{g} = g\chi_{8\tau B} + \operatorname{Lip}(\tilde{u},\cdot)\chi_{X\setminus 8\tau B} \in L^p(X)$$

is a p-weak upper gradient of $\tilde{u}$, cf. the proof of [1, Theorem 2.6]. Define $v(x) = \tilde{u}(x)d_{E_B}(x)^{\beta/p}$ for every $x \in X$. Then $v(x) = u(x)d_{E_B}(x)^{\beta/p}$ for every $x \in 8\tau B$ and in particular v vanishes in E_B. Moreover, by applying the assumptions on u and g in combination with the Leibniz and chain rules of Theorems 2.15 and 2.16 in [1], we find that v has a p-weak upper gradient g_v such that

$$g_v(x) \le g(x)d_{E_B}(x)^{\beta/p} + \frac{\beta}{p}|u(x)|d_{E_B}(x)^{\beta/p-1}$$

for every $x \in 8\tau B$; in particular, also g_v vanishes on the set E_B. Using estimate (28) for the pair v and g_v, we obtain

$$\begin{aligned}\int_{2B\setminus E_B} \frac{|u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x) &= \int_{2B\setminus E_B} \frac{|v(x)|^p}{d_{E_B}(x)^{p+\beta}}\,d\mu(x) \le \frac{C_{X,p}}{c_0\beta} \int_{8\tau B\setminus E_B} \frac{g_v(x)^p}{d_{E_B}(x)^\beta}\,d\mu(x) \\ &\le \frac{C_{X,p}}{c_0\beta} \int_{8\tau B} g(x)^p\,d\mu(x) + \frac{C_{X,p}}{c_0}\beta^{p-1} \int_{8\tau B\setminus E_B} \frac{|u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x)\,.\end{aligned}$$

We can now apply Lemma 5.3 with parameters

$$\varsigma = 2\,, \qquad \sigma = 4\tau\,, \qquad q = p\,, \qquad C_1 = \frac{C_{X,p}}{c_0}\beta^{p-1}\,, \qquad C_2 = \frac{C_{X,p}}{c_0\beta}\,.$$

Recall our convention in §2.5 and choose $0 < \beta < 1$, depending on C_X, p, and c_0, such that

$$C_3 = 1 - C_1(1 + C_{C_D,\sigma,\varsigma,p}) \ge \frac{1}{2}\,.$$

Then, Lemma 5.3 yields that

$$\int_{8\tau B\setminus E_B} \frac{|u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x) \le C_{X,p,c_0} \int_{8\tau^2 B} g(x)^p\,d\mu(x)\,,$$

and this concludes the proof. □

Before entering the final stage in our proof of the self-improvement of uniform p-fatness, we take a small side step and give a proof for Theorem 1.3 that was stated in the introduction.

Proof of Theorem 1.3. Fix $w \in E$ and $0 < r < (1/8)\operatorname{diam}(X)$, and let E_B be as in Lemma 5.1 for the ball $B = B(w,r)$. Fix $u \in \operatorname{Lip}_0(X \setminus E)$ and a p-weak upper gradient g of u. Since we first aim to prove estimate (29) below, we can assume that $g \in L^p(8\tau^2 B)$.

For every $\delta > 0$, we define a Lipschitz function $u_\delta = \max\{0, |u| - \delta\}$. Since g is clearly a p-weak upper gradient of u_δ, it is straightforward to show that the function

$$h = g\chi_{8\tau^2 B} + \operatorname{Lip}(u_\delta, \cdot)\chi_{X\setminus 8\tau^2 B}$$

is a p-weak upper gradient of u_δ, cf. the proof of [1, Theorem 2.6]. Since u_δ vanishes in the set $U_\delta = \{|u| < \delta\}$ we can apply the local version of the glueing lemma [1, Lemma 2.19] with u_δ and h. From this we can deduce that $g_\delta = h\chi_{X\setminus U_\delta}$ is a p-weak upper gradient of u_δ. Observe that both u_δ and g_δ vanish in the open neighbourhood U_δ of E and $\operatorname{dist}(E_B, X \setminus U_\delta) > 0$ if $U_\delta \neq X$. Since $E \cap (1/2)B \subset E_B$ and $w \in E_B$, we see that $d_E = d_{E_B}$ in $(1/4)B$. Hence, by monotone convergence and Proposition 5.4, we conclude that

$$\begin{aligned} \int_{(1/4)B\setminus E} \frac{|u(x)|^p}{d_E(x)^p}\, d\mu(x) &= \lim_{\delta\to 0} \int_{(1/4)B\setminus E} \frac{|u_\delta(x)|^p}{d_{E_B}(x)^p}\, d\mu(x) \\ &\le C_H \liminf_{\delta\to 0} \int_{8\tau^2 B} g_\delta(x)^p\, d\mu(x) \le C_H \int_{8\tau^2 B} g(x)^p\, d\mu(x)\,. \end{aligned} \tag{29}$$

The desired inequality (3) now follows by a simple change of variables. □

5.3. **Improvement.** In this section we improve the 'local integral Hardy inequality', that was established in Proposition 5.4, by adapting ideas from Koskela–Zhong [13] to the present setting and applying again the absorption Lemma 5.3. This improvement argument constitutes the final step in the proof of Theorem 3.1.

Proposition 5.7. *Let $1 < p < \infty$ and suppose that X supports the improved (q,q)-Poincaré inequalities (PI) for $p_0 \le q \le p$. Assume that $E \subset X$ is a uniformly p-fat closed set, let $w \in E$ and $0 < r < (1/8)\operatorname{diam}(X)$, and let E_B be as in Lemma 5.1 for $B = B(w,r)$. Then there exists constants $0 < \varepsilon = \varepsilon_{X,p_0,p,C_H} < p - p_0$ and $C > 0$ such that the inequality*

$$\int_{12\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) \le C \int_{12\tau^3 B} \operatorname{Lip}(u,x)^{p-\varepsilon}\, d\mu(x) \tag{30}$$

holds whenever $u \in \operatorname{Lip}_0(X \setminus E)$. Here $C_H = C_{X,p,c_0}$ is the constant from Proposition 5.4.

In the proof of Proposition 5.7, we use the restricted maximal function $M_R u$ at $x \in X$, for $R : X \to [0,\infty)$ and a locally integrable function u on X, that is defined by $M_R u(x) = |u(x)|$ if $R(x) = 0$, and otherwise by

$$M_R u(x) = \sup_r \fint_{B(x,r)} |u(y)|\, d\mu(y)\,,$$

where the supremum is taken over all radii $0 < r < R(x)$.

Proof of Proposition 5.7. Without loss of generality, we may assume that $C_H \ge 1$ in (23). We will first prove inequality (30) under the additional assumption that $u \in \operatorname{Lip}(X)$ is such that $u = 0$ in an open set $U \subsetneq X$ for which $\operatorname{dist}(E, X \setminus U) > 0$. Throughout this proof, we write $g = \operatorname{Lip}(u,\cdot)$; in particular, also $g = 0$ in U.

Fix a number $\lambda > 0$, and define $F_\lambda = H_\lambda \cap G_\lambda$, where

$$\begin{aligned} H_\lambda &= \{x \in 8\tau^2 B \,:\, |u(x)| \le \lambda d_{E_B}(x)\}\,, \\ G_\lambda &= \Big\{x \in 8\tau^2 B \,:\, \big(M_{d_{E_B}(x)/2}\, g^{p_0}(x)\big)^{1/p_0} \le \lambda\Big\}\,. \end{aligned}$$

We claim that the restriction of u to F_λ is $(C_{X,p_0}\lambda)$-Lipschitz. Indeed, let $x,y\in F_\lambda$, $x\neq y$, be such that $d_{E_B}(y)\le d_{E_B}(x)$. If $d_{E_B}(x)\ge 5\tau d(x,y)$, then

$$d_{E_B}(y)\ge d_{E_B}(x)-d(x,y)\ge 4\tau d(x,y)\,.$$

Thus a standard chaining argument [5, Theorem 3.2], which is based on the facts that μ is doubling and that the $(1,p_0)$-Poincaré inequality holds for the pair u and $g=\operatorname{Lip}(u,\cdot)$, yields that

$$\begin{aligned}|u(x)-u(y)| &\le C_{X,p_0}\, d(x,y)\Big(\big(M_{2\tau d(x,y)}\, g^{p_0}(x)\big)^{1/p_0}+\big(M_{2\tau d(x,y)}\, g^{p_0}(y)\big)^{1/p_0}\Big)\\ &\le C_{X,p_0}\, d(x,y)\Big(\big(M_{d_{E_B}(x)/2}\, g^{p_0}(x)\big)^{1/p_0}+\big(M_{d_{E_B}(y)/2}\, g^{p_0}(y)\big)^{1/p_0}\Big)\\ &\le C_{X,p_0}\lambda d(x,y)\,.\end{aligned}$$

On the other hand, if $d_{E_B}(y)\le d_{E_B}(x)\le 5\tau d(x,y)$, then

$$|u(x)-u(y)|\le |u(x)|+|u(y)|\le \lambda(d_{E_B}(x)+d_{E_B}(y))\le 10\tau\lambda d(x,y).$$

These two estimates show that u is Lipschitz on F_λ.

Next we use the McShane extension (5) and extend the restriction of u on $A=F_\lambda$ to a $(C_{X,p_0}\lambda)$-Lipschitz function $\tilde u$ on X. Then also $\tilde u$ vanishes on an open set $\widetilde U\subset U$ such that $\operatorname{dist}(E_B,X\setminus\widetilde U)>0$; indeed, if $x\in\widetilde U=\{x\in 8\tau^2B : d_{E_B}(x)<\operatorname{dist}(E_B,X\setminus U)/2\}$ then $u(x)=0$ and $x\in F_\lambda$ (here we use the fact that $g=\operatorname{Lip}(u,\cdot)=0$ in U), whence $\tilde u(x)=0$.

By [1, Lemma 2.19], the bounded function

$$\tilde g(x)=\chi_{F_\lambda}(x)g(x)+C_{X,p_0}\lambda\chi_{X\setminus F_\lambda}(x)$$

is a p-weak upper gradient of $\tilde u$ that vanishes on $\widetilde U$. Hence, applying Proposition 5.4 to the pair $\tilde u$ and $\tilde g$, we find that

$$\begin{aligned}\int_{(8\tau B\setminus E_B)\cap F_\lambda}\frac{|u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x) &\le \int_{8\tau B\setminus E_B}\frac{|\tilde u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x)\\ &\le C_H\int_{F_\lambda} g(x)^p\,d\mu(x)+C_H C_{X,p_0}^p\lambda^p\mu(8\tau^2B\setminus F_\lambda)\end{aligned}$$

and, since $C_H\ge 1$, that

$$\begin{aligned}&\int_{(8\tau B\setminus E_B)\cap H_\lambda}\frac{|u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x)\\ &\qquad\le C_H\int_{F_\lambda} g(x)^p\,d\mu(x)+C_H C_{X,p_0}^p\lambda^p\mu(8\tau^2B\setminus F_\lambda)+\int_{(H_\lambda\setminus E_B)\setminus G_\lambda}\frac{|u(x)|^p}{d_{E_B}(x)^p}\,d\mu(x)\\ &\qquad\le C_H\int_{G_\lambda} g(x)^p\,d\mu(x)+C_H C_{X,p_0,p}\lambda^p\big(\mu(8\tau^2B\setminus F_\lambda)+\mu(H_\lambda\setminus G_\lambda)\big)\\ &\qquad\le C_H\int_{G_\lambda} g(x)^p\,d\mu(x)+C_H C_{X,p_0,p}\lambda^p\big(\mu(8\tau^2B\setminus H_\lambda)+\mu(8\tau^2B\setminus G_\lambda)\big)\,.\end{aligned}$$

The above estimate holds for all $\lambda>0$. We multiply it by $\lambda^{-1-\varepsilon}$ (here $0<\varepsilon<(p-p_0)/2$ is a parameter to be fixed later) and integrate the resulting estimate over $(0,\infty)$. This gives

$$\begin{aligned}&\varepsilon^{-1}\int_{8\tau B\setminus E_B}\frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\,d\mu(x)\\ &\qquad\le C_H\int_0^\infty \lambda^{-1-\varepsilon}\int_{G_\lambda} g(x)^p\,d\mu(x)\,d\lambda\\ &\qquad\qquad + C_H C_{X,p_0,p}\int_0^\infty \lambda^{p-1-\varepsilon}\big(\mu(8\tau^2B\setminus H_\lambda)+\mu(8\tau^2B\setminus G_\lambda)\big)\,d\lambda\,.\end{aligned}$$

By the definition of G_λ, and the observation that $g(x) \le \big(M_{d_{E_B}(x)/2}\, g^{p_0}(x)\big)^{1/p_0}$ for a.e. $x \in 8\tau^2 B$, we find that the first term on the right-hand side is dominated by

$$C_H \varepsilon^{-1} \int_{8\tau^2 B} g(x)^{p-\varepsilon}\, d\mu(x)\,.$$

Using the definitions of H_λ and G_λ, the second term on the right-hand side can be estimated from above by

$$\frac{C_H C_{X,p_0,p}}{p-\varepsilon}\bigg(\int_{8\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) + \int_{8\tau^2 B} \big(M_{d_{E_B}(x)/2}\, g^{p_0}(x)\big)^{\frac{p-\varepsilon}{p_0}}\, d\mu(x)\bigg)\,.$$

Since $d_{E_B}(x)/2 \le 4\tau^2 r$ for all $x \in 8\tau^2 B$, we have by the Hardy–Littlewood maximal theorem, see e.g. [1, Theorem 3.13], that

$$\begin{aligned}\int_{8\tau^2 B} \big(M_{d_{E_B}(x)/2}\, g^{p_0}(x)\big)^{\frac{p-\varepsilon}{p_0}}\, d\mu(x) &\le \int_X \big(M(\chi_{12\tau^2 B}\, g^{p_0})(x)\big)^{\frac{p-\varepsilon}{p_0}}\, d\mu(x)\\ &\le C_{C_D,p_0,p,\varepsilon} \int_{12\tau^2 B} g(x)^{p-\varepsilon}\, d\mu(x)\,;\end{aligned}$$

here M denotes the usual unrestricted maximal operator.

By combining the estimates above, we obtain

$$\begin{aligned}&\int_{8\tau B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x)\\ &\qquad \le C_1 \int_{8\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) + C_2 \int_{12\tau^2 B} g(x)^{p-\varepsilon}\, d\mu(x)\,,\end{aligned}$$

where

$$C_1 = C_H C_{X,p_0,p}\,\varepsilon (p-\varepsilon)^{-1} \qquad \text{and} \qquad C_2 = C_H\big(1 + C_{X,p_0,p}\,\varepsilon(p-\varepsilon)^{-1} C_{C_D,p_0,p,\varepsilon}\big)\,.$$

In order to apply Lemma 5.3, we write

$$\varsigma = 8\tau\,, \qquad \sigma = 3\tau/2\,, \qquad q = p-\varepsilon\,.$$

Recall our convention in §2.5 and choose $0<\varepsilon<(p-p_0)/2$, depending on X, p_0, p, and C_H, in such a way that

$$C_3 = 1 - C_1(1 + C_{C_D,\sigma,\varsigma,p}) \ge 1/2\,.$$

Then, Lemma 5.3 yields that

$$\int_{12\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) \le 2C_3 \int_{12\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) \le 2C_4 \int_{12\tau^3 B} g(x)^{p-\varepsilon}\, d\mu(x)\,,$$

where $C_4 = (1+C_2) C_{X,\sigma,\varsigma,p-\varepsilon}$. This proves the claim in the case where $u=0$ in an open set $U \subset X$ with $\operatorname{dist}(E, X\setminus U) > 0$.

To prove the general case, let $u \in \operatorname{Lip}_0(X\setminus E)$. Clearly, we may assume that u does not vanish everywhere in X. For every $\delta>0$, we define a Lipschitz function $u_\delta = \max\{0, |u|-\delta\}$. Now $\operatorname{Lip}(u_\delta,\cdot) \le \operatorname{Lip}(u,\cdot)$ and u_δ vanishes in the open neighbourhood $U_\delta = \{|u|<\delta\}$ of E. Thus, by monotone convergence and the special case of inequality (30) that was established above we conclude that

$$\begin{aligned}\int_{12\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x) &= \lim_{\delta\to 0} \int_{12\tau^2 B\setminus E_B} \frac{|u_\delta(x)|^{p-\varepsilon}}{d_{E_B}(x)^{p-\varepsilon}}\, d\mu(x)\\ &\le C \liminf_{\delta\to 0} \int_{12\tau^3 B} \operatorname{Lip}(u_\delta, x)^{p-\varepsilon}\, d\mu(x)\\ &\le C \int_{12\tau^3 B} \operatorname{Lip}(u, x)^{p-\varepsilon}\, d\mu(x)\,.\end{aligned}$$

This proofs the claim in the general case $u \in \operatorname{Lip}_0(X\setminus E)$. □

References

[1] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[2] J. Björn, P. MacManus, and N. Shanmugalingam. Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces. *J. Anal. Math.*, 85:339–369, 2001.

[3] S. M. Buckley and P. Koskela. Orlicz-Hardy inequalities. *Illinois J. Math.*, 48(3):787–802, 2004.

[4] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[5] P. Hajłasz and P. Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[6] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1993.

[7] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.

[8] S. Keith and X. Zhong. The Poincaré inequality is an open ended condition. *Ann. of Math. (2)*, 167(2):575–599, 2008.

[9] T. Kilpeläinen and P. Koskela. Global integrability of the gradients of solutions to partial differential equations. *Nonlinear Anal.*, 23(7):899–909, 1994.

[10] J. Kinnunen and R. Korte. Characterizations for the Hardy inequality. In *Around the research of Vladimir Maz'ya. I*, volume 11 of *Int. Math. Ser. (N. Y.)*, pages 239–254. Springer, New York, 2010.

[11] R. Korte, J. Lehrbäck, and H. Tuominen. The equivalence between pointwise Hardy inequalities and uniform fatness. *Math. Ann.*, 351(3):711–731, 2011.

[12] R. Korte and N. Shanmugalingam. Equivalence and self-improvement of p-fatness and Hardy's inequality, and association with uniform perfectness. *Math. Z.*, 264(1):99–110, 2010.

[13] P. Koskela and X. Zhong. Hardy's inequality and the boundary size. *Proc. Amer. Math. Soc.*, 131(4):1151–1158 (electronic), 2003.

[14] J. L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Soc.*, 308(1):177–196, 1988.

[15] J. L. Lewis. On very weak solutions of certain elliptic systems. *Comm. Partial Differential Equations*, 18(9-10):1515–1537, 1993.

[16] O. E. Maasalo and A. Zatorska-Goldstein. Stability of quasiminimizers of the p-Dirichlet integral with varying p on metric spaces. *J. Lond. Math. Soc. (2)*, 77(3):771–788, 2008.

[17] V. G. Maz'ya. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[18] P. Mikkonen. On the Wolff potential and quasilinear elliptic equations involving measures. *Ann. Acad. Sci. Fenn. Math. Diss.*, (104):71, 1996.

[19] A. Wannebo. Hardy inequalities. *Proc. Amer. Math. Soc.*, 109(1):85–95, 1990.

(J.L.) University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyvaskyla, Finland
E-mail address: juha.lehrback@jyu.fi

(H.T.) University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyvaskyla, Finland
E-mail address: heli.m.tuominen@jyu.fi

(A.V.V.) University of Jyvaskyla, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyvaskyla, Finland
E-mail address: antti.vahakangas@iki.fi