Assouad dimensions: characterizations and applications

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This talk is based on parts of the following works (in chronological order):

[LT] J. Lehrbäck and H. Tuominen. A note on the dimensions of Assouad and Aikawa, *J. Japan Math. Soc.* 65(2): 343–356, 2013.

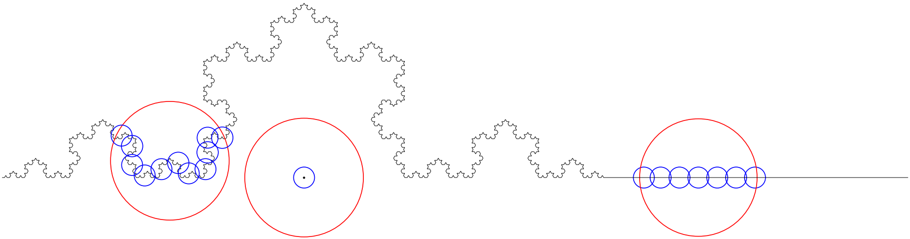
[LS] J. Lehrbäck and N. Shanmugalingam. Quasiadditivity of variational capacity, *Potential Anal.* (to appear) arXiv:1211.6933

[KLV] A. Käenmäki, J. Lehrbäck and M. Vuorinen. Dimensions, Whitney covers, and tubular neighborhoods. *Indiana Univ. Math. J.* (to appear) arXiv:1209.0629

[L] J. Lehrbäck. Hardy inequalities and Assouad dimensions, in preparation.

1. Metric spaces

Dimensions via local covers



Let X be a metric space.

How to describe the (local) dimension of a set $E \subset X$?

Take a piece of a the set, i.e. $E \cap B(w, R)$, where $w \in E$,
cover this with balls of radius $0 < r < R$,
and count how many balls are needed.

(upper) Assouad dimension

Let $E \subset X$. We consider all exponents $\lambda \geq 0$ for which there is $C = C(E, \lambda) \geq 1$ s.t. $E \cap B(w, R)$ can be covered by *at most* $C(r/R)^{-\lambda}$ balls of radius r for all $0 < r < R < \operatorname{diam}(E)$ and $w \in E$.

The infimum of such exponents λ is the (upper) Assouad dimension $\overline{\dim}_{\mathrm{A}}(E)$.

Recall that a metric space (X, d) is *doubling* if there is $N = N(X) \in \mathbb{N}$ so that any closed ball $B(x, r) \subset X$ can be covered by at most N balls of radius $r/2$. Iteration of this doubling condition shows that then $\overline{\dim}_{\mathrm{A}}(E) \leq \overline{\dim}_{\mathrm{A}}(X) \leq \log_2 N$ for all $E \subset X$. In particular:

Lemma

A metric space X is doubling if and only if $\overline{\dim}_{\mathrm{A}}(X) < \infty$.

lower Assouad dimension

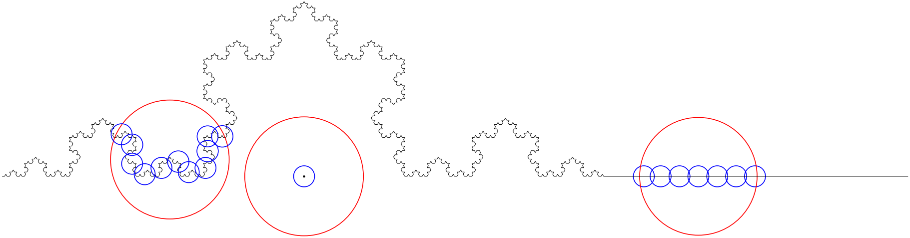
Conversely: consider all $\lambda \geq 0$ for which there is $c > 0$ s.t. if $0 < r < R < \operatorname{diam}(E)$, then for every $w \in E$ *at least* $c(r/R)^{-\lambda}$ balls of radius r are needed to cover $E \cap B(w, R)$. The supremum of all such λ is the *lower Assouad dimension* of E.

Recall that a set $E \subset X$ is *uniformly perfect* if $\#E \geq 2$ and there is $C \geq 1$ s.t. for every $w \in E$ and $r > 0$ we have $\big(B(w,r) \cap E\big) \setminus B(w, r/C) \neq \emptyset$ whenever $E \setminus B(w,r) \neq \emptyset$.

Lemma

A set E is uniformly perfect if and only if $\underline{\dim}_{\mathrm{A}}(E) > 0$.

the example



In our example

- $\overline{\dim}_{\mathrm{A}}(E) = \log 4/\log 3$ because of the snowflake part
- $\underline{\dim}_{\mathrm{A}}(E) = 0$ because of the isolated point
- (without the isolated point would have $\underline{\dim}_{\mathrm{A}}(E) = 1$)

some comments on Assouad dimensions

(Upper) Assouad dimension was introduced by P. Assouad around 1980 in connection to bi-Lipschitz embedding problem between metric and Euclidean spaces. However, equivalent (or closely related) concepts have appeared under different names, e.g. *(uniform) metric dimension*, some dating back (at least) to [Bouligand 1928]. See [Luukkainen 1998] for a nice account on the basic properties of (upper) Assouad dimension as well as some historical comments.

Lower Assouad dimension has (essentially) appeared under names *lower dimension*, *minimal dimensional number*, and *uniformity dimension*. Some basic properties of this are recently established in [Fraser 2013].

Other concepts of dimension: Minkowski

So once again:
$\overline{\dim}_{\mathrm{A}}(E)$ is the infimum of $\lambda \geq 0$ s.t. $E \cap B(w,R)$ can (always) be covered by at most $C(r/R)^{-\lambda}$ balls of radius $0 < r < R < \operatorname{diam}(E)$

$\underline{\dim}_{\mathrm{A}}(E)$ is the supremum of $\lambda \geq 0$ s.t. (always) at least $C(r/R)^{-\lambda}$ balls of radius $0 < r < R < \operatorname{diam}(E)$ are needed to cover $E \cap B(w,R)$

For comparison, recall the *upper and lower Minkowski dimensions* of a compact $E \subset X$:

$\overline{\dim}_{\mathrm{M}}(E)$ is the infimum of $\lambda \geq 0$ s.t. E can be covered by at most $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$

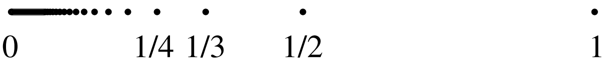
$\underline{\dim}_{\mathrm{M}}(E)$ is the supremum of $\lambda \geq 0$ s.t. at least $Cr^{-\lambda}$ balls of radius $0 < r < \operatorname{diam}(E)$ are needed to cover E.

Thus $\quad \underline{\dim}_{\mathrm{A}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E).$

More examples (1)

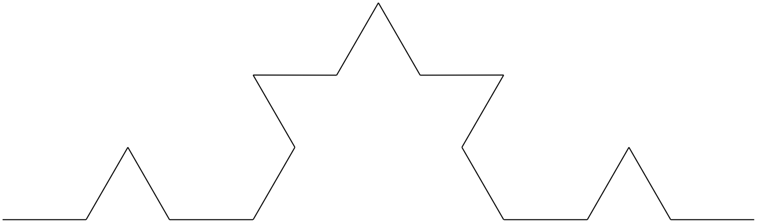
General idea: Assouad dimensions reflect the 'extreme' behavior of sets and take into account all scales $0 < r < d(E)$.

- If $E = \{0\} \cup [1,2] \subset \mathbb{R}$, then $\underline{\dim}_A(E) = 0$ and $\overline{\dim}_A(E) = 1$ $(\underline{\dim}_M(E) = \overline{\dim}_M(E) = 1)$.
- $\underline{\dim}_A(\mathbb{Z}) = 0$ and $\overline{\dim}_A(\mathbb{Z}) = 1$.
- If $E = \{(1/j, 0, \ldots, 0) : j \in \mathbb{N}\} \cup \{(0, 0, \ldots, 0)\} \subset \mathbb{R}^n$, then then $\underline{\dim}_A(E) = 0$ and $\overline{\dim}_A(E) = 1$ $(\underline{\dim}_M(E) = \overline{\dim}_M(E) = 1/2)$.

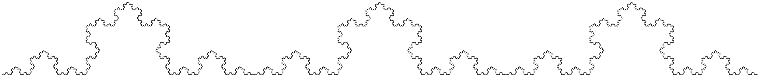


More examples (2)

- If $S \subset \mathbb{R}^2$ is an unbounded, locally rectifiable von Koch snowflake -type curve consisting of unit intervals, then $\underline{\dim}_A(S) = 1$ and $\overline{\dim}_A(E) = \log 4/\log 3$ (flat on small scales, fractal on large scales)



- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\underline{\dim}_A(S) = 1$ and $\overline{\dim}_A(E) = \log 4/\log 3$ (fractal on small scales, flat on large scales).



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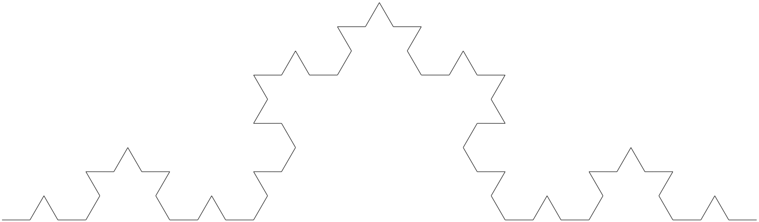

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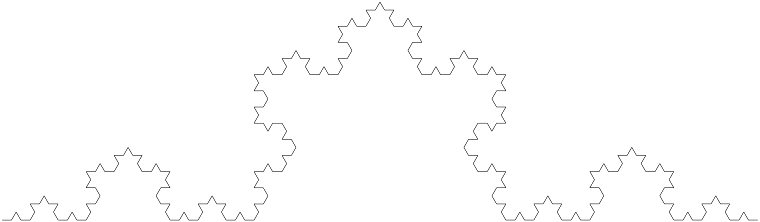

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Other concepts of dimension: Hausdorff

Recall that the *Hausdorff (r-)content* of dimension λ, for $E \subset X$, is

$$\mathcal{H}^\lambda_r(E) = \inf\left\{ \sum_k r_k^\lambda : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \leq r \right\}.$$

The λ-*Hausdorff measure* of E is $\mathcal{H}^\lambda(E) = \lim_{r\to 0} \mathcal{H}^\lambda_r(E)$ and the *Hausdorff dimension* of E is

$$\dim_{\mathrm{H}}(A) = \inf\{\lambda \geq 0 : \mathcal{H}^\lambda(A) = 0\}.$$

Lower Assouad and Hausdorff

Lemma

If $E \subset X$ is closed, then $\underline{\dim}_{\mathrm{A}}(E) \le \dim_{\mathrm{H}}(E \cap B)$ for all balls B centered at E.

The proof is based on the fact (obtained by iteration), that for each $0 < t < \underline{\dim}_{\mathrm{A}}(E)$ it holds that

$$\mathcal{H}^t_R\big(E \cap B(w,R)\big) \ge cR^t \text{ for all } w \in E,\ 0 < r < R < \operatorname{diam}(E) \qquad (1)$$

(see e.g. [L. 2009] for details). Therefore in particular $\dim_{\mathrm{H}}(E \cap B(w,R)) \ge t$ and the claim follows.

In fact, for closed $E \subset X$ we have $\underline{\dim}_A(E) = \sup\{t \ge 0 : (1) \text{ holds}\}$.

(Note however that e.g. $\underline{\dim}_{\mathrm{A}}(\mathbb{Q}) = 1$ but $\dim_{\mathrm{H}}(\mathbb{Q}) = 0$)

Whitney covers

An open set $\Omega \subsetneq X$ can be covered with a countable collection $\mathcal{W}(\Omega)$ of closed balls $B_i = B\big(x_i, \frac{1}{8}\operatorname{dist}(x_i, X \setminus \Omega)\big)$, $x_i \in \Omega$, such that the overlap of these balls is uniformly bounded (the factor $\frac{1}{8}$ is not special).

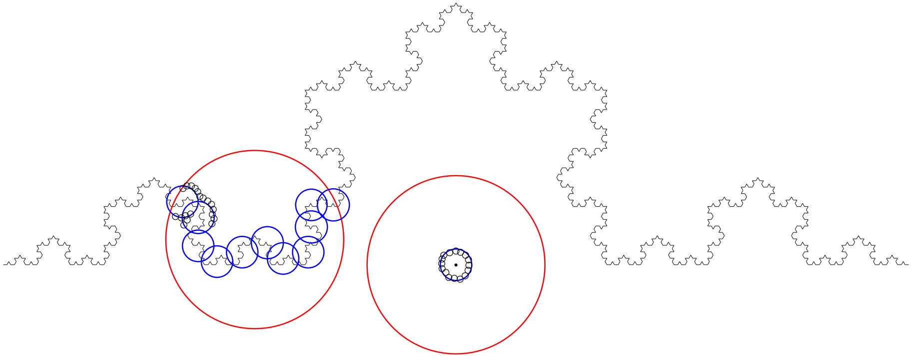
For $k \in \mathbb{Z}$ and $A \subset X$ we set
$\mathcal{W}_k(\Omega; A) = \{B(x_i, r_i) \in \mathcal{W}(\Omega) : 2^{-k-1} < r_i \leq 2^{-k} \text{ and } A \cap B(x_i, r_i) \neq \emptyset\}$
and $\mathcal{W}_k(\Omega) = \mathcal{W}_k(\Omega; X)$.

In [Martio–Vuorinen 1987], the relation between upper Minkowski dimension and upper bounds for Whitney *cube* count was considered for compact $E \subset \mathbb{R}^n$. In particular, if $\mathcal{H}^n(E) = 0$, then

$$\overline{\dim}_{\mathrm{M}}(E) = \inf\{\lambda \geq 0 : \#\mathcal{W}_k(\mathbb{R}^n \setminus E) \leq C2^{\lambda k} \text{ for all } k \geq k_0\}.$$

In [KLV] we established similar results for Assouad (and Minkowski) dimensions in metric spaces.

Assouad dimensions and Whitney ball count



A (blue) ball $B(x,r)$, $x \in E$, in the cover of E intersects *always* at most a fixed number of Whitney balls of $\Omega = X \setminus E$ with a comparable radius.

Conversely, each $B(x,r)$, contains *usually* at least one Whitney ball of a comparable radius. (The latter is not true in general but under some geometric assumptions:)

Geometric conditions

A metric space X is *q-quasiconvex* if there is $q \geq 1$ such that for every $x, y \in X$ there is a curve $\gamma\colon [0,1] \to X$ so that $x = \gamma(0)$, $y = \gamma(1)$, and $\text{length}(\gamma) \leq q d(x,y)$.

We say that a set $E \subset X$ is *(uniformly) ρ-porous* (for $0 \leq \rho \leq 1$), if for every $x \in E$ and all $0 < r < d(E)$ there is $y \in X$ such that $B(y, \rho r) \subset B(x, r) \setminus E$.

Under these conditions balls covering E always contain Whitney balls of comparable radius.

The porosity assumption is more or less crucial in this context, but quasiconvexity (as such) is not that essential; in particular, the existence of rectifiable curves is not necessary. However, without any local connectivity assumptions some generations $\mathcal{W}_k$ of Whitney balls might be empty.

The relation between Assouad dimensions and Whitney covers (from [KLV]) can be summarized as follows; here $E \subset X$ is closed and $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$:

If $\overline{\dim}_{\mathsf{A}}(E) < \lambda$, then $\quad \#\mathcal{W}_k(X \setminus E; B_0) \le C2^{\lambda k} R^{\lambda}$
for all B_0 and all $k > -\log_2 R$.

If we have for all B_0 and all $k \ge -\log_2 R + \ell$
$\#\mathcal{W}_k(X \setminus E; B_0) \ge c2^{\lambda k} R^{\lambda}$, then $\underline{\dim}_{\mathsf{A}}(E) \ge \lambda$.

If X is q-convex and $E \subset X$ is porous, and $\underline{\dim}_{\mathsf{A}}(E) > \lambda$, then
$\#\mathcal{W}_k(X \setminus E; B_0) \ge c2^{\lambda k} R^{\lambda}$ for all B_0 and all $k > -\log_2 R + \ell$.

If X is q-convex and $E \subset X$ is porous, and for all B_0 and all $k \ge -\log_2 R$
$\#\mathcal{W}_k(X \setminus E; B_0) \le C2^{\lambda k} R^{\lambda}$, then $\overline{\dim}_{\mathsf{A}}(E) \le \lambda$.

Thus Assouad dimensions (of porous sets $E \subset X$) can be characterized in terms of $\#\mathcal{W}_k(X \setminus E)$.

Assouad dimensions and r-boundaries in $\mathbb{R}^n$

Let us also mention the following Euclidean results from [KLV]; here $E \subset \mathbb{R}^n$ is closed, $E_r = \{x \in \mathbb{R}^n : d(x, E) < r\}$, and $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$.

$\overline{\dim}_A(E) < \lambda$
$$\implies \quad \mathcal{H}^{n-1}(\partial E_r \cap B_0) \le C r^{n-1}(r/R)^{-\lambda} \quad \text{for all } B_0 \text{ and } 0 < r < R.$$

$\mathcal{H}^{n-1}(\partial E_r \cap B_0) \ge c r^{n-1}(r/R)^{-\lambda}$ for all B_0 and $0 < r < \delta R$
$$\implies \underline{\dim}_A(E) \ge \lambda.$$

If E is porous, then $\underline{\dim}_A(E) > \lambda$
$$\implies \quad \mathcal{H}^{n-1}(\partial E_r \cap B_0) \ge c r^{n-1}(r/R)^{-\lambda} \quad \text{for all } B_0 \text{ and } 0 < r < \delta R.$$

If $\mathcal{H}^n(E) = 0$ (weaker than porosity), then
$\mathcal{H}^{n-1}(\partial E_r \cap B_0) \le C r^{n-1}(r/R)^{-\lambda}$ for all B_0 and $0 < r < R$
$$\implies \quad \overline{\dim}_A(E) \le \lambda.$$

Thus Assouad dimensions (of porous sets $E \subset \mathbb{R}^n$) can be characterized in terms of $\mathcal{H}^{n-1}(\partial E_r)$.

2. Metric measure spaces

doubling measures I

A measure μ on X is *doubling* if there is $C \geq 1$ so that $0 < \mu(2B) \leq C\mu(B)$ for all closed balls $B \subset X$.

Iterating, we find $C > 0$ and $s > 0$ s.t.

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\left(\frac{r}{R}\right)^s \tag{2}$$

for all $y \in B(x,R)$ and $0 < r < R < d(X)$. The infimum of s satisfying (2) is called the *upper regularity dimension* of μ, $\overline{\dim}_{\mathrm{reg}}(\mu)$.

It is easy to see that $\overline{\dim}_{\mathrm{A}}(X) \leq \overline{\dim}_{\mathrm{reg}}(\mu)$ whenever μ is doubling on X. In particular, if X has a doubling measure, then X is doubling.

On the other hand, if X is doubling and complete, then there is a doubling measure μ on X [Luukkainen–Saksman 1998; Vol'berg–Konyagin 1987 (for compact sets)].

doubling measures II

Conversely, if X is uniformly perfect and μ is doubling then there are $t > 0$ and $C \geq 1$ s.t.

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq C\Big(\frac{r}{R}\Big)^t \tag{3}$$

whenever $0 < r < R < d(X)$ and $y \in B(x,R)$. The supremum of t satisfying (3) is called the *lower regularity dimension* of μ, $\underline{\dim}_{\mathrm{reg}}(\mu)$.

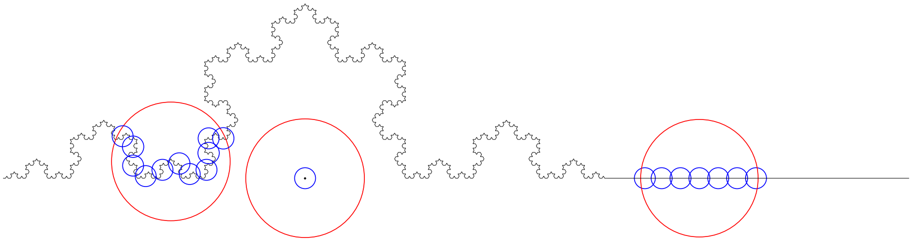
Thus $\underline{\dim}_{\mathrm{reg}}(\mu) > 0$ if μ is doubling and X is uniformly perfect, and in fact $\underline{\dim}_{\mathrm{reg}}(\mu) \leq \underline{\dim}_{\mathrm{A}}(X)$.

Measure μ (and the space X) is called *(Ahlfors) s-regular*, if there is $C > 0$ so that

$$\tfrac{1}{C} r^s \leq \mu(B(x,r)) \leq Cr^s$$

for every $x \in X$ and all $0 < r < d(X)$. Then $\underline{\dim}_{\mathrm{reg}}(\mu) = \overline{\dim}_{\mathrm{reg}}(\mu) = s$.

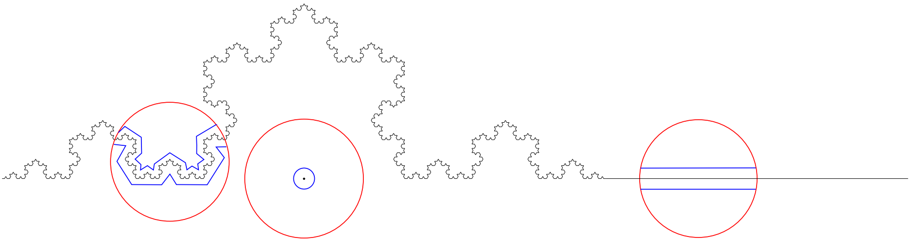
Dimensions via measures of neighborhoods



Let $X = (X, \mu, d)$ be a metric measure space and let $E \subset X$.

Instead covering $E \cap B(w, R)$ with balls of radius $0 < r < R$,

Dimensions via measures of neighborhoods



Let $X = (X, \mu, d)$ be a metric measure space and let $E \subset X$.

Instead covering $E \cap B(w, R)$ with balls of radius $0 < r < R$,

we may consider the measure $\mu(E_r \cap B(x, R))$, where
$E_r = \{x \in X : d(x, E) < r\}$ is the r-neighborhood of E.

This leads to the concepts of *Assouad codimension*.

Assouad revisited

Let μ is doubling and $E \subset X$. In [KLV] we introduce the following concepts:

The *lower Assouad codimension* $\underline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(E)$ is the supremum of $t \geq 0$ for which there is $C > 0$ s.t.

$$\frac{\mu(E_r \cap B(x,R))}{\mu(B(x,R))} \leq C\Big(\frac{r}{R}\Big)^t$$

for every $x \in E$ and all $0 < r < R < \operatorname{diam}(E)$.

The *upper Assouad codimension* $\overline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(E)$ is the infimum of $s \geq 0$ for which there is $C > 0$ s.t.

$$\frac{\mu(E_r \cap B(x,R))}{\mu(B(x,R))} \geq C\Big(\frac{r}{R}\Big)^s$$

for every $x \in E$ and all $0 < r < R < \operatorname{diam}(E)$.

Assoud vs. co-Assouad

Lemma (KLV)

If μ is a doubling measure on X and $E \subset X$, then

$$\begin{aligned} \underline{\dim}_{\mathrm{reg}}(\mu) \le \underline{\operatorname{co}\dim}_{\mathrm{A}}^{\mu}(E) + \overline{\dim}_{\mathrm{A}}(E) \le \overline{\dim}_{\mathrm{reg}}(\mu), \\ \underline{\dim}_{\mathrm{reg}}(\mu) \le \overline{\operatorname{co}\dim}_{\mathrm{A}}^{\mu}(E) + \underline{\dim}_{\mathrm{A}}(E) \le \overline{\dim}_{\mathrm{reg}}(\mu). \end{aligned} \tag{4}$$

In particular, if μ is s-regular, then the above lemma implies

$$\begin{aligned} \overline{\dim}_{\mathrm{A}}(E) = s - \underline{\operatorname{co}\dim}_{\mathrm{A}}^{\mu}(E), \\ \underline{\dim}_{\mathrm{A}}(E) = s - \overline{\operatorname{co}\dim}_{\mathrm{A}}^{\mu}(E) \end{aligned}$$

for all $E \subset X$. The first equation was also proven in [LT]. On the other hand, it is not hard to give examples where μ is doubling and any given inequality in (4) is strict for a set $E \subset X$.

Porosity and Assouad dimensions

Porous sets have upper bounds for their (upper) Assouad dimension in regular spaces:

Proposition (KLV, strongly based on [JJKRRS 2010])

If X is s-regular, then there is a constant $c > 0$ such that $\overline{\dim}_{\mathrm{A}}(E) \leq s - c\rho^s$ for all ρ-porous sets $E \subset X$.

If μ is (only) doubling, then it is still true that each ϱ-porous set $E \subset X$ satisfies $\underline{\operatorname{co}\dim}_{\mathrm{A}}^{\mu}(E) \geq t$, where $t > 0$ only depends on ϱ and the doubling constant of μ (again observed in [KLV] but based on [JJKRRS 2010]).

Aikawa

In [LT] it was shown that the lower Assouad codimension $\underline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(E)$ (and thus $s - \overline{\dim}_{\mathrm{A}}(E)$ in an s-regular space) can be characterized as the supremum of $q \geq 0$ for which there is $C \geq 1$ s.t.

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \operatorname{dist}(y,E)^{-q} \, d\mu(y) \leq C r^{-q} \tag{5}$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(E)$. (We interpret the integral to be $+\infty$ if $q > 0$ and E has positive measure.)

A concept of dimension defined via integrals as in (5) was first used in [Aikawa 1991] for subsets of $\mathbb{R}^n$ in connection to the so-called quasiadditivity property of (Riesz) capacity.

(Thus in [LT] the lower Assouad codimension is actually called the *Aikawa codimension*.)

upper co-Assouad and co-Hausdorff

The *Hausdorff content of codimension q* for $E \subset X$ can be defined as

$$\mathcal{H}^{\mu,q}_R(E) = \inf\left\{ \sum_k \operatorname{rad}(B_k)^{-q}\mu(B_k) : E \subset \bigcup_k B_k,\ \operatorname{rad}(B_k) \le R \right\}.$$

The *Hausdorff codimension* is $\operatorname{co\,dim}_{\mathrm{H}}(E) = \sup\left\{q \ge 0 : \mathcal{H}^{\mu,q}_R(E) = 0\right\}$.

It was recently established in [L] that if $q > \overline{\operatorname{co\,dim}}^{\mu}_{\mathrm{A}}(E)$, then there is $C > 0$ s.t.

$$\mathcal{H}^{\mu,q}_R\big(E \cap B(w,R)\big) \ge CR^{-q}\mu(B(w,R)) \tag{6}$$

for every $w \in E$ and all $0 < R < \operatorname{diam}(E)$. (Recall that we had a similar condition for $0 < t < \underline{\dim}_{\mathrm{A}}(E)$ and $\mathcal{H}^t_R(E)$.)

In fact, we have that $\overline{\operatorname{co\,dim}}^{\mu}_{\mathrm{A}}(E) = \inf\{q \ge 0 : \text{(6) holds}\}$.

Let us remark here that the uniform estimate (6) for an exponent $1 < q < p$ (and for all $0 < R < \infty$) is equivalent to the set E being *uniformly p-fat* (a capacity condition).

3. Applications: Hardy inequalities

Hardy inequalities

In an open set $\Omega \subset \mathbb{R}^n$ the (p, β)-Hardy inequality, for $1 < p < \infty$ and $\beta \in \mathbb{R}$, reads as

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta - p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx,$$

where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

If there exists a constant $C > 0$ such that this holds for all $u \in C_0^\infty(\Omega)$, we say that Ω admits a (p, β)-Hardy inequality.

In a metric space X, with a doubling measure μ, smooth functions are replaced with Lipschitz functions with compact support in Ω, and $|\nabla u(x)|$ is replaced with *an upper gradient* g of u:

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta - p} \, d\mu \leq C \int_\Omega g(x)^p d_\Omega(x)^\beta \, d\mu.$$

Sufficient conditions I

We have the following recent result from [L]:

Theorem

Let $1 \leq p < \infty$, $\beta < p - 1$, *and assume that* X *is an unbounded doubling metric space. If* $\beta \leq 0$, *we further assume that* X *supports a p-Poincaré inequality, and if* $\beta > 0$ *we assume that* X *supports a* $(p - \beta)$*-Poincaré inequality. If* $\Omega \subset X$ *is an open set satisfying*

$$\underline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(X \setminus \Omega) > p - \beta,$$

then Ω *admits a* (p, β)*-Hardy inequality.*

This has been previously known in $\mathbb{R}^n$ (with different terminology) in the case $\beta = 0$ by [Aikawa 1991] and [Koskela–Zhong 2003], and for general β under some additional geometric assumptions [L. 2008].

Sufficient conditions II

Conversely, a combination of some previously known results (e.g. [L. PAMS (to apper)]) based on Hausdorff content density / uniform fatness and the link between these and the upper Assouad codimension gives the following formulation:

Theorem

Let $1 \leq p < \infty$, $\beta < p - 1$, and assume that X is a doubling metric space which supports a p-Poincaré inequality if $\beta \leq 0$, and a $(p - \beta)$-Poincaré inequality if $\beta > 0$. Let $\Omega \subset X$ be an open set satisfying

$$\overline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(X \setminus \Omega) < p - \beta\,;$$

in case Ω is unbounded, we require in addition that $X \setminus \Omega$ is unbounded as well. Then Ω admits a (p, β)-Hardy inequality.

Sufficient conditions in $\mathbb{R}^n$

In the Euclidean case, we can reformulate the previous results as follows:

Corollary

Let $1 \leq p < \infty$ and $\beta < p - 1$, and let $\Omega \subset \mathbb{R}^n$ be an open set. If

$$\overline{\dim}_A(\Omega^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_A(\Omega^c) > n - p + \beta,$$

then Ω admits a (p, β)-Hardy inequality; in the latter case, if Ω is unbounded, then we require that also Ω^c is unbounded.

In [LS] we established an equivalence between p-Hardy inequalities ($\beta = 0$) and the quasiadditivity of the variational p-capacity (in metric spaces). This provides a link between the work of Aikawa (where essentially the condition $\overline{\dim}(\Omega^c) < n - p$ was used) and our recent considerations.

Necessary conditions

The above sufficient conditions (i.e. $\underline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(\Omega^c) > p - \beta$ or $\overline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(\Omega^c) < p - \beta$) are rather natural for (p, β)-Hardy inequalities. In fact, the following necessary conditions hold as well:

Theorem (LT ($\beta = 0$), L)

Let $1 < p < \infty$ and $\beta \neq p$, and suppose that a domain $\Omega \subset X$ admits a (p, β)-Hardy inequality. Then

$$\operatorname{co\,dim}_{\mathrm{H}}(\Omega^c) < p - \beta \quad \text{or} \quad \underline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(\Omega^c) > p - \beta.$$

Moreover, such a dichotomy also holds locally, i.e. for each ball $B_0 \subset X$

$$\operatorname{co\,dim}_{\mathrm{H}}(4B_0 \cap \Omega^c) < p - \beta \quad \text{or} \quad \underline{\operatorname{co\,dim}}_{\mathrm{A}}^{\mu}(B_0 \cap \Omega^c) > p - \beta.$$

A blast from the past

In my talk in the Finnish Mathematical Days 2010 I asked:
... samaa ideaa käyttäen saadaan $\mathbb{R}^n$:ssä esimerkkejä, joissa [reunan osan dimensio] $\mu \geq n-1$. Tällöin paksu osa reunasta saadaan 'piiloon' pienen osan taakse, eikä (p,β)-Hardy päde millekään $\beta \geq p-n+\mu$ [vaikka siis olisi $\overline{\dim}_A(\Omega^c) = \mu < n-p+\beta$].

Toisaalta, jos pieni osa reunaa on μ-ulotteinen ($0 \leq \mu < n$) ja tämän osan läheltä päästään λ-paksun reunan osan lähelle ($\mu < \lambda$), pätee (p,β)-Hardy, kun $\quad p-n+\mu < \beta < p-n+\lambda$.

Kysymys: Päteekö edellä (p,β)-Hardy kaikille

$$p-n+\mu < \beta < p-n+\lambda \text{ ilman lisäehtoa, jos } \mu < n-1\,?$$

Edellisten tulosten perusteella osaan nyt vastata:
KYLLÄ, kunhan $\beta < p-1$
(ja jos $\lambda = \underline{\dim}_A(\Omega^c) > n-1$ niin ei välttämättä kun $\beta \geq p-1$.)

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