Some developments on Hardy inequalities in $\mathbb{R}^n$

Juha Lehrbäck

University of Jyväskylä

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Original inequalities

Hardy 1925:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p\, dx,$$

when $1 < p < \infty$ and $f \geq 0$ is measurable.

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when $1 < p < \infty$ and $f \ge 0$ is measurable.
Another form:

$$\int_0^\infty |u(x)|^p x^{-p}\,dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |u'(x)|^p\,dx,$$

where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.

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$$\int_\Omega |u(x)|^p \, d_\Omega(x) \;\; ^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \qquad dx \tag{2}$$

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$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta - p} \, dx \leq C \int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta \, dx \qquad (2)$$

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This is the (p,β)-*Hardy inequality* for $u \in C_0^\infty(\Omega)$.
If (2) (resp. (1)) holds for all $u \in C_0^\infty(\Omega)$ with the same constant $C = C(\Omega, p, \beta) > 0$, we say that the domain $\Omega \subset \mathbb{R}^n$ *admits* the (p,β)-Hardy inequality (resp. p-Hardy inequality).

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- . . . mathematical curiosity . . .

Sufficient conditions

Theorem (Nečas 1962)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$.

In particular, a Lipschitz domain admits the p-Hardy inequality for all $1 < p < \infty$. Also, the bound $\beta < p - 1$ is sharp.

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Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

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A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, if it satisfies a uniform capacity density condition, which is equivalent to the following uniform Hausdorff content density condition:

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There exists some $\lambda > n - p$ and a constant $C > 0$ such that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap B(w,r)\big) \geq Cr^{\lambda} \quad \text{for all } w \in E \text{ and all } r > 0. \tag{3}$$

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Recall that the λ-Hausdorff content of $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^{\lambda}_{\infty}(A) = \inf\left\{ \sum_{i=1}^{\infty} r_i^{\lambda} : A \subset \bigcup_{i=1}^{\infty} B(z_i, r_i) \right\}.$$

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For instance, every non-empty $E \subset \mathbb{R}^n$ is unif. p-fat for all $p > n$, and the complement of a Lipschitz domain $\Omega \subset \mathbb{R}^n$ is unif. p-fat for all $p > 1$.

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Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exist some exponent $1 < q < p$ and a constant $C > 0$ such that the inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2 d_\Omega(x)} \big(|\nabla u|^q \big)(x) \big)^{1/q} \tag{4}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

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Here $M_R f$ is the usual restricted Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, defined by
$M_R f(x) = \sup_{r \leq R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$

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Idea of $\Rightarrow$: Let $B(x, 2d_\Omega(x)) \cap \partial\Omega \subset \bigcup_{i=1}^{N} B(z_i, r_i)$ and use the pointwise p-Hardy for test function

$$\varphi(y) = \min_{1 \leq i \leq N} \big\{1,\ r_i^{-1} d(y, B(z_i, 2r_i))\big\} \cdot (\text{cut-off})$$

Inner boundary density and uniform fatness

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Q: Could condition 4. be used in the context of variable exponent spaces to give a characterization of "$p(x)$"-fatness of Ω^c?

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Q: Could condition 4. be used in the context of variable exponent spaces to give a characterization of "$p(x)$"-fatness of Ω^c?

A weighted pointwise inequality

There is also a pointwise version of the weighted (p, β)-Hardy inequality:

$$|u(x)| \leq C d_\Omega(x)^{1} \quad \Big(M_{2 d_\Omega(x)} \big(|\nabla u|^{q} \qquad \big)(x) \Big)^{1/q}, \qquad (5)$$

where again $1 < q < p$.

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$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{2d_\Omega(x)}\big(|\nabla u|^q d_\Omega{}^{\frac{\beta}{p}q}\big)(x)\Big)^{1/q}, \qquad (5)$$

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where again $1 < q < p$.
A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality if there exist some $1 < q < p$ and a constant $C > 0$ so that (5) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with these q and C.

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As in the unweighted case, the pointwise (p, β)-Hardy inequality implies the usual weighted (p, β)-Hardy inequality.

summary and questions

Recall some previously known results for weighted Hardy inequalities:

- Ω^c unif. p-fat $\implies$ Ω admits (p, β)-Hardy for all $\beta < \beta_0$, $\beta_0 > 0$ small (Wannebo)

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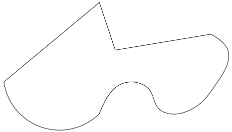

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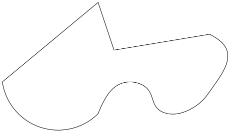

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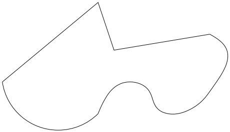

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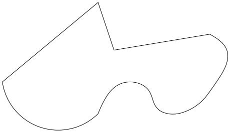

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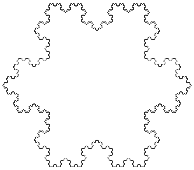

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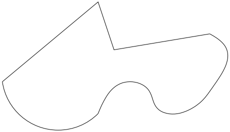

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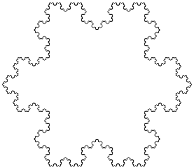

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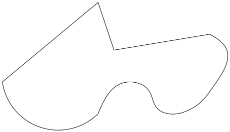

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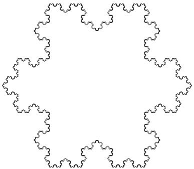

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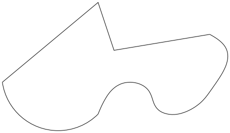

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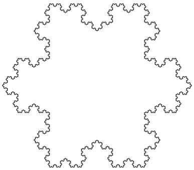

Notice that the boundaries of both Lipschitz and snowflake domains are well accessible from the points inside the domain

A counterexample

Let $\Omega \subset \mathbb{R}^2$ be as below, with $\lambda = \dim(\partial\Omega) > 1$. Then Ω satisfies

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega,$$

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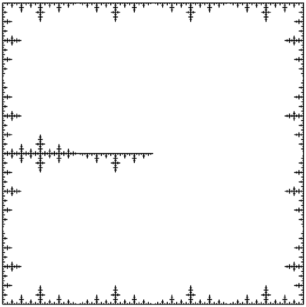

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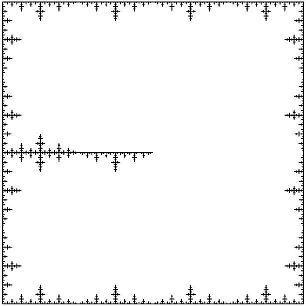

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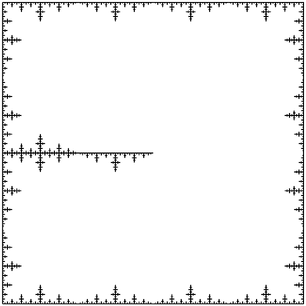

Main Thorem

Theorem (Koskela-L. 2007(preprint))

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that there exist $0 \leq \lambda \leq n$, $c \geq 1$, and $C_0 > 0$ so that

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A point $w \in \partial\Omega$ is in the set $v_x(c)\text{–}\partial\Omega$, if w is *accessible* from x by a *c-John curve*, that is, there exists a curve $\gamma = \gamma_{w,x} \colon [0, l] \to \Omega$, parametrized by arc length, with $\gamma(0) = w$, $\gamma(l) = x$, and satisfying $d(\gamma(t), \partial\Omega) \geq t/c$ for every $t \in [0, l]$.

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- Intergate (7) w.r.t. a Frostman measure μ on $v_x(c)$–$\partial\Omega$, use Fubini to change the order of integration and summation

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- Finally, $|u(x)| \leq |u(x) - u_{Q_0}| + |u_{Q_0}|$, use standard $\int M(|\nabla u|)$ -estimate for the first term to obtain the pointwise (p, β)-Hardy

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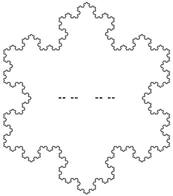

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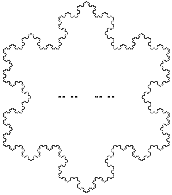

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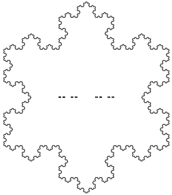

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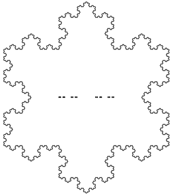

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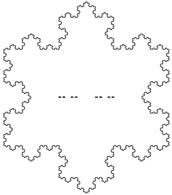

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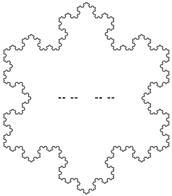

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However, (8) alone is not sufficient for the pointwise (p, β)-Hardy inequality (need also the "John-accessibility")

A global necessary condition for (p, β)-Hardy

Theorem (L. 2008 (manuscripta math))

If a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality, and $\beta \neq p$, then there exists $\delta > 0$ such that either

$$\dim \quad (\Omega^c) > n - p + \beta + \delta$$

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The case $\beta = 0$ is essentially due to Koskela-Zhong (2003).

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Here $\dim_{\mathcal{H}}$ is the Hausdorff dimension and $\dim_{\mathcal{A}}$ is a concept of dimension, introduced by Aikawa, such that $\dim_{\mathcal{A}}(E) \geq \dim_{\mathcal{H}}(E)$ for all closed $E \subset \mathbb{R}^n$.

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If a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality, and $\beta \neq p$, then there exists $\delta > 0$ such that for each ball $B \subset \mathbb{R}^n$ either

$$\dim_{\mathcal{H}}(4B \cap \Omega^c) > n - p + \beta + \delta$$

or

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The case $\beta = 0$ is again due to Koskela-Zhong (2003).

An example

Let $E = \{(j^{-1}, 0, \dots, 0) : j \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}^n$, and take $\Omega = \mathbb{R}^n \setminus E$. Then $\dim_{\mathcal{H}}(E) = 0$ and $\dim_{\mathcal{A}}(E) = 1$ (and the Minkowski dimension is $\dim_{\mathcal{M}}(E) = 1/2$).

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By the necessary conditions, the domain $\Omega = \mathbb{R}^n \setminus E$ can not admit the (p, β)-Hardy inequality if

$$0 = \dim_{\mathcal{H}}(E) \leq n - p + \beta \leq \dim_{\mathcal{A}}(E) = 1,$$

i.e. if $p - n \leq \beta \leq p - n + 1$.

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On the other hand, every proper subdomain of $\mathbb{R}^n$ admits the (p, β)-Hardy if $\beta < p - n$, and one can also show (using the sufficient conditions for general domains) that Ω admits the (p, β)-Hardy if $\beta > p - n + 1$.

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Thus we conclude that Ω admits the (p, β)-Hardy inequality *if and only if* $\beta < p - n$ or $\beta > p - n + 1$.

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The End

Thank you for your attention