IN BETWEEN THE INEQUALITIES OF SOBOLEV AND HARDY

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Abstract. We establish both sufficient and necessary conditions for the validity of the so-called Hardy–Sobolev inequalities on open sets of the Euclidean space. These inequalities form a natural interpolating scale between the (weighted) Sobolev inequalities and the (weighted) Hardy inequalities. The Assouad dimension of the complement of the open set turns out to play an important role in both sufficient and necessary conditions.

1. Introduction

The *Sobolev inequality* is a fundamental tool in all analysis related to partial differential equations and variational problems, see e.g. [7, 29]. When $G \subset \mathbb{R}^n$ is an open set and $1 \le p < n$, this inequality states that

$$\left(\int_G |f|^{np/(n-p)}\,dx\right)^{(n-p)/np} \le C\left(\int_G |\nabla f|^p\,dx\right)^{1/p} \qquad \text{for all } f \in C_0^\infty(G)\,, \tag{1}$$

where the constant $C > 0$ depends only on n and p. If G is bounded (or of finite measure) and $1 \le q \le np/(n-p) := p^*$, a simple use of Hölder's inequality yields a corresponding inequality where on the left-hand side of (1) the p^*-norm is replaced by the q-norm; the constant in the inequality then depends on the measure of G as well. In particular, for $q = p$ this gives the so-called *Friedrichs' inequality*

$$\int_G |f|^p\,dx \le C\int_G |\nabla f|^p\,dx \qquad \text{for all } f \in C_0^\infty(G)\,.$$

However, if $p > 1$ and the open set G satisfies some additional properties, e.g. G is a Lipschitz domain or more generally the complement of G is uniformly p-fat, then Friedrichs' inequality can be improved into a *p-Hardy inequality*

$$\int_G |f|^p \delta_{\partial G}^{-p}\,dx \le C\int_G |\nabla f|^p\,dx \qquad \text{for all } f \in C_0^\infty(G), \tag{2}$$

where $\delta_{\partial G}(x) = \operatorname{dist}(x, \partial G)$ denotes the distance from $x \in G$ to the boundary of G; see e.g. Lewis [26] and Wannebo [35]. Unlike Friedrichs' inequality, this p-Hardy inequality can be valid even if the open set G has infinite measure. A weighted (p,β)-Hardy inequality is obtained from inequality (2) by replacing dx with $\delta_{\partial G}^{\beta}\,dx$, $\beta \in \mathbb{R}$, on both sides of (2). Such an inequality holds, for instance, in a Lipschitz domain G for $1 < p < \infty$ if (and only if) and $\beta < p - 1$, as was shown by Nečas [30]. On the other hand, if, roughly speaking, ∂G contains an isolated part of dimension $n - p + \beta$, then the (p,β)-Hardy inequality can not be valid in $G \subset \mathbb{R}^n$; we refer to [20, 23].

In this paper, we are interested in certain inequalities forming a natural interpolating scale in between the (weighted) Sobolev inequalities and the (weighted) Hardy inequalities.

2010 *Mathematics Subject Classification.* 35A23 (26D15, 46E35).

Key words and phrases. Sobolev inequality, Hardy inequality, Assouad dimension.

More precisely, we say that an open set $G \subsetneq \mathbb{R}^n$ *admits a* (q,p,β)*-Hardy–Sobolev inequality* if there is a constant $C > 0$ such that the inequality

$$\left(\int_G |f|^q \, \delta_{\partial G}^{(q/p)(n-p+\beta)-n} \, dx\right)^{1/q} \leq C\left(\int_G |\nabla f|^p \, \delta_{\partial G}^{\beta} \, dx\right)^{1/p} \tag{3}$$

holds for all $f \in C_0^\infty(G)$. Notice how the Sobolev inequality (1) is obtained as the case $q = p^* = np/(n-p)$, $\beta = 0$ in (3); and the weighted (p,β)-Hardy inequality is exactly the case $q = p$ in (3).

We begin in Section 2 by showing that if an open set $G \subset \mathbb{R}^n$ admits a (p,β)-Hardy inequality, then also the (q,p,β)-Hardy–Sobolev inequality holds for all $p \leq q \leq p^*$, see Theorem 2.1. Thus, for these q, sufficient conditions for Hardy inequalities always yield sufficient conditions for Hardy–Sobolev inequalities, see Corollary 4.1. We recall that there are in principle two separate classes of open sets in which the (p,β)-Hardy inequality can hold: either the complement $G^c = \mathbb{R}^n \setminus G$ is 'thick', as in the case of Lipschitz domains or uniformly fat complements, or then the complement is 'thin', corresponding to an upper bound on the Assouad dimension $\dim_{\mathrm{A}}(G^c)$. (We refer to Section 3 for definitions and preliminary results related to various notions of dimension and to Section 4 for a more precise formulation of this 'dichotomy' between thick and thin cases.) It turns out that actually in the thick case the sufficient conditions emerging from Hardy inequalities are rather sharp for Hardy–Sobolev inequalities as well; see the discussion after Corollary 4.1 and Theorem 4.6.

On the other hand, the $(p^*,p,0)$-Hardy–Sobolev inequality (that is, the Sobolev inequality) holds without any extra assumptions on G, and this is not the case with the p-Hardy inequality. Hence it is natural to expect that for exponents $p < q < p^*$ one could at least in some cases relax the assumptions required for Hardy inequalities and still obtain the Hardy–Sobolev inequalities. We show that this is indeed possible in the case when the complement G^c is thin. More precisely, in [22] it was shown that if $\dim_{\mathrm{A}}(G^c) < n-p+\beta$ and $\beta < p-1$, then G admits a (p,β)-Hardy inequality. Now, in Theorem 4.2 we show that for the (q,p,β)-Hardy–Sobolev inequality, with $p \leq q \leq p^*$, it is actually sufficient that $\dim_{\mathrm{A}}(G^c) < \min\bigl\{\frac{q}{p}(n-p+\beta), n-1\bigr\}$. For $q = p$, this result gives an improvement for the sufficient condition for the (p,β)-Hardy inequality as well, see Remark 4.3 for details.

The proof of Theorem 4.2 relies heavily on the work of Horiuchi [14], where the main interest was in the existence of embeddings between weighted Sobolev spaces. Our main contribution to this part is the observation that the so-called $P(s)$-property (see Definition 3.2) that Horiuchi is using as a sufficient condition can actually be characterized using the Assouad dimension; this is done in our Theorem 3.4. It is worth a mention that this adds one more item to the already long list of notions equivalent to the Assouad dimension, see, for instance [17, 25, 27], and also gives a wealth of new examples where Horiuchi's original results can be applied. In Section 5 we present Horiuchi's proof, adapted to our setting, for the sake of clarity and completeness.

It should be noted that in the above results involving a 'thin' complement the test functions do not have to vanish near ∂G, but the inequalities actually hold for all functions in $C_0^\infty(\mathbb{R}^n)$. We call such inequalities *global* Hardy–Sobolev inequalities. In Section 6 we establish necessary conditions for these global inequalities, which in particular yield, together with the sufficient condition from Section 5, the following characterization in the unweighted case $\beta = 0$.

Theorem 1.1. *Let $E \neq \emptyset$ be a closed set in $\mathbb{R}^n$ and let $1 \le p \le q < np/(n-p) < \infty$. Then there is a constant $C > 0$ such that the global $(q,p,0)$-Hardy–Sobolev inequality*

$$\left(\int_{\mathbb{R}^n} |f|^q \, \delta_E^{(q/p)(n-p)-n} \, dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{1/p}$$

holds for every $f \in C_0^\infty(\mathbb{R}^n)$ if and only if $\dim_{\mathrm{A}}(E) < \frac{q}{p}(n-p)$.

Sections 5 and 6 contain, respectively, both sufficient and necessary conditions for more general global Hardy–Sobolev inequalities as well. In the weighted case $\beta \neq 0$ these do not always give full characterizations, but they nevertheless complement each other and show that our sufficient conditions are not too far from being optimal. For instance, when $\beta < 0$, we need to require in the necessary condition of Theorem 6.2 that E is either compact (and porous) or that E satisfies an *a priori* dimensional bound. The necessity of some extra assumption is shown by an example, whose justification requires a closer look on the sufficient conditions for the Hardy–Sobolev inequality (3) in the case when the complement of the domain G is 'thick' and $\beta \le 0$. Such a result is established in Section 7.

Finally, in Section 8 we prove that an open set which admits the Hardy–Sobolev inequality (3) has to satisfy both local and global dimensional dichotomies: Either the complement has (locally) a large Hausdorff (or lower Minkowski) dimension or a small Assouad dimension. (The global result is formulated earlier in the paper in Theorem 4.6.) Contrary to the sufficient conditions, as far as we know no general necessary conditions for (q,p,β)-Hardy–Sobolev inequalities have been considered in the literature when $q > p$. For $q = p$, i.e. for (p,β)-Hardy inequalities, the corresponding dichotomy is well known, see [20, 23].

We end this introduction with a brief overview of the previously known sufficient conditions for Hardy–Sobolev inequalities. In the case when $E \subset \mathbb{R}^n$ is an m-dimensional subspace, $1 \le m \le n-1$, and $G = \mathbb{R}^n \setminus E$, it is due to Maz'ya [29, Section 2.1.6] that the global version of the (q,p,β)-Hardy–Sobolev inequality (3) holds for all functions $f \in C_0^\infty(\mathbb{R}^n)$ if $m < \frac{q}{p}(n-p+\beta)$; notice how this corresponds to the dimensional bound given above, since here $\dim_{\mathrm{A}}(E) = m$. Badiale and Tarantello [3] (essentially) rediscovered Maz'ya's result for $\beta = 0$, and applied these inequalities to study the properties of the solutions for a certain elliptic partial differential problem related to the dynamics of galaxies. See also Gazzini–Musina [12] and the references therein for other applications of Hardy–Sobolev inequalities where the distances are taken to subspaces of $\mathbb{R}^n$. For $m = 0$, i.e. $E = \{0\}$, the corresponding Hardy–Sobolev inequality is known as Caffarelli–Kohn–Nirenberg inequality, since this case first appeared in [4].

For bounded domains with Lipschitz (or Hölder) boundary, Hardy-Sobolev inequalities have been discussed in [31, Section 21]. Let us also mention that unweighted (q,p)-Hardy–Sobolev inequalities follow from the more general 'improved Hardy inequalities' of Filippas, Maz'ya and Tertikas [9], under the assumptions that $2 \le p < n$, $p < q \le p^*$, $G \subset \mathbb{R}^n$ is a bounded domain with a C^2-smooth boundary, and the distance function satisfies the condition $-\Delta \delta_{\partial G} \ge 0$ (this is the case $k = 1$ of [9, Theorem 1.1]). Moreover, the inequalities in [9] contain an additional 'Hardy' term (with the best constant) on the left-hand side of the inequality (3), so these inequalities are much stronger than than the (q,p)-Hardy–Sobolev inequality—but of course the results in [9] are subject to much stronger assumptions, as well. For irregular domains satisfying a 'plumpness' condition, Hardy–Sobolev inequalities have also been studied by Edmunds and Hurri-Syrjänen in [5].

Notation. Throughout the paper we assume that G is a non-empty open set in $\mathbb{R}^n$, $n \geq 2$, with a non-empty boundary. The open ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ is $B(x,r)$. The Euclidean distance from $x \in \mathbb{R}^n$ to a given set E in $\mathbb{R}^n$ is written as $\operatorname{dist}(x,E) = \delta_E(x)$. The diameter of E is $d(E)$. We write χ_E for the characteristic function of a set E. The boundary of E is written as ∂E, its closure is written as $\overline{E}$, and the complement of E is $E^c = \mathbb{R}^n \setminus E$. The Lebesgue n-measure of a measurable set $E \subset \mathbb{R}^n$ is $|E|$. If $0 < |E| < \infty$, the integral average of a function $f \in L^1(E)$ is $f_E = \fint_E f\,dx = |E|^{-1} \int_E f\,dx$.

All cubes we use are closed and have their sides parallel to the coordinate axes. For a $\lambda > 0$ and a cube Q in $\mathbb{R}^n$, we denote by λQ the cube with the same center as Q but with side length λ times that of Q. The letters C and c will denote positive constants whose values are not necessarily the same at each occurrence. If there exists a constant $C > 0$ such that $a \leq Cb$, we sometimes write $a \lesssim b$, and if $a \lesssim b \lesssim a$ we write $a \simeq b$ and say that a and b are comparable.

Acknowledgments. J.L. wishes to thank Petteri Harjulehto for inspiring questions and discussions related to Hardy–Sobolev inequalities. J. L. has been supported by the Academy of Finland, grant no. 252108.

2. Interpolation

We show in this section how (weighted) Hardy–Sobolev inequalities can be obtained by interpolating between (weighted) Hardy inequalities and (unweighted) Sobolev inequalities. Recall that the unweighted Sobolev inequalities are valid for all open sets.

Theorem 2.1. *Assume that $1 \leq p < n$ and $\beta \in \mathbb{R}$. If G admits a (p,p,β)-Hardy–Sobolev inequality (i.e., a (p,β)-Hardy inequality), then G admits (q,p,β)-Hardy–Sobolev inequalities for all exponents $p \leq q \leq p^* = np/(n-p)$.*

Let us first prove the following special case; all the other inequalities can then be obtained with the help of Hölder's inequality.

Lemma 2.2. *Let $1 \leq p < n$ and $\beta \in \mathbb{R}$. If G admits a (p,p,β)-Hardy–Sobolev inequality, then G admits a (p^*,p,β)-Hardy–Sobolev inequality.*

Proof. Let $f \in C_0^\infty(G)$ and write $g = |f|\delta_{\partial G}^{\beta/p}$. Then g is a Lipschitz function with a compact support in G, and the gradient of g satisfies (almost everywhere)

$$|\nabla g| \leq |\nabla f|\delta_{\partial G}^{\beta/p} + \tfrac{|\beta|}{p}|f|\delta_{\partial G}^{\beta/p-1}\,.$$

The Sobolev inequality for g (which holds by approximation), the above estimate for $|\nabla g|$, and the (p,p,β)-Hardy–Sobolev inequality (i.e. (p,β)-Hardy inequality) for f imply that

$$\begin{aligned}
\bigg(\int_G |f|^{np/(n-p)}\delta_{\partial G}^{n\beta/(n-p)}\,dx\bigg)^{(n-p)/np} &= \bigg(\int_G |g|^{np/(n-p)}\,dx\bigg)^{(n-p)/np}\\
&\leq C_1\bigg(\int_G |\nabla g|^p\,dx\bigg)^{1/p}\\
&\leq C_1\bigg\{\bigg(\int_G |\nabla f|^p\delta_{\partial G}^{\beta}\,dx\bigg)^{1/p} + \frac{|\beta|}{p}\bigg(\int_G |f|^p\delta_{\partial G}^{\beta-p}\,dx\bigg)^{1/p}\bigg\}\\
&\leq C_2\bigg(\int_G |\nabla f|^p\delta_{\partial G}^{\beta}\,dx\bigg)^{1/p},
\end{aligned}$$

which yields the (p^*, p, β)-Hardy–Sobolev inequality for f. $\square$

Proof of Theorem 2.1. Let $p < q < p^*$ and write $\alpha = p^2/(np - nq + qp)$, $\alpha' = \alpha/(\alpha - 1)$. Assume that $f \in C_0^\infty(G)$. A straightforward computation for the exponents and Hölder's inequality (for exponents α and α') yields

$$\begin{aligned}\left(\int_G |f|^q\, \delta_{\partial G}^{(q/p)(n-p+\beta)-n}\, dx\right)^{1/q} &= \left(\int_G |f|^{\frac{p}{\alpha}+\frac{p^*}{\alpha'}}\, \delta_{\partial G}^{\frac{\beta-p}{\alpha}+\frac{n\beta}{(n-p)\alpha'}}\, dx\right)^{1/q} \\ &\le \left(\int_G |f|^p\, \delta_{\partial G}^{\beta-p}\, dx\right)^{\frac{1}{q\alpha}} \left(\int_G |f|^{p^*}\, \delta_{\partial G}^{\frac{n\beta}{n-p}}\, dx\right)^{\frac{1}{q\alpha'}} .\end{aligned} \tag{4}$$

By the assumptions and Lemma 2.2, we now have available the two ‘extreme’ Hardy–Sobolev-inequalities, i.e. (p, p, β)- and (p^*, p, β)-Hardy–Sobolev inequalities. Using these to the two integrals on the last line of (4), respectively, and noting that $\frac{1}{q\alpha} + \frac{n}{n-p}\frac{1}{q\alpha'} = \frac{1}{p}$, we obtain from (4) that

$$\begin{aligned}\left(\int_G |f|^q\, \delta_{\partial G}^{(q/p)(n-p+\beta)-n}\, dx\right)^{1/q} &\le C_3 \left(\int_G |\nabla f|^p\, \delta_{\partial G}^{\beta}\, dx\right)^{\frac{1}{q\alpha}} \left(\int_G |\nabla f|^p\, \delta_{\partial G}^{\beta}\, dx\right)^{\frac{n}{n-p}\frac{1}{q\alpha'}} \\ &= C_3 \left(\int_G |\nabla f|^p\, \delta_{\partial G}^{\beta}\, dx\right)^{1/p} ,\end{aligned} \tag{5}$$

as desired. $\square$

Remark 2.3. If G admits a (q, p, β)-Hardy–Sobolev inequality, we use the notation $\kappa_{q,p,\beta}$ for the best constant appearing in (3); recall that $\kappa_{p^*,p,0} < \infty$ for all open sets G in $\mathbb{R}^n$. In the proof of Lemma 2.2 we have $C_1 = \kappa_{p^*,p,0}$, and so we obtain for $\kappa_{p^*,p,\beta}$ the following upper bound:

$$\kappa_{p^*,p,\beta} \le C_2 = \kappa_{p^*,p,0}\big(1 + \tfrac{|\beta|}{p}\kappa_{p,p,\beta}\big) .$$

On the other hand, the constant in the proof of Theorem 2.1 is $C_3 = \kappa_{p,p,\beta}^{p/(q\alpha)}\kappa_{p^*,p,\beta}^{p^*/(q\alpha')}$, where we have written (as in the proof of Theorem 2.1) $\alpha = p^2/(np - nq + qp)$ and $\alpha' = \alpha/(\alpha - 1)$. Thus our interpolation yields the following estimate for the best constant in the (q, p, β)-Hardy–Sobolev inequality, in terms of the constants in the Sobolev and (p, β)-Hardy inequalities:

$$\kappa_{q,p,\beta} \le \kappa_{p,p,\beta}^{p/(q\alpha)} \Big(\kappa_{p^*,p,0}\big(1 + \tfrac{|\beta|}{p}\kappa_{p,p,\beta}\big)\Big)^{p^*/(q\alpha')} .$$

3. Concepts of dimension and the $P(s)$-property

The λ-dimensional Hausdorff measure and Hausdorff content of $E \subset \mathbb{R}^n$ are denoted by $\mathcal{H}^\lambda(E)$ and $\mathcal{H}^\lambda_\infty(E)$, respectively, and the Hausdorff dimension of E is $\dim_{\mathrm{H}}(E)$; see [28, Chapter 4]. Besides this well-known notion, we will need several other concepts of dimension in our results to describe various geometric properties of sets.

When $A \subset \mathbb{R}^n$ is bounded and $r > 0$, we let $N(A, r)$ denote the minimal number of (open) balls of radius r and centered at A that are needed to cover the set A. The λ-dimensional *Minkowski content* of a bounded set $E \subset \mathbb{R}^n$ is then defined to be

$$\mathcal{M}^\lambda_r(E) = N(E, r)r^\lambda ,$$

and the *upper* and *lower Minkowski dimensions* of E are

$$\overline{\dim}_{\mathrm{M}}(E) = \inf\big\{\lambda \ge 0 : \limsup_{r\to 0} \mathcal{M}^\lambda_r(E) = 0\big\}$$

and

$$\underline{\dim}_{\mathrm{M}}(E) = \inf\left\{\lambda \geq 0 : \liminf_{r\to 0} \mathcal{M}_r^{\lambda}(E) = 0\right\},$$

respectively.

For general $E \subset \mathbb{R}^n$ we define the following 'localized' versions of Minkowski dimensions: The *(upper) Assouad dimension* of E is defined by setting

$$\begin{aligned}
&\dim_{\mathrm{A}}(E) = \overline{\dim}_{\mathrm{A}}(E) \\
&= \inf\left\{\lambda \geq 0 : N(E \cap B(x,R), r) \leq C_\lambda \left(\frac{r}{R}\right)^{-\lambda} \text{ for all } x \in E,\ 0 < r < R < d(E)\right\}.
\end{aligned}$$

This upper Assouad dimension is the 'usual' Assouad dimension found in the literature, and usually only the notation $\dim_{\mathrm{A}}(E)$ is used. Conversely, we define the *lower Assouad dimension* of E to be

$$\begin{aligned}
&\underline{\dim}_{\mathrm{A}}(E) \\
&= \sup\left\{\lambda \geq 0 : N(E \cap B(x,R), r) \geq c_\lambda \left(\frac{r}{R}\right)^{-\lambda} \text{ for all } x \in E,\ 0 < r < R < d(E)\right\}.
\end{aligned}$$

It is clear from the definitions that for a bounded set E in $\mathbb{R}^n$ we always have

$$\underline{\dim}_{\mathrm{A}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E).$$

In addition, if $E \subset \mathbb{R}^n$ is closed, then $\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}(E \cap B)$ for all balls centered in E; see [17, Lemma 2.2]. We refer to [2, 10, 17, 21, 27] for more information on these and closely related concepts.

A closed set $E \subset \mathbb{R}^n$ is said to be (Ahlfors) λ-regular (or a λ-set), for $0 \leq \lambda \leq n$, if there is a constant $C \geq 1$ such that

$$C^{-1} r^{\lambda} \leq \mathcal{H}^{\lambda}(E \cap B(x,r)) \leq C r^{\lambda}$$

for every $x \in E$ and all $0 < r < d(E)$. If E is a λ-regular set, then all of the above dimensions coincide and are equal to λ, and so in particular $\underline{\dim}_{\mathrm{A}}(E) = \overline{\dim}_{\mathrm{A}}(E) = \lambda$; see e.g. [17] for details.

Next, we recall the following 'Aikawa condition' for the intergability of the distance function, which is closely related to the (upper) Assouad dimension: When $\emptyset \neq E \subset \mathbb{R}^n$ is a closed set, we let $\mathcal{A}(E)$ be the set of all $s \geq 0$ for which there is a constant $C > 0$ such that inequality

$$\int_{B(x,r)} \operatorname{dist}(y,E)^{s-n}\, dy \leq C r^s \tag{6}$$

holds whenever $x \in E$ and $0 < r < d(E)$. This condition was used by Aikawa [1] in connection to the so-called quasiadditivity of capacity, which has subsequently turned out to be intimately related to Hardy inequalities, we refer to [22, 23, 25].

The following lemma collects useful properties related to the Aikawa condition and the (upper) Assouad dimension. Recall that a set $E \subset \mathbb{R}^n$ is said to be *porous*, if there is a constant $0 < c < 1$ such that for every $x \in E$ and all $0 < r < d(E)$ there exists a point $y \in \mathbb{R}^n$ such that $B(y, cr) \subset B(x,r) \setminus E$.

Lemma 3.1. *Let $E \neq \emptyset$ be a closed set.*

(A) *We have $\overline{\dim}_{\mathrm{A}}(E) = \inf \mathcal{A}(E) \leq n$.*
(B) *If $\overline{\dim}_{\mathrm{A}}(E) < s$ or $n \leq s$, then $s \in \mathcal{A}(E)$.*

(C) *The Aikawa condition is self-improving: If* $s \in \mathcal{A}(E)$ *with* $0 < s < n$*, then there is* $0 < s' < s$ *such that* $s' \in \mathcal{A}(E)$*; in particular* $\overline{\dim}_{\mathrm{A}}(E) < s$*.*
(D) *The set* E *is porous if and only if* $\overline{\dim}_{\mathrm{A}}(E) < n$*. In particular, we have that* $|E| = 0$ *if* $\overline{\dim}_{\mathrm{A}}(E) < n$*.*
(E) *Let* $E \neq \emptyset$ *be a compact set in* $\mathbb{R}^n$ *and let* $s > 0$*. Then* $s \in \mathcal{A}(E)$ *if and only if for every (or, equivalently, for some)* $0 < R \leq \infty$ *there exists a constant* $C > 0$ *such that inequality* (6) *holds whenever* $x \in E$ *and* $0 < r < R$*.*

Proof. (A) This is proven in [25].
(B) This is easy to see from the definitions and property (A).
(C) The proof is based on the Gehring lemma; see e.g. [22, Lemma 2.2] for details.
(D) See [27, Theorem 5.2].
(E) Let us outline the proof; the reader will find it straightforward to fill in the details. We fix a number $0 < R < 2\,d(E)$ and assume that (6) holds whenever $x \in E$ and $0 < r < R$. It suffices to prove that there exists a constant $C > 0$ such that inequality (6) holds whenever $x \in E$ and $0 < r < \infty$; clearly, we may also assume that $0 < s < n$. If $0 < r < R$, inequality (6) holds by the assumption. If $R \leq r \leq 2\,d(E)$, we use the compactness of E to find points $x_1, \ldots, x_K \in E$ such that

$$\{y \in \mathbb{R}^n \,:\, \operatorname{dist}(y, E) < R/4\} \subset \bigcup_{j=1}^{K} B(x_j, R/2)\,. \tag{7}$$

To estimate the left-hand side of (6), we split the ball $B(x, r)$ in two parts: $A = B(x, r) \cap \{y \,:\, \operatorname{dist}(y, E) < R/4\}$ and $B(x, r) \setminus A$. The integral over the set A is estimated by using (7) and the validity of inequality (6) for all radii up to R, and the integral over the set $B(x, r) \setminus A$ is easy to estimate since therein the distances to E are larger than $R/4$ and r is dominated by $2\,d(E)$. Finally, if $2\,d(E) < r < \infty$, we split $B(x, r)$ as $D = B(x, 2\,d(E))$ and $B(x, r) \setminus D$. The integral over the set D is treated as in the case $R \leq r \leq 2\,d(E)$ above, and the integral over the set $B(x, r) \setminus D$ is estimated by using the fact that therein the distance of a point to the set E is comparable to its distance to the point x. We conclude that (6) holds for all $0 < r < \infty$. □

In [14], Horiuchi introduced the following $P(s)$-property in order to study imbeddings for weighted Sobolev spaces. This property was subsequently applied also in [15]. Here we denote $E_\eta = \{x \in \mathbb{R}^n : \delta_E(x) < \eta\}$, that is, E_η is the (open) η-neighborhood of E.

Definition 3.2. Let $0 \leq s \leq n$. A closed set $E \subset \mathbb{R}^n$ has the property $P(s)$ if $|E| = 0$ and there is a constant $C > 0$ such that

$$|B \cap (E_{\eta_2} \setminus E_{\eta_1})| \leq C\eta_2^{s-1}(\eta_2 - \eta_1)\, d(B)^{n-s} \quad \text{if } 1 \leq s \leq n$$

and

$$|B \cap (E_{\eta_2} \setminus E_{\eta_1})| \leq C(\eta_2 - \eta_1)^s\, d(B)^{n-s} \quad \text{if } 0 \leq s < 1\,,$$

for all balls B and numbers η_1, η_2 satisfying $0 \leq \eta_1 < \eta_2 \leq d(B)$.

Remark 3.3. Horiuchi required the (respective) inequality in Definition 3.2 to hold only for balls B and numbers η_1, η_2 satisfying inequalities $0 \leq \eta_1 < \eta_2 \leq d(B) \leq A_0$ with a fixed $A_0 \in (0, \infty]$. For our purposes the given formulation is more suitable. Horiuchi also excludes the case $s = 0$ in Definition 3.2; observe that all closed sets with zero Lebesgue measure have the property $P(0)$.

We now have the following theorem, which characterizes the (upper) Assouad dimension in terms of the $P(s)$-property. This result also clarifies the $P(s)$-property and immediately gives numerous examples of sets having this property and thus satisfying the main assumption in Horiuchi's papers [14, 15].

Theorem 3.4. *Let $E \subset \mathbb{R}^n$ be a closed set with $|E| = 0$. Then*

$$\overline{\dim}_{\mathrm{A}}(E) = n - \sup\big\{0 \le s \le n \,:\, E \text{ has the property } P(s)\big\}.$$

In particular, the $P(s)$-property holds for all $0 \le s < n - \overline{\dim}_{\mathrm{A}}(E)$.

Proof. Assume first that E has the property $P(s)$. Fix a ball B so that $d(B) \le d(E)$. Choosing $\eta_1 = 0$ and writing $\eta = \eta_2$, we find that

$$|B \cap E_\eta| \le C\eta^s\, d(B)^{n-s}$$

for all numbers $0 < \eta \le d(B)$, regardless of whether $s \ge 1$ or $s < 1$. From this it follows, by [27, Theorem A.12], that (in the language of [27]) the set E is $(n-s)$-homogeneous, and thus $\overline{\dim}_{\mathrm{A}}(E) \le n - s$ (see also [25, Theorem 5.1] whose proof is a short argument based on the Aikawa condition). This proves one direction ('$\le$') of the claimed equality.

The converse inequality is somewhat more involved. Since E always has the property $P(0)$, we may assume that $\overline{\dim}_{\mathrm{A}}(E) < n$. Let us fix $0 < s < n$ such that $\overline{\dim}_{\mathrm{A}}(E) < n-s$. It suffices to show that then E has the property $P(s)$, so let $B = B(w,R)$ be a ball in $\mathbb{R}^n$. Without loss of generality, we may assume that $w \in E$. Let us fix $0 \le \eta_1 < \eta_2 \le d(B) = 2R$. If $\eta_1 < \eta_2/2$, then $\eta_2 - \eta_1 \simeq \eta_2$, and the desired estimate follows from Lemma 3.1(B,E). Indeed,

$$|B \cap (E_{\eta_2} \setminus E_{\eta_1})| \le |B \cap E_{\eta_2}| \le \eta_2^s \int_B \operatorname{dist}(y,E)^{-s}\, dy \lesssim \eta_2^s\, R^{n-s}\,.$$

(Alternatively, this estimate could be obtained from [27, Theorem A.12] or the proof of [25, Theorem 5.1]).

We may hence assume that $\eta_2/2 \le \eta_1 < \eta_2$. Let $\mathcal{W}(E^c) = \{B_i\}$ be a Whitney-type cover of E^c with balls $B_i = B\big(x_i, \frac18 \delta_E(x_i)\big)$. In particular, the overlap of these balls is uniformly bounded. We also write

$$\mathcal{W}(E^c; 2B; \eta_1) = \{B_i \in \mathcal{W}(E^c) : B_i \cap 2B \cap \partial E_{\eta_1} \neq \emptyset\}.$$

It follows that $\operatorname{dist}(B_i, E) \simeq d(B_i) \simeq \eta_1$ for all $B_i \in \mathcal{W}(E^c; 2B; \eta_1)$. Thus, the assumption $\overline{\dim}_{\mathrm{A}}(E) < n - s$ and Lemma 3.1(B,E) yield

$$\begin{aligned} \#\mathcal{W}(E^c; 2B; \eta_1) &\lesssim \eta_1^{s-n} \sum_{B_i \in \mathcal{W}(E^c;2B;\eta_1)} |B_i|^{1-s/n} \\ &\lesssim \eta_1^{s-n} \int_{cB} \operatorname{dist}(y,E)^{-s}\, dy \lesssim \eta_1^{s-n} R^{n-s}\,; \end{aligned} \tag{8}$$

in particular, we obtain that $\#\mathcal{W}(E^c; 2B; \eta_1) \lesssim \eta_1^{s-n} R^{n-s}$. (See also [17, Lemma 4.4] for another proof of this estimate.)

In addition, the proof of [17, Lemma 5.4] shows that, for each ball $B_i \in \mathcal{W}(E^c; 2B; \eta_1)$, there exists a 2-Lipschitz mapping from a subset of ∂B_i *onto* $B_i \cap \partial E_{\eta_1}$, and thus

$$\mathcal{M}_\varepsilon^{n-1}(B_i \cap \partial E_{\eta_1}) \le 4^{n-1} \mathcal{M}_{C\varepsilon}^{n-1}(\partial B_i) \simeq \eta_1^{n-1}, \tag{9}$$

where we write $\varepsilon = \eta_2 - \eta_1$. On the other hand, for each $B_i \in \mathcal{W}(E^c; 2B; \eta_1)$, let $\{B^i_j\}_j$ be a cover of $B_i \cap \partial E_{\eta_1}$ with balls $B^i_j = B(y^i_j, \varepsilon)$, where $y^i_j \in B_i \cap \partial E_{\eta_1}$ are chosen so that

$$\mathcal{M}^{n-1}_{\varepsilon}(B_i \cap \partial E_{\eta_1}) \geq C_n \varepsilon^{-1} \sum_j |B^i_j|. \tag{10}$$

Then $B \cap (E_{\eta_2} \setminus E_{\eta_1}) \subset \bigcup_i \bigcup_j 2B^i_j$, and thus it follows from (10), (9), and (8) that

$$\begin{aligned} |B \cap (E_{\eta_2} \setminus E_{\eta_1})| &\leq \sum_{B_i \in \mathcal{W}(E^c;2B;\eta_1)} \sum_j |2B^i_j| \lesssim \varepsilon \sum_{B_i \in \mathcal{W}(E^c;2B;\eta_1)} \varepsilon^{-1} \sum_j |B^i_j| \\ &\lesssim \varepsilon \sum_{B_i \in \mathcal{W}(E^c;2B;\eta_1)} \mathcal{M}^{n-1}_{\varepsilon}(B_i \cap \partial E_{\eta_1}) \lesssim \varepsilon \sum_{B_i \in \mathcal{W}(E^c;2B;\eta_1)} \eta_1^{n-1} \\ &\lesssim \varepsilon \eta_1^{n-1} \eta_1^{s-n} R^{n-s} \leq \eta_2^{s-1}(\eta_2 - \eta_1) R^{n-s}. \end{aligned}$$

This shows that E has property $P(s)$, provided that $1 \leq s < n$. Moreover, if $0 < s < 1$, then (under the assumption $\eta_2/2 \leq \eta_1 < \eta_2$) $\eta_1^{s-1} \leq C\eta_2^{s-1} \leq C(\eta_2 - \eta_1)^{s-1}$, and thus

$$|B \cap (E_{\eta_2} \setminus E_{\eta_1})| \lesssim \eta_1^{s-1}(\eta_2 - \eta_1) R^{n-s} \lesssim (\eta_2 - \eta_1)^s R^{n-s},$$

proving that E has property $P(s)$ in this case as well. □

Remark 3.5. In particular, (the proof of) Theorem 3.4 shows that for our purposes we could equivalently define the property $P(s)$ in the case $0 < s < 1$ in the same way as in the case $1 \leq s < n$.

We also record the following lemma that will be useful in the proofs of Section 5.

Lemma 3.6. *Let $E \subset \mathbb{R}^n$ be a closed porous set. If $s > \overline{\dim}_{\mathrm{A}}(E)$, then*

$$\int_B \delta_E(x)^{s-n}\, dx \simeq d(B)^n \big(d(B) + \operatorname{dist}(B, E)\big)^{s-n} \tag{11}$$

for all balls B in $\mathbb{R}^n$, where the constants of comparison are independent of the ball B.

Proof. This follows from Theorem 3.4 and [14, Proposition 6.1]. However, it is also easy to give a direct proof using the Aikawa condition and the porosity of E. □

4. Main results

By Theorem 2.1, sufficient conditions for a (p, β)-Hardy inequality are also sufficient for (q, p, β)-Hardy–Sobolev inequalities, for all $p \leq q \leq p^*$. Hence we obtain the following corollary by combining the previously known sufficient conditions for (p, β)-Hardy inequalities—more precisely, Corollary 1.3 in [22]—and Theorem 2.1.

Corollary 4.1. *Let $1 < p \leq q \leq np/(n-p) < \infty$ and $\beta < p - 1$. If $G \subset \mathbb{R}^n$ is an open set and*

$$\overline{\dim}_{\mathrm{A}}(G^c) < n - p + \beta \quad \text{or} \quad \underline{\dim}_{\mathrm{A}}(G^c) > n - p + \beta\,,$$

then G admits a (q, p, β)-Hardy–Sobolev inequality; in the latter case, if G is unbounded, then we require that also G^c is unbounded.

The second sufficient condition in Corollary 4.1 for the (q, p, β)-Hardy–Sobolev inequality in terms of the lower Assouad dimension $\underline{\dim}_{\mathrm{A}}(G^c)$, which corresponds to the case where the complement is 'thick', turns out to be rather sharp; we refer to Theorem 4.6 below and also to Section 8. Let us remark that, for $p - \beta > 1$, the condition $\underline{\dim}_{\mathrm{A}}(G^c) > n-p+\beta$ is equivalent to G^c being *uniformly* $(p-\beta)$*-fat* (see [17, Remark 3.2]).

For the unweighted p-Hardy inequality (i.e., the $(p,p,0)$-Hardy–Sobolev inequality), uniform p-fatness is a well-known sufficient condition, see e.g. [26]. The bound $\beta < p-1$ in Corollary 4.1 is sharp and necessary for this generality, but it can be removed for instance under additional accessibility conditions for ∂G (cf. [19, 23]).

On the other hand, the following Theorem 4.2 shows that the first sufficient condition in Corollary 4.1, given in terms of the (upper) Assouad dimension $\overline{\dim}_{\mathrm{A}}(G^c)$ and corresponding to the 'thin' case, can be weakened in two ways: The factor $q/p \geq 1$ appears, and the upper bound $\beta < p-1$ can be changed to the weaker assumption that $\overline{\dim}_{\mathrm{A}}(G^c) < n-1$ (cf. Remark 4.3).

Theorem 4.2. *Let $1 \leq p \leq q \leq np/(n-p) < \infty$ and $\beta \in \mathbb{R}$. If $G \subset \mathbb{R}^n$ is an open set and*

$$\overline{\dim}_{\mathrm{A}}(G^c) < \min\left\{\frac{q}{p}(n-p+\beta)\,,\, n-1\right\}, \tag{12}$$

then G admits a (q,p,β)-Hardy–Sobolev inequality.

Theorem 4.2 is a consequence of a more general result for functions $f \in C_0^\infty(\mathbb{R}^n)$ that we will establish in Theorem 5.1.

Remark 4.3. In the case $q = p$, Theorem 4.2 in particular improves the sufficient condition for the (p,β)-Hardy inequality from [22, Corollary 1.3]. In both of these results it is assumed that $\overline{\dim}_{\mathrm{A}}(G^c) < n-p+\beta$, but in [22, Corollary 1.3] the extra assumption is $\beta < p-1$ instead of $\overline{\dim}_{\mathrm{A}}(G^c) < n-1$ in Theorem 4.2, and clearly $\overline{\dim}_{\mathrm{A}}(G^c) < n-p+\beta$ and $\beta < p-1$ together imply that $\overline{\dim}_{\mathrm{A}}(G^c) < n-1$.

Let us give an easy example which shows that the assumption $\overline{\dim}_{\mathrm{A}}(G^c) < n-1$ in Theorem 4.2 can not be removed:

Example 4.4. Let $G = \mathbb{R}^n \setminus \partial B(0,1)$, and consider functions $f_j \in C_0^\infty(G)$ such that $f_j(x) = 1$ when $|x| \leq 1-2^{1-j}$, $f_j(x) = 0$ when $|x| \geq 1-2^{-j}$, and $|\nabla f_j| \leq C2^j$ when $1-2^{1-j} < |x| < 1-2^{-j}$. Then, for any $1 \leq p \leq q \leq np/(n-p) < \infty$ and $\beta \in \mathbb{R}$, the left-hand side of the (q,p,β)-Hardy–Sobolev inequality (3) is uniformly bounded away from zero for these functions f_j if $j > 1$ but, on the other hand, when $\beta > p-1$ we have for the right hand side of (3) that

$$\int_G |\nabla f_j|^p\, \delta_{\partial G}^\beta\, dx \lesssim \big|\{x \in \mathbb{R}^n \,:\, 1-2^{1-j} < |x| < 1-2^{-j}\}\big| 2^{jp} 2^{-j\beta} \lesssim 2^{-j(\beta-p+1)} \xrightarrow{j\to\infty} 0.$$

Hence the (q,p,β)-Hardy–Sobolev inequality fails in G when $\beta > p-1 = p-n+\overline{\dim}_{\mathrm{A}}(G^c)$, even though in this case $\overline{\dim}_{\mathrm{A}}(G^c) < n-p+\beta \leq \frac{q}{p}(n-p+\beta)$, which is exactly the first bound given by (12).

See also [23, Sect. 6.3] for a another example where the open set G is connected. The calculation is given there only for $p = q$, but the example works, just as above, for all $1 \leq p \leq q \leq np/(n-p) < \infty$ and $\beta > p-1$. Nevertheless, some positive results can also be given for the case $\overline{\dim}_{\mathrm{A}}(G^c) \geq n-1$, see Theorem 5.1 for details, but these always require an upper bound for β.

For the complements of λ-regular sets (for $0 < \lambda < n-1$) the 'thick' and 'thin' cases overlap when $p < q$, and hence all Hardy–Sobolev inequalities hold in this case:

Corollary 4.5. *Let $1 < p < q \le np/(n-p) < \infty$ and $0 < \lambda < n-1$, and assume that $E \subset \mathbb{R}^n$ is an unbounded λ-regular set. Then the open set $G = \mathbb{R}^n \setminus E$ admits (q,p,β)-Hardy–Sobolev inequalities for every $\beta \in \mathbb{R}$.*

Proof. For $\beta < p - n + \lambda$ this follows from the 'thick' case of Corollary 4.1 (observe that $G^c = E$ is assumed to be unbounded), and for $\beta > p - n + \frac{p}{q}\lambda$ from Theorem 4.2. Since $p - n + \frac{p}{q}\lambda < p - n + \lambda$, these two cases cover all $\beta \in \mathbb{R}$. $\square$

Note that Corollary 4.5 is not true when $p = q$, since then the $(p,p,p-n+\lambda)$-Hardy–Sobolev inequality fails for all $1 < p < \infty$ by [23, Thm 1.1]. Contrary to Corollary 4.5, as soon as the Hausdorff (or lower Minkowski) and (upper) Assouad dimensions of the complement G^c differ, some Hardy–Sobolev inequalities fail in G by the following Theorem 4.6, which gives a 'dichotomy' condition for the dimension of the complement G^c when G admits a (q,p,β)-Hardy–Sobolev inequality. Local versions of these statements, as well as the proof of Theorem 4.6, are given in Section 8.

Theorem 4.6. *Let $G \subset \mathbb{R}^n$ be an open set and assume that $1 \le p \le q < np/(n-p) < \infty$ and $\beta \in \mathbb{R}$ are such that G admits a (q,p,β)-Hardy–Sobolev inequality* (3).

If $\beta \ge 0$ and $q(n-p+\beta)/p \ne n$, then either

$$\overline{\dim}_{\mathrm{A}}(G^c) < \frac{q}{p}(n-p+\beta) \quad \text{or} \quad \dim_{\mathrm{H}}(G^c) \ge n-p+\beta\,.$$

If $\beta < 0$ and G^c is compact and porous, then either

$$\overline{\dim}_{\mathrm{A}}(G^c) < \frac{q}{p}(n-p+\beta) \quad \text{or} \quad \underline{\dim}_{\mathrm{M}}(G^c) \ge n-p+\beta\,.$$

In particular, Theorem 4.6 shows that the numbers $\frac{q}{p}(n-p+\beta)$ and $n-p+\beta$, which bound the dimension of G^c from above or below in the sufficient conditions in Theorem 4.2 and in the thick case of Corollary 4.1, respectively, are sharp, although the notions of dimension are not the same in the lower bounds. Recall, however, that when $E \subset \mathbb{R}^n$ is a closed set, then $\underline{\dim}_{\mathrm{A}}(E) \le \dim_{\mathrm{H}}(E \cap B)$ for all balls B centered at E, see [17, Lemma 2.2]. Moreover, for many sets it holds that $\underline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{H}}(E)$ (and even $\underline{\dim}_{\mathrm{A}}(E) = \underline{\dim}_{\mathrm{M}}(E)$), so in this sense the sufficient condition $\underline{\dim}_{\mathrm{A}}(G^c) > n-p+\beta$ in Corollary 4.1 is not that far from being optimal.

Let us also illustrate the applicability and sharpness of our results with the following example:

Example 4.7. Consider a closed unbounded set $E \subset \mathbb{R}^n$ with

$$0 \le \underline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{H}}(E) = \lambda_1 < \lambda_2 = \overline{\dim}_{\mathrm{A}}(E) < n-1. \tag{13}$$

Then for $\beta \ge 0$ and $1 < p \le q < np/(n-p) < \infty$, with $q(n-p+\beta)/p \ne n$, the open set $G = \mathbb{R}^n \setminus E$ admits a (q,p,β)-Hardy–Sobolev inequality if

$$n-p+\beta < \lambda_1 \qquad \text{or} \qquad \lambda_2 < \frac{q}{p}(n-p+\beta)\,,$$

while if

$$\lambda_1 < n-p+\beta \le \frac{q}{p}(n-p+\beta) \le \lambda_2\,,$$

then the (q,p,β)-Hardy–Sobolev inequality fails in G; notice that the latter inequalities always hold for some parameters q,p,β when $0 \le \lambda_1 < \lambda_2 \le n$. Moreover, sets satisfying the dimensional bounds in (13) do exist. An easy example would be a closed, unbounded, non-porous, and countable set $E \subset \mathbb{R}^m$, $m \le n-2$, which is embedded to $\mathbb{R}^n$; then it is well-known that $\underline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{H}}(E) = 0$ and $\overline{\dim}_{\mathrm{A}}(E) = m < n-1$. A concrete example of such a set, for $m = 1$ and $n \geq 3$, is obtained by embedding a copy of the set $E_0 = \{1/j : j \in \mathbb{N}\} \cup \{0\} \subset [0,1]$ to each unit interval $[k, k+1]$, $k \in \mathbb{Z}$, in the x_1 axis of $\mathbb{R}^n$.

Notice that we do not know in Example 4.7 what happens when $n - p + \beta = \lambda_1$. However, in the case $p = q$ it is known that the (p,β)-Hardy inequality does not hold when $n - p + \beta = \dim_{\mathrm{H}}(G^c)$, i.e., the lower bound in terms of the Hausdorff dimension is strict in Theorem 4.6 for $p = q$, see [23, Thm 1.1]. The proof of the strict inequality is heavily based on the known self-improvement of Hardy inequalities (cf. e.g. [20]). We do not know if such self-improvement holds in general for Hardy–Sobolev inequalities, but such a property would certainly yield strict inequalities to the lower bounds of Theorem 4.6 for all $p < q < p^*$, as well.

Nevertheless, let us also point out that the strict inequality plays a much bigger role in the case $p = q$, since for instance when the complement of G is λ-regular, the strict inequality shows that the $(p, p - n + \lambda)$-Hardy inequality does not hold in G; compare this to Corollary 4.5.

5. Sufficient conditions for global inequalities

For the next two sections, we change the perspective a little bit and consider the validity of the global Hardy–Sobolev inequalities for all functions $f \in C_0^\infty(\mathbb{R}^n)$ when the distance is taken to a closed set $E \subset \mathbb{R}^n$ with $|E| = 0$. In this section we establish the following sufficient condition for such inequalities. Note that Theorem 4.2 is an immediate consequence of the case $\overline{\dim}_{\mathrm{A}}(E) < n-1$ of this more general result.

Theorem 5.1. *Let $E \neq \emptyset$ be a closed porous set in $\mathbb{R}^n$, and let $1 \leq p \leq q \leq np/(n-p) < \infty$ and $\beta \in \mathbb{R}$ be such that*

$$\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n - p + \beta)\,.$$

In addition, assume that either $\overline{\dim}_{\mathrm{A}}(E) < n-1$ or that

$$\beta \leq \frac{(p-1)(qp + np - nq)}{qp + p - q}\,. \tag{14}$$

Then, there is a constant $C > 0$ such that inequality

$$\biggl(\int_{\mathbb{R}^n} |f(x)|^q \,\delta_E(x)^{(q/p)(n-p+\beta)-n}\,dx\biggr)^{1/q} \leq C\biggl(\int_{\mathbb{R}^n} |\nabla f(x)|^p \,\delta_E(x)^{\beta}\,dx\biggr)^{1/p} \tag{15}$$

holds for every $f \in C_0^\infty(\mathbb{R}^n)$.

The proof of Theorem 5.1 follows from the results of Horiuchi [14] and our characterization of the (upper) Assouad dimension, Theorem 3.4. Nevertheless, since the proof of inequality (15) is somewhat implicit in [14], where the interest is mainly in the corresponding non-homogeneous inequalities, we have chosen to include here a detailed proof. This also allows us to clarify some rather subtle points in the proof and to distinguish during the proof when the full power of the $P(s)$-property is needed and when the claims follow more directly from the dimensional bound $\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n - p + \beta)$.

We first reduce Theorem 5.1 to the following lemma, which is essentially [14, Proposition 5.1], corresponding to the case $p = 1$. The proof of Lemma 5.2 is given at the end of this section after we have established an important technical tool in Lemma 5.4.

Throughout this section we say that a set M in $\mathbb{R}^n$ is *admissible* if M is both open and bounded and ∂M is a closed $(n-1)$-dimensional manifold of class C^∞.

Lemma 5.2. *Let $E \neq \emptyset$ be a closed porous set in $\mathbb{R}^n$, and let $1 \le q \le n/(n-1)$ and $\beta \in \mathbb{R}$ be such that $\overline{\dim}_{\mathrm{A}}(E) < q(n-1+\beta)$. If $n-1 \le \overline{\dim}_{\mathrm{A}}(E) < n$, then we assume in addition that $\beta \le 0$. Then there exists a positive constant C_1 such that*

$$\left(\int_M \delta_E(x)^{q(n-1+\beta)-n}\, dx \right)^{1/q} \le C_1 \int_{\partial M} \delta_E(x)^\beta d\mathcal{H}^{n-1}(x)\,,$$

whenever $M \subset \mathbb{R}^n$ is an admissible set.

Also the following result, which shows that the Assouad dimension of E is closely related to the Muckenhoupt A_1-properties of the powers of the distance function, will be needed in the proof of Theorem 5.1.

Lemma 5.3. *Let $E \neq \emptyset$ be a closed set in $\mathbb{R}^n$ and let $\omega = \delta_E^{s-n}$, where $\overline{\dim}_{\mathrm{A}}(E) < s \le n$. Then ω is a Muckenhoupt A_1-weight, i.e., there is a constant $c \ge 1$ such that inequality*

$$\fint_B \omega(x)\, dx \le c \operatorname*{ess\,inf}_{x\in B} \omega(x)$$

holds whenever B is any ball in $\mathbb{R}^n$.

This result is not difficult to see from Lemma 3.1 (see also [15, Lemma 2.2]), and hence we omit the straightforward but somewhat tedious details. For more information on Muckenhoupt weights, we refer to [11, Chapter IV].

Proof of Theorem 5.1. Let us first consider the case $p = 1$. Fix a non-negative function $f \in C_0^\infty(\mathbb{R}^n)$. By Sard's theorem [32], and the implicit function theorem, the following two statements hold simultaneously for almost every $t > 0$. First, the boundary of the compact set

$$M_t = \{x \in \mathbb{R}^n \,:\, f(x) > t\}$$

coincides with the level set $\{x \in \mathbb{R}^n \,:\, f(x) = t\}$ and, second, this level set is a compact $(n-1)$-dimensional manifold of class C^∞.

By Minkowski's integral inequality [33, p. 271], the co-area formula (see e.g. [8, Theorem 3.2.12] and Lemma 5.2, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n} f(x)^q\, \delta_E(x)^{q(n-1+\beta)-n}\, dx \right)^{1/q} &\le \int_0^\infty \left(\int_{M_t} \delta_E(x)^{q(n-1+\beta)-n}\, dx \right)^{1/q} dt \\ &\le C_1 \int_0^\infty \int_{\partial M_t} \delta_E(x)^\beta d\mathcal{H}^{n-1}(x)\, dt = C_1 \int_{\mathbb{R}^n} |\nabla f(x)| \delta_E(x)^\beta\, dx\,.\end{aligned}$$

(Note here that since $\overline{\dim}_{\mathrm{A}}(E) < q(n-1+\beta) \le n + \beta q$, both $\delta_E^{q(n-1+\beta)-n}$ and δ_E^β are locally integrable by Lemma 3.1(B).) An approximation argument via mollification of $|f|$ concludes the proof of Theorem 5.1 when $p = 1$; the dominated convergence theorem (in the case of distance weights with positive powers) or Lemma 5.3 and [34, Theorem 2.1.4] (in case of negative powers) can be used to make the argument rigorous.

Let then q, p and β be as in the statement of Theorem 5.1. We assign

$$\hat{q} = \frac{1}{1 - 1/p + 1/q}\,, \qquad \hat{\beta} = \frac{q(n-p+\beta)}{\hat{q}p} - n + 1\,, \qquad \text{and} \qquad \hat{f} = |f|^{q/\hat{q}}\,.$$

It is straightforward to verify that then $\hat{q}$ and $\hat{\beta}$ satisfy the assumptions of the case $p = 1$ of the theorem (in particular, if β satisfies the additional bound (14), then $\hat{\beta} \le 0$). Hence we obtain, using the case $p = 1$ for $\hat{f}$ with exponents $\hat{q}$ and $\hat{\beta}$, that

$$\begin{aligned} &\biggl(\int_{\mathbb{R}^n} |f(x)|^q \, \delta_E(x)^{(q/p)(n-p+\beta)-n} \, dx \biggr)^{1-1/p+1/q} \\ &= \biggl(\int_{\mathbb{R}^n} |\hat{f}(x)|^{\hat{q}} \, \delta_E(x)^{\hat{q}(n-1+\hat{\beta})-n} \, dx \biggr)^{1/\hat{q}} \lesssim \int_{\mathbb{R}^n} |\nabla \hat{f}(x)| \, \delta_E(x)^{\hat{\beta}} \, dx \, ; \end{aligned} \tag{16}$$

the standard mollification of $\hat{f}$ can be used to justify the last step (see above). Since

$$|\nabla \hat{f}(x)| = |\nabla |f|^{q/\hat{q}}(x)| \le C(p,q) |f(x)|^{q(p-1)/p} |\nabla f(x)|$$

almost everywhere, we can dominate the last integral in (16) with the help of Hölder's inequality by

$$\begin{aligned} &C(p,q) \int_{\mathbb{R}^n} |\nabla f(x)| |f(x)|^{q(p-1)/p} \, \delta_E(x)^{\hat{\beta}} \, dx \\ &\lesssim \biggl(\int_{\mathbb{R}^n} |\nabla f(x)|^p \, \delta_E(x)^{\beta} \, dx \biggr)^{1/p} \biggl(\int_{\mathbb{R}^n} |f(x)|^q \, \delta_E(x)^{(q/p)(n-p+\beta)-n} \, dx \biggr)^{1-1/p} . \end{aligned} \tag{17}$$

The (q,p,β)-Hardy–Sobolev inequality now follows from estimates (16) and (17). □

The proof of Lemma 5.2 is in turn based on the following result (cf. [14, Proposition 6.3]), which utilizes a weighted Poincaré inequality in the case $\beta \le 0$ and a relative isoperimetric inequality when $\beta > 0$.

Lemma 5.4. *Let $E \ne \emptyset$ be a closed porous set in $\mathbb{R}^n$, and let $\beta \in \mathbb{R}$ be such that $\overline{\dim}_{\mathrm{A}}(E) < n + \beta$; if $n - 1 \le \overline{\dim}_{\mathrm{A}}(E) < n$, then we assume in addition that $\beta \le 0$. If M is an admissible set in $\mathbb{R}^n$ and B is a ball in $\mathbb{R}^n$ such that*

$$\int_{M \cap B} \delta_E(x)^{\beta} \, dx = \tfrac{1}{2} \int_B \delta_E(x)^{\beta} \, dx \, , \tag{18}$$

then

$$\int_{M \cap B} \delta_E(x)^{\beta} \, dx \le C \, d(B) \int_{\partial M \cap B} \delta_E(x)^{\beta} \, d\mathcal{H}^{n-1}(x) \, .$$

Here the constant $C > 0$ is independent of M and B.

Proof. We give an outline of the proof and, moreover, provide some additional details that are not very explicit in [14].

Let B be ball a which satisfies the assumption (18). We consider first the case $\beta \le 0$. The key tools in this case are: (A) the inequality $|M \cap B| \le (1 - \kappa)|B|$ with a constant $\kappa > 0$ independent of M and B (this inequality is not explicitly stated in [14] but is used there) and (B) the weighted Poincaré inequality

$$\int_B |u(x) - u_B| \, \delta_E(x)^{\beta} \, dx \le C \, d(B) \int_B |\nabla u(x)| \, \delta_E(x)^{\beta} \, dx \tag{19}$$

that is valid for Lipschitz functions on B (see e.g. [14, pp. 387–388]). We remark that the assumption $\overline{\dim}_{\mathrm{A}}(E) < n + \beta$ is used to establish both (A) and (B). The key observation is that then δ_E^{β} satisfies an A_1-condition by Lemma 5.3, and so property (A) follows from [11, Lemma 2.2 and Theorem 2.9].

Having (A) and (B), the idea is to approximate the characteristic function of M with the Lipschitz functions

$$u_\varepsilon(x) = \begin{cases} 1 & x \in M^\varepsilon \\ \operatorname{dist}(x, \partial M)/\varepsilon\,, & x \in M \setminus M^\varepsilon \\ 0\,, & x \in M^c\,, \end{cases}$$

where $M^\varepsilon = \{x \in M : \operatorname{dist}(x, \partial M) > \varepsilon\}$. Indeed, by using (A), (B), and the coarea formula [8, Theorem 3.2.12], we obtain

$$\begin{aligned} \int_{M\cap B} \delta_E(x)^\beta\, dx &= \lim_{\varepsilon\to 0+} \int_{M^\varepsilon \cap B} \delta_E(x)^\beta\, dx \le C\kappa^{-1}\, d(B) \lim_{\varepsilon\to 0+} \int_B |\nabla u_\varepsilon(x)|\, \delta_E(x)^\beta\, dx \\ &= C\kappa^{-1}\, d(B) \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{B\cap\{x\in M\,:\,\operatorname{dist}(x,\partial M)=t\}} \delta_E(x)^\beta\, d\mathcal{H}^{n-1}(x)\, dt \\ &= C\kappa^{-1}\, d(B) \int_{\partial M\cap B} \delta_E(x)^\beta\, d\mathcal{H}^{n-1}(x)\,. \end{aligned}$$

Actually, in order to make the last limiting step rigorous, one should first perform the estimates above with the family of truncations

$$\delta_E(x)^{\beta,\lambda} = \min\{\delta_E(x)^\beta, \lambda\}\,, \qquad \lambda > 0\,,$$

as weights. The resulting estimates turn out to be uniform in λ; namely, the truncations satisfy an A_1-condition with the same constant as δ_E^β does. Consequently, the weighted Poincaré inequality (19) holds for these truncated weights with a constant independent of the truncation parameter $\lambda > 0$, and this yields the desired estimate also for the weight δ_E^β.

The case where $\beta > 0$ and $d(B) < \operatorname{dist}(B, E)$ is quite straightforward, since in this case we have $\delta_E(x)^\beta \simeq \operatorname{dist}(B, E)^\beta$ for every $x \in B$. This equivalence, together with the assumption (18), used for both M and M^c, allows us to employ the isoperimetric inequality [29, p. 163] as follows:

$$\begin{aligned} \int_{M\cap B} \delta_E(x)^\beta\, dx &\le \operatorname{dist}(B, E)^\beta\, d(B) \min\{|M\cap B|, |M^c\cap B|\}^{(n-1)/n} \\ &\lesssim \operatorname{dist}(B, E)^\beta\, d(B)\, \mathcal{H}^{n-1}(\partial M \cap B) \\ &\simeq d(B) \int_{\partial M\cap B} \delta_E(x)^\beta\, d\mathcal{H}^{n-1}(x)\,. \end{aligned}$$

Finally, we consider the case where $\beta > 0$ and $d(B) \ge \operatorname{dist}(B, E)$. In this case we have the additional assumption that $\overline{\dim}_{\mathrm{A}}(E) < n-1$; this permits us to fix a number $s > 1$ such that $\overline{\dim}_{\mathrm{A}}(E) < n - s < n - 1$. By Theorem 3.4, the set E then has the $P(s)$-property. Set $\eta = d(B)/N$ for some large $N > 1$ that is to be determined later, and write

$$M_1 = M\,, \qquad M_2 = E_\eta^c = \{x \in \mathbb{R}^n : \delta_E(x) \ge \eta\}\,.$$

For simplicity, we first assume that ∂M_2 is a closed $(n-1)$-dimensional manifold of class C^∞; the modifications required in the general case are discussed at the end of the proof.

Since E is a closed porous set and $\overline{\dim}_{\mathrm{A}}(E) < n+\beta$ (the latter is actually a triviality since $\beta > 0$), we have by Lemma 3.6 that

$$
\begin{aligned}
d(B)^n \sup_{x\in B} \delta_E(x)^\beta &\lesssim d(B)^{n+\beta} \simeq \int_B \delta_E(x)^\beta\,dx \le 2\int_{M\cap B} \delta_E(x)^\beta\,dx \\
&\le 2 \sup_{x\in B}\delta_E(x)^\beta\,|M_1\cap M_2\cap B| + 2\int_{M_1\cap M_2^c\cap B}\delta_E(x)^\beta\,dx\,.
\end{aligned}\tag{20}
$$

Using the $P(s)$-property, the last term in (20) can be estimated as follows:

$$
\int_{M_1\cap M_2^c\cap B}\delta_E(x)^\beta\,dx \le \sup_{x\in B}\delta_E(x)^\beta\,|E_\eta\cap B| \le CN^{-s}\sup_{x\in B}\delta_E(x)^\beta\;\,d(B)^n\,.
$$

From the $P(s)$-property, as in [14, Proposition 6.1(4)], or from the estimate $\overline{\dim}_{\mathrm{A}}(E) < n-s$, as in [17, Corollary 5.10], we obtain that

$$
\mathcal{H}^{n-1}(\partial E_\eta\cap B) \le C\eta^{s-1}\,d(B)^{n-s} = CN^{1-s}\,d(B)^{n-1}\,,
$$

where the constant C is independent of both B and η. Using inequalities (11) and (18), it is not hard to show that

$$
|M_1\cap M_2\cap B| \le C|(M_1\cap M_2)^c\cap B|
$$

where C is again independent B and η. An isoperimetric inequality [29, p. 163] (we also refer to [14, Lemma 4.5]) then yields that

$$
\begin{aligned}
|M_1\cap M_2\cap B| &\le C\,d(B)\min\{|M_1\cap M_2\cap B|, |(M_1\cap M_2)^c\cap B|\}^{(n-1)/n} \\
&\le C\,d(B)\big(\mathcal{H}^{n-1}((M_1\cap\partial M_2)\cap B) + \mathcal{H}^{n-1}((\partial M_1\cap M_2)\cap B)\big) \\
&\le C\,d(B)\bigg(\mathcal{H}^{n-1}(\partial E_\eta\cap B) + \eta^{-\beta}\int_{\partial M_1\cap B}\delta_E(x)^\beta\,d\mathcal{H}^{n-1}(x)\bigg) \\
&\le CN^{1-s}\,d(B)^n + CN^\beta\,d(B)^{1-\beta}\int_{\partial M\cap B}\delta_E(x)^\beta\,d\mathcal{H}^{n-1}(x)\,.
\end{aligned}\tag{21}
$$

Since $\sup_{x\in B}\delta_E(x)^\beta \le C\,d(B)^\beta$ and $-s < 1-s$, we conclude from inequalities above that

$$
d(B)^n\sup_{x\in B}\delta_E(x)^\beta \le CN^{1-s}\sup_{x\in B}\delta_E(x)^\beta\,d(B)^n + CN^\beta\;\,d(B)\int_{\partial M\cap B}\delta_E(x)^\beta\,d\mathcal{H}^{n-1}(x)\,.
$$

As $s > 1$, we can now choose N to be so large that $CN^{1-s} < \frac{1}{2}$, whence the first term on the right-hand side can be absorbed to the left-hand side, and the claimed inequality follows as an easy consequence.

When ∂M_2 is not a closed $(n-1)$-dimensional C^∞-manifold, the isoperimetric type inequality (the second step in (21)) might fail. In this case, the following approximation is deployed. First, we fix a function $g\in C^\infty(\mathbb{R}^n\setminus\partial E_\eta)$ such that, for each $x\in\mathbb{R}^n\setminus\partial E_\eta$,

$$
c_1\,\delta_{\partial E_\eta}(x) \le g(x) \le c_2\,\delta_{\partial E_\eta}(x)
$$

and $|\nabla g(x)| \le c_3$. Here the constants c_1, c_2 and c_3 can be chosen to be independent of ∂E_η, c.f. [33, VI.2.1]. For $\varepsilon > 0$, we write

$$
M_2^\varepsilon = M_2\cup\{x\in\mathbb{R}^n\,:\,g(x)\le\varepsilon\}\,.
$$

By applying the assumption $s > 1$ and the $P(s)$-property, we obtain a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ of positive non-critical values of g, converging to zero, such that each $G_j = \{x : g(x) = \varepsilon_j\}$ is an $(n-1)$-dimensional manifold of class C^∞ and

$$\limsup_{j\to\infty} \mathcal{H}^{n-1}(G_j \cap B) \le C\eta^{s-1} d(B)^{n-s}\,,$$

where C is independent of B and η, we refer to [14, p. 385]. Since every ε_j is a non-critical value of g, the set $\partial M_2^{\varepsilon_j}$ is an $(n-1)$-dimensional manifold of class C^∞ and $\partial M_2^{\varepsilon_j} \subset \{x \in \mathbb{R}^n : g(x) = \varepsilon_j\}$. Indeed, the boundary is locally represented by the latter level set which, in turn, is given in terms of a non-critical value of g. We can now adapt the estimates starting from (20), first replacing M_2 by each $M_2^{\varepsilon_j}$, where $j \in \mathbb{N}$, and then applying a limiting argument. □

Remark 5.5. The last case in the proof of Lemma 5.4, where $\beta > 0$ and $d(B) \ge \operatorname{dist}(B, E)$, is the only instance in the proof of Theorem 5.1 where validity of the $P(s)$-property is needed for all $0 \le \eta_1 < \eta_2$, in particular for η_1 close to η_2. Moreover, here it suffices that the $P(s)$-property holds for some $s > 1$.

To conclude the proof of Theorem 5.1, we prove Lemma 5.2 with the help of Lemma 5.4.

Proof of Lemma 5.2. Let us fix an admissible set M in $\mathbb{R}^n$. First we notice that $\overline{\dim}_{\mathrm{A}}(E) < n + \beta$. Indeed, for $\beta \ge 0$ this is trivial by Lemma 3.1(D), and for $\beta < 0$ follows from the assumptions since $0 \le \overline{\dim}_{\mathrm{A}}(E) < q(n-1+\beta) \le n(n-1+\beta)/(n-1) < n+\beta$. Thus $n+\beta \in \mathcal{A}(E)$ by Lemma 3.1(B), and therefore δ_E^β is locally integrable. When $x \in M$, the function

$$\Lambda : [0,\infty) \to \mathbb{R}\,, \qquad \Lambda(r) = \int_{M\cap B(x,r)} \delta_E(y)^\beta\, dy - \tfrac{1}{2}\int_{B(x,r)} \delta_E(y)^\beta\, dy\,,$$

is continuous in r, $\Lambda(r) > 0$ for small values of r, and $\Lambda(r) \to -\infty$ as $r \to \infty$ by Lemma 3.6. Hence, by the intermediate value theorem, for each $x \in M$ there exists a ball $B(x, r(x))$ such that

$$\int_{M\cap B(x,r(x))} \delta_E(y)^\beta\, dy = \tfrac{1}{2}\int_{B(x,r(x))} \delta_E(y)^\beta\, dy\,;$$

see also [14, Proposition 6.2]. By Besicovitch's covering theorem, see e.g. [28, p. 30], we thus find a sequence of balls $\{B_j\}$ in $\{B(x,r(x)) : x \in M\}$ such that M is covered by the union of the balls $\overline{B_j}$ and the overlap of these balls if uniformly bounded.

Using estimate (11) with $s = q(n-1+\beta) > \overline{\dim}_{\mathrm{A}}(E)$, we obtain

$$\begin{aligned}
\Big(\int_M \delta_E(x)^{q(n-1+\beta)-n}\,dx\Big)^{1/q} &\le \sum_j \Big(\int_{B_j\cap M} \delta_E(x)^{q(n-1+\beta)-n}\,dx\Big)^{1/q}\\
&\lesssim \sum_j d(B_j)^{n/q}\big(d(B_j)+\operatorname{dist}(B_j,E)\big)^{n-1+\beta-n/q}\\
&\le \sum_j d(B_j)^{n/q}\big(d(B_j)+\operatorname{dist}(B_j,E)\big)^{\beta}\, d(B_j)^{n-1-n/q}\\
&= \sum_j d(B_j)^{n-1}\big(d(B_j)+\operatorname{dist}(B_j,E)\big)^{\beta},
\end{aligned} \tag{22}$$

where the penultimate step holds since $n-1-n/q \le 0$. We continue the estimate in (22), first using (11) with $s = n+\beta > \overline{\dim}_{\mathrm{A}}(E)$, and then Lemma 5.4, and conclude that

$$\begin{aligned}\sum_j d(B_j)^{n-1}\big(d(B_j)+\operatorname{dist}(B_j,E)\big)^{\beta} &\lesssim \sum_j d(B_j)^{-1}\int_{B_j}\delta_E(x)^{\beta}\,dx\\ &\le 2\sum_j d(B_j)^{-1}\int_{M\cap B_j}\delta_E(x)^{\beta}\,dx\\ &\lesssim \sum_j \int_{\partial M\cap B_j}\delta_E(x)^{\beta}\,d\mathcal{H}^{n-1}(x) \lesssim \int_{\partial M}\delta_E(x)^{\beta}\,d\mathcal{H}^{n-1}(x)\,.\end{aligned}$$

This proves Lemma 5.2. □

6. Necessary conditions for global inequalities

We turn to the necessary conditions for the global Hardy–Sobolev inequality (15). The case $\beta \ge 0$ is straightforward, and in particular yields the necessity part of Theorem 1.1.

Theorem 6.1. *Suppose that $E \neq \emptyset$ is a closed set in $\mathbb{R}^n$. Let $1 \le p, q < \infty$ and $\beta \ge 0$ be such that $q(n-p+\beta)/p \neq n$ and that inequality*

$$\left(\int_{\mathbb{R}^n}|f(x)|^q\,\delta_E(x)^{(q/p)(n-p+\beta)-n}\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|\nabla f(x)|^p\,\delta_E(x)^{\beta}\,dx\right)^{1/p} \tag{23}$$

holds for every $f \in C_0^\infty(\mathbb{R}^n)$. Then

$$\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n-p+\beta)\,.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function which is supported in the ball $B(0,2)$ and satisfies $\varphi(y) = 1$ if $y \in B(0,1)$. Fix a point $x \in E$ and a radius $0 < r < d(E)$. We write

$$f(y) = \varphi((y-x)/r)$$

for every $y \in \mathbb{R}^n$. Since $\beta \ge 0$, we obtain from Lemma 3.1(B) that $\beta + n \in \mathcal{A}(E)$, and since $f(y) = 1$ for all $y \in B(x,r)$ and f is supported in $B(x,2r)$, it follows from Lemma 3.1(E) that

$$\begin{aligned}\int_{B(x,r)}\delta_E(y)^{q(n-p+\beta)/p-n}\,dy &\le \int_{B(x,2r)}|f(y)|^q\,\delta_E(y)^{(q/p)(n-p+\beta)-n}\,dy\\ &\lesssim \left(\int_{B(x,2r)}|\nabla f(y)|^p\,\delta_E(y)^{\beta}\,dy\right)^{q/p}\\ &\lesssim r^{-q}\left(\int_{B(x,2r)}\delta_E(y)^{\beta+n-n}\;dy\right)^{q/p} \lesssim r^{q(n-p+\beta)/p}\,.\end{aligned}$$

This estimate shows that $q(n-p+\beta)/p > 0$ and $q(n-p+\beta)/p \in \mathcal{A}(E)$, but then, by the self-improvement of Aikawa condition (see Lemma 3.1(C)), we have $\overline{\dim}_{\mathrm{A}}(E) < q(n-p+\beta)/p$. (Note that if $q(n-p+\beta)/p > n$ then we are already done). □

The case $\beta < 0$ is more technical, and based upon the self-improvement of reverse Hölder inequalities. In this context, we need to assume that E is both porous and compact. We do not know if it is possible to remove the porosity assumption, but at least the compactness assumption cannot be entirely omitted; for further details, we refer to Example 6.4 and Remark 6.5 below.

Theorem 6.2. *Suppose that $E \neq \emptyset$ is a compact and porous set in $\mathbb{R}^n$. Let $\beta < 0$ and $1 \le p < q < np/(n-p) < \infty$ be such that the Hardy–Sobolev inequality* (23) *holds for every $f \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\overline{\dim}_{\mathrm{A}}(E) < \frac{q}{p}(n-p+\beta)\,.$$

For the proof of Theorem 6.2, we need the following result due to Iwaniec–Nolder [16, Theorem 2], which shows that the exponent on the right-hand side of the reverse Hölder inequality (24) can actually be improved to any $t > 0$.

Proposition 6.3. *Let $0 < s < p$ and $f \in L^p_{\mathrm{loc}}(G)$, where $G \subset \mathbb{R}^n$ is an open set. Suppose that for each cube Q with $2Q \subset G$,*

$$\left(\fint_Q |f(x)|^p\,dx\right)^{1/p} \le A\left(\fint_{2Q} |f(x)|^s\,dx\right)^{1/s}, \tag{24}$$

where the constant $A > 0$ is independent of Q. Then for each $t > 0$, $\sigma > 1$ and each cube Q with $\sigma Q \subset G$,

$$\left(\fint_Q |f(x)|^p\,dx\right)^{1/p} \le C\left(\fint_{\sigma Q} |f(x)|^t\,dx\right)^{1/t},$$

where the constant $C > 0$ depends only on σ, n, p, s, t and A.

Proof of Theorem 6.2. Since E is porous, we may assume that $q(n-p+\beta)/p \neq n$. By the proof of Theorem 6.1, it suffices to show that $\beta + n \in \mathcal{A}(E)$, and by the assumptions and Lemma 3.1(A,B) for this it suffices that $(q/p)\beta + n > 0$ and $(q/p)\beta + n \in \mathcal{A}(E)$.

To this end, let Q be a cube in $\mathbb{R}^n$. If $2Q \cap E \neq \emptyset$, we have for every $y \in Q$ that

$$\delta_E(y)^{(q/p)\beta} = \delta_E(y)^{n+q-nq/p+(q/p)(n-p+\beta)-n} \le C\ell(Q)^{n+q-nq/p}\delta_E(y)^{(q/p)(n-p+\beta)-n}\,;$$

notice that it follows from the assumptions that $n + q - nq/p \ge 0$. Hence, inequality (23) with an appropriate test function that is adapted to Q shows that

$$\begin{aligned}\left(\fint_Q \delta_E(y)^{(q/p)\beta}\,dy\right)^{p/q} &\le \left(\ell(Q)^{n+q-nq/p}\fint_Q \delta_E(y)^{(q/p)(n-p+\beta)-n}\,dy\right)^{p/q}\\ &\le C\fint_{2Q} \delta_E(y)^{\beta}\,dy\,.\end{aligned} \tag{25}$$

On the other hand, if $2Q \cap E = \emptyset$, then $\delta_E(y) \simeq \operatorname{dist}(Q,E)$ for every $y \in Q$ which easily gives inequality (25).

Observe that $(q/p)(n-p+\beta) - n < \beta$; indeed, if $n - p + \beta \le 0$ this is immediate, and if $n - p + \beta > 0$, we obtain

$$\frac{q}{p}(n-p+\beta) < \frac{n}{n-p}(n-p+\beta) < n+\beta\,. \tag{26}$$

Fix $x \in E$ and $R > 0$ such that $E \subset B(x, R/2)$ (recall that E was assumed to be compact). By applying the assumed inequality (23) to a function $f \in C_0^\infty(\mathbb{R}^n)$ which satisfies the condition $f(y) = 1$ if $y \in B(x,R)$, we see that δ_E^β is locally integrable. From this fact and inequality (25) it follows that $\delta_E^\beta \in L^{q/p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $(q/p)\beta + n > 0$. Choose $\varepsilon > 0$ in such a way that $\overline{\dim}_{\mathrm{A}}(E) < \varepsilon\beta + n$. Then, since $0 < 1 < q/p$, we may apply the self-improvement of the reverse Hölder inequality (25), given by Proposition 6.3, to conclude that inequality

$$\left(\fint_Q \delta_E(y)^{(q/p)\beta}\, dy \right)^{p/q} \le C \left(\fint_{2Q} \delta_E(y)^{\varepsilon\beta}\, dy \right)^{1/\varepsilon} \tag{27}$$

holds for all cubes Q in $\mathbb{R}^n$. Here C depends on n, p, q, ε and the constant appearing in the inequality (25). But inequality (27) and the fact that $\varepsilon\beta + n \in \mathcal{A}(E)$ clearly imply that $(q/p)\beta + n \in \mathcal{A}(E)$ as was required. □

To see that some additional assumption is needed for the set E in Theorem 6.2, let us consider the following example.

Example 6.4. Let E be an $(n-k)$-dimensional subspace in $\mathbb{R}^n$, where $1 \le k < n$; then E is a closed and porous set. Let us fix numbers $1 < p < q < np/(n-p) < \infty$ and $\beta = -k$. Then

$$\underline{\dim}_{\mathrm{A}}(E) = n - k > n - p + \beta\,.$$

By Theorem 7.1, which we have postponed to the following section, the (q,p,β)-Hardy–Sobolev inequality (23) actually holds for all functions $f \in C_0^\infty(\mathbb{R}^n)$ satisfying $f(x) = 0$ whenever $x \in E$; note that Corollary 4.1 is not enough to guarantee this. On the other hand, if $f \in C_0^\infty(\mathbb{R}^n)$ and $f(x) \neq 0$ for some $x \in E$, then there is a point $y \in E$ such that $\nabla f(y) \neq 0$. In particular,

$$\int_{\mathbb{R}^n} |\nabla f|^p\, \delta_E^\beta\, dx = \infty\,,$$

and consequently the (q,p,β)-Hardy–Sobolev inequality (23) holds trivially also for such functions. Thus we conclude that the (q,p,β)-Hardy–Sobolev inequality (23) holds for all $f \in C_0^\infty(\mathbb{R}^n)$, but nevertheless $\overline{\dim}_{\mathrm{A}}(E) > (q/p)(n-p+\beta)$; indeed, if $n-p+\beta \le 0$ this is immediate, and if $n-p+\beta > 0$, we can repeat the estimate (26) to see that

$$\frac{q}{p}(n-p+\beta) < n + \beta = \overline{\dim}_{\mathrm{A}}(E)\,.$$

Thus we have shown that, contrary to the case $\beta \ge 0$, the conclusion of Theorem 6.2 does not hold for all closed (and porous) $E \subset \mathbb{R}^n$.

Remark 6.5. While Example 6.4 shows that the compactness assumption in Theorem 6.2 cannot be completely removed, it can still be relaxed. Indeed, the proof of the theorem reveals that we may replace the assumption that E is a compact set by the assumption that E is a closed set such that, for every $x_0 \in E$,

$$\inf \int_{\mathbb{R}^n} |\nabla f|^p\, \delta_E^\beta\, dx < \infty\,,$$

where the infimum ranges over all $f \in C_0^\infty(\mathbb{R}^n)$ such that $f(x_0) = 1$. This weighted p-capacity condition is clearly satisfied by all compact sets E. On the other hand, when the set E is non-compact, it clearly suffices to assume that δ_E^β is locally integrable, which in turn follows, for instance, if we assume that $\overline{\dim}_{\mathrm{M}}(E \cap B) < n + \beta$ for all balls B centered at E. Note that Example 6.4 shows the sharpness of this last condition.

7. The case of thick complements revisited

We have the following slight generalization for the 'thick' case of Corollary 4.1 when $\beta \le 0$ and $p < q < p^*$. While Theorem 7.1 has also independent interest, the main reason why we have included it here is that Example 6.4 relies on this result.

Theorem 7.1. *Let $\beta \le 0$ and $1 < p < q < np/(n-p) < \infty$. Suppose that G is a proper open set in $\mathbb{R}^n$, $n \ge 2$, such that $\underline{\dim}_{\mathrm{A}}(G^c) > n - p + \beta$; if G is unbounded, we assume that G^c is unbounded, as well. Then there is a constant $C > 0$ such that*

$$\left(\int_G |f(x)|^q \delta_{\partial G}(x)^{\frac{q}{p}(n-p+\beta)-n}\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |\nabla f(x)|^p\,\delta_{\partial G}(x)^\beta\,dx\right)^{1/p} \tag{28}$$

whenever $f \in C_0^\infty(\mathbb{R}^n)$ and $f(x) = 0$ for all $x \in G^c$.

The proof of Theorem 7.1 is based upon a general scheme, built in [6], in combination with 'pointwise Hardy' techniques, developed in [18, 19, 24]. The latter approach yields the following lemma, which is a modification of the results in [24].

By $\mathcal{W}(G)$ we denote a Whitney decomposition (as in [33, Section VI.1]) of a proper open set G in $\mathbb{R}^n$. That is, the union of these dyadic cubes (whose interiors are pairwise disjoint) is the whole of G and, moreover,

$$d(Q) \le \operatorname{dist}(Q, \partial G) \le 4\,d(Q) \tag{29}$$

whenever $Q \in \mathcal{W}(G)$.

Lemma 7.2. *Let $\beta \le 0$ and $1 < p < q < np/(n-p) < \infty$, and write $L = 10\sqrt{n}$. Suppose that G is a proper open set in $\mathbb{R}^n$, $n \ge 2$, such that $\underline{\dim}_{\mathrm{A}}(G^c) > n - p + \beta$; if G is unbounded, we assume that G^c is unbounded, as well. Then there exists an exponent $1 < r_0 < p$ as follows: For every $r_0 < r < p$ there is a constant $C > 0$ such that*

$$|f_Q|^q \lesssim \ell(Q)^{q-\frac{\beta q}{p}}\left(\fint_{LQ} |\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx\right)^{q/r} \tag{30}$$

whenever $Q \in \mathcal{W}(G)$ and $f \in C_0^\infty(\mathbb{R}^n)$ is such that $f(x) = 0$ for all $x \in G^c$.

Proof. The assumption $\underline{\dim}_{\mathrm{A}}(G^c) > n - p + \beta$ implies that there exists a positive number $\lambda > n - p + \beta$ such that $\ell(Q)^\lambda \lesssim \mathcal{H}^\lambda_\infty(G^c \cap LQ)$ for all $Q \in \mathcal{W}(G)$; see e.g. [17, Remark 2.3] and notice that here we need to know the unboundedness of G^c if G is unbounded. On the other hand, by a simple modification of [24, Lemma 3.1(a)] there exists $1 < r_0 < p$ such that we have for all $r_0 < r < p$ that

$$\mathcal{H}^\lambda_\infty(G^c \cap LQ)|f_Q|^r \lesssim \ell(Q)^{r-\frac{\beta r}{p}-n+\lambda}\int_{LQ} |\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx, \tag{31}$$

and actually the proof of [24, Lemma 3.1(a)] shows that for (31) it is enough to assume that $f(x) = 0$ for all $x \in G^c$. Combining (31) with the above estimate $\ell(Q)^\lambda \lesssim \mathcal{H}^\lambda_\infty(G^c \cap LQ)$, and taking everything to power q/r yields the desired estimate (30). $\square$

Proof of Theorem 7.1. Write $L = 10\sqrt{n}$, and let $1 < r_0 < p$ be as in Lemma 7.2. Fix a function $f \in C_0^\infty(\mathbb{R}^n)$ such that $f(x) = 0$ for all $x \in G^c$. By the covering property of Whitney cubes and inequality (29),

$$\int_G |f(x)|^q\,\delta_{\partial G}(x)^{\frac{q}{p}(n-p+\beta)-n}\,dx \lesssim \sum_{Q\in\mathcal{W}(G)} \ell(Q)^{\frac{q}{p}(n-p+\beta)}\left\{\fint_Q |f(x)-f_Q|^q\,dx + |f_Q|^q\right\}. \tag{32}$$

We choose $r_0 < r < p$ such that $1/r - 1/q < 1/n$. Then the integral on the right-hand side of (32) can be estimated using a (q,r)-Poincaré inequality for cubes:

$$\begin{aligned}\fint_Q |f(x)-f_Q|^q\,dx &\lesssim \ell(Q)^q\biggl(\fint_Q |\nabla f(x)|^r\,dx\biggr)^{q/r}\\ &\lesssim \ell(Q)^{q-\frac{\beta q}{p}}\biggl(\fint_{LQ}|\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx\biggr)^{q/r}.\end{aligned}$$

On the other hand, a corresponding estimate for the second integral on the right-hand side of (32) follows from Lemma 7.2 since $r_0 < r < p$. That is,

$$|f_Q|^q \lesssim \ell(Q)^{q-\frac{\beta q}{p}}\biggl(\fint_{LQ}|\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx\biggr)^{q/r}.$$

Insertion of these two estimates to (32) yields that

$$\begin{aligned}&\int_G |f(x)|^q\,\delta_{\partial G}(x)^{\frac{q}{p}(n-p+\beta)-n}\,dx\\ &\qquad\lesssim \sum_{Q\in\mathcal{W}(G)} \ell(Q)^{\frac{q}{p}(n-p+\beta)}\ell(Q)^{q-\frac{\beta q}{p}}\biggl(\fint_{LQ}|\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx\biggr)^{q/r}\\ &\qquad\lesssim \sum_{Q\in\mathcal{W}(G)} |Q|^{q/p}\biggl(\fint_{LQ}|\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx\biggr)^{q/r}.\end{aligned}\tag{33}$$

Since $1 < r < p < q < \infty$, we have for every $g\in L^{p/r}(\mathbb{R}^n)$ that

$$\sum_{Q\in\mathcal{W}(G)} |Q|^{q/p}\biggl(\fint_{LQ}|g(x)|\,dx\biggr)^{q/r} \lesssim \biggl(\int_{\mathbb{R}^n}|g(x)|^{p/r}\,dx\biggr)^{q/p}.\tag{34}$$

Indeed, to obtain inequality (34), one first dominates the left-hand side of (34) by (a constant multiple of)

$$\sum_{Q\in\mathcal{W}(G)}\int_{\mathbb{R}^n}\chi_Q(x)\bigl(I_\sigma|g|(x)\bigr)^{q/r}\,dx$$

and then applies the pairwise disjointedness of the interiors of Whitney cubes and the boundedness of the Riesz potential $I_\sigma\colon h\mapsto |x|^{\sigma-n}*h$, where $\sigma = nr(q/p-1)/q\in(0,n)$, from $L^{p/r}(\mathbb{R}^n)$ to $L^{q/r}(\mathbb{R}^n)$; we refer to [13, Theorem 1].

Now estimates (33) and (34) (the latter with $g = |\nabla f|^r\delta_{\partial G}^{\beta r/p}$; if $g\notin L^{p/r}(\mathbb{R}^n)$ the claim is trivial) show that, indeed

$$\begin{aligned}\int_G |f(x)|^q\,\delta_{\partial G}(x)^{\frac{q}{p}(n-p+\beta)-n}\,dx &\lesssim \sum_{Q\in\mathcal{W}(G)} |Q|^{q/p}\biggl(\fint_{LQ}|\nabla f(x)|^r\,\delta_{\partial G}(x)^{\beta r/p}\,dx\biggr)^{q/r}\\ &\lesssim \biggl(\int_{\mathbb{R}^n}|\nabla f(x)|^p\,\delta_{\partial G}(x)^{\beta}\,dx\biggr)^{q/p},\end{aligned}$$

and inequality (28) follows. □

8. Dimensional dichotomy for the complement

In this final section, we establish dimensional dichotomy results for the complements of open sets which admit Hardy–Sobolev inequalities. In particular, the global dichotomy result, stated in Theorem 4.6, is proved at the end of this section. Before that, we study local versions of these results in the following Propositions 8.1 and 8.2 corresponding to the cases $\beta \geq 0$ and $\beta < 0$, respectively. Similar results for the unweighted p-Hardy inequality were proven in [20], and for the weighted (p,β)-Hardy inequality in [23]. Recall that in the 'Hardy' case $q = p$, both of the dimensional bounds for the complement G^c are strict, and that we do not know if this is true for the lower bounds also when $q > p$ (see the discussion after Example 4.7).

Proposition 8.1. *Let $G \subset \mathbb{R}^n$ be an open set and assume that $1 \leq p \leq q < np/(n-p) < \infty$ and $\beta \geq 0$ are such that $q(n-p+\beta)/p \neq n$ and G admits a (q,p,β)-Hardy–Sobolev inequality* (3)*. Then for each (closed) ball $B = \overline{B}(x,R) \subset \mathbb{R}^n$ either*

$$\overline{\dim}_{\mathrm{A}}(G^c \cap B) < \frac{q}{p}(n-p+\beta) \quad \textit{or} \quad \dim_{\mathrm{H}}(G^c \cap 2B) \geq n-p+\beta\,.$$

Proof. The proof is based on the approach in [20], and uses also ideas from the proofs of Lemmas 6.4 and 6.6 in [22]. Fix $x_0 \in \mathbb{R}^n$ and $R > 0$, and write $B_0 = \overline{B}(x_0,R) \subset \mathbb{R}^n$. It suffices to show that if $\dim_{\mathrm{H}}(G^c \cap 2B_0) < n-p+\beta$, then $\overline{\dim}_{\mathrm{A}}(G^c \cap B_0) < (q/p)(n-p+\beta)$, and without loss of generality we may also assume that $0 < q(n-p+\beta)/p < n$.

If, indeed, $\dim_{\mathrm{H}}(G^c \cap 2B_0) < n-p+\beta$, then $\mathcal{H}^{n-p+\beta}_\infty(G^c \cap 2B_0) = 0$. Thus, assuming that $f \in C_0^\infty(G \cup B(x_0,2R))$ is fixed (and $f \not\equiv 0$), we can find balls $B_i^j = B(w_{ij}, r_{ij})$ with $w_{ij} \in G^c \cap 2B_0$, $i = 1,\ldots,N_j$, so that $G^c \cap 2B_0 \subset \bigcup_{i=1}^{N_j} B_i^j$, and

$$\sum_{i=1}^{N_j} r_{ij}^{n-p+\beta} \leq \|f\|_\infty^{-p}\, 2^{-j}$$

for all $j \in \mathbb{N}$. Define cut-off functions $\psi_j(y) = \min_i\{1, r_{ij}^{-1}\delta_{2B_i^j}(y)\}$ and set $f_j = \psi_j f$. Then f_j is clearly a Lipschitz function with a compact support in G and

$$|\nabla f_j|^p \lesssim \sum_i r_{ij}^{-p}\chi_{3B_i^j}|f|^p + |\nabla f|^p$$

almost everywhere. Moreover, we have that $\lim_{j\to\infty} f_j = f$ in G.

Since $1 \leq p,q < \infty$ and $\delta_{\partial G}$ is bounded and away from zero in the support of f_j, the (q,p,β)-Hardy–Sobolev inequality holds also for the function f_j by the standard approximation, and this with the choice of the balls B_i^j implies that

$$\begin{aligned}
\left(\int_G |f_j|^q \delta_{\partial G}^{(q/p)(n-p+\beta)-n}\,dx\right)^{p/q} &\leq C\int_G |\nabla f_j|^p \delta_{\partial G}^\beta\,dx \\
&\leq C\left\{\|f\|_\infty^p \sum_{i=1}^{N_j} |B_i^j| r_{ij}^{-p+\beta} + \int_G |\nabla f|^p \delta_{\partial G}^\beta\,dx\right\} \\
&\leq C2^{-j} + C\int_G |\nabla f|^p \delta_{\partial G}^\beta\,dx\,.
\end{aligned}$$

By Fatou's lemma this argument shows that the (q,p,β)-Hardy–Sobolev inequality (3) holds for all functions $f \in C_0^\infty(G \cup B(x_0,2R))$ with a constant $C = C(C_1,q,p,\beta,n) > 0$, where $C_1 > 0$ is the constant for which the (q,p,β)-Hardy–Sobolev inequality was assumed to hold for all $f \in C_0^\infty(G)$.

Let then $w \in G^c \cap B_0$ and $0 < r < R/2$. By the above reasoning, we can now use the (q,p,β)-Hardy–Sobolev inequality for the function $f(y) = \varphi((y-w)/r)$, where φ is as in the proof of Theorem 6.1; indeed, $f \in C_0^\infty(G \cup B(x_0, 2R))$. Since $|G^c \cap 2B_0| = 0$, a calculation similar to the one given in the proof of Theorem 6.1 shows that

$$\int_{B(w,r)} \delta_{G^c}(x)^{(q/p)(n-p+\beta)-n}\,dx = \int_{B(w,r)} \delta_{\partial G}(x)^{(q/p)(n-p+\beta)-n}\,dx \le C r^{(q/p)(n-p+\beta)}. \tag{35}$$

Here the constant C is independent of both w and r. Using Lemma 3.1(E) and the fact that $\delta_{G^c} \le \delta_{G^c \cap B_0}$, we infer from (35) that $(q/p)(n-p+\beta) \in \mathcal{A}(G^c \cap B_0)$, and thus Lemma 3.1(C) yields the claim $\overline{\dim}_{\mathrm{A}}(G^c \cap B_0) < (q/p)(n-p+\beta)$. □

For $\beta < 0$, the case $q > p$ involves additional difficulties compared to the case $q = p$ that was considered in [23].

Proposition 8.2. *Let $G \subset \mathbb{R}^n$ be an open set. Assume that $1 \le p < q < np/(n-p) < \infty$ and $\beta < 0$ are such that G admits a (q,p,β)-Hardy–Sobolev inequality* (3)*. Then, by writing $\lambda = 8\sqrt{n}$, we have either*

$$\overline{\dim}_{\mathrm{A}}(G^c \cap B) < \frac{q}{p}(n-p+\beta) \quad \text{or} \quad \underline{\dim}_{\mathrm{M}}(G^c \cap \lambda B) \ge n-p+\beta$$

whenever $B = \overline{B}(x,R) \subset \mathbb{R}^n$ is a ball such that $\delta_{G^c}^\beta \in L^1(\lambda B)$ and $G^c \cap \lambda B$ is porous.

Proof. The proof follows the lines of the proof of Proposition 8.1, but many of the details are different and hence we include here a complete proof. Fix a ball $B_0 = \overline{B}(x_0,R)$ such that $\delta_{G^c}^\beta \in L^1(\lambda B_0)$ and $\overline{\dim}_{\mathrm{A}}(G^c \cap \lambda B_0) < n$, i.e. $G^c \cap \lambda B_0$ is porous; recall Lemma 3.1(D). It suffices to show $\overline{\dim}_{\mathrm{A}}(G^c \cap B_0) < (q/p)(n-p+\beta)$ while assuming that $\underline{\dim}_{\mathrm{M}}(G^c \cap \lambda B_0) < n-p+\beta$. Without loss of generality, we may also assume that $0 < q(n-p+\beta)/p < n$.

Fix $f \in C_0^\infty(G \cup B(x_0, 6\sqrt{n}R))$ such that $f \not\equiv 0$. Since $\underline{\dim}_{\mathrm{M}}(G^c \cap \lambda B_0) < n-p+\beta$, there is a sequence $(r_j)_{j\in\mathbb{N}}$ of positive numbers, converging to zero and satisfying the following two conditions for each $j \in \mathbb{N}$: (i) $r_j \le 2\sqrt{n}R$ and $r_j \le \delta_{G^c}(y)$ whenever $y \in G \cap (B(x_0, 6\sqrt{n}R))^c$ is such that $f(y) \ne 0$, and (ii) there are balls $B_i^j = B(w_{ij}, r_j)$ with $w_{ij} \in G^c \cap \lambda B_0$, $i = 1, \ldots, N_j$, so that $G^c \cap \lambda B_0 \subset \bigcup_{i=1}^{N_j} B_i^j$, and

$$N_j r_j^{n-p+\beta} \le 2^{-j} \|f\|_\infty^{-p}.$$

Define cut-off functions $\psi_j(y) = \min_i\{1, r_j^{-1}\delta_{2B_i^j}(y)\}$ and let $f_j = \psi_j f$ for each $j \in \mathbb{N}$. Then f_j is a Lipschitz function that is compactly supported in G. Moreover, a careful inspection shows that

$$|\nabla f_j|^p \lesssim \sum_i r_j^{-p} \chi_{3B_i^j} |f|^p \chi_{\{r_j \le \delta_{G^c}\}} + |\nabla f|^p \tag{36}$$

almost everywhere, and $\lim_{j\to\infty} f_j = f$ in G. Let us remark that compared to the proof of Proposition 8.1, the new factor $\chi_{\{r_j \le \delta_{G^c}\}}$ appears. This will be needed in the subsequent arguments due to the assumption that $\beta < 0$, and this is the reason why the upper bound is now given in terms of the lower Minkowski dimension instead of the Hausdorff dimension.

Using approximation and Fatou's lemma, and applying the assumed (q,p,β)-Hardy–Sobolev inequality and inequality (36), we obtain

$$\begin{aligned}\left(\int_G |f|^q \delta_{\partial G}^{(q/p)(n-p+\beta)-n}\,dx\right)^{p/q} &\le \liminf_{j\to\infty}\left(\int_G |f_j|^q \delta_{\partial G}^{(q/p)(n-p+\beta)-n}\,dx\right)^{p/q}\\ &\le \lim_{j\to\infty} C\left(2^{-j}+\int_G |\nabla f|^p \delta_{\partial G}^{\beta}\,dx\right) = C\int_G |\nabla f|^p\delta_{\partial G}^{\beta}\,dx\,.\end{aligned}$$

Hence the (q,p,β)-Hardy–Sobolev inequality (3) holds in fact for all functions $f\in C_0^\infty(G\cup B(x_0,6\sqrt{n}R))$ with a constant $C=C(C_1,q,p,\beta,n)>0$, where $C_1>0$ is the constant for which the (q,p,β)-Hardy–Sobolev inequality holds for all $f\in C_0^\infty(G)$.

Arguing as in the proof of Theorem 6.2 and using the fact that $|G^c\cap\lambda B_0|=0$, we obtain a constant $\kappa>0$ such that

$$\left(\fint_Q \delta_{G^c}(y)^{(q/p)\beta}\,dy\right)^{p/q}\le \kappa\fint_{2Q}\delta_{G^c}(y)^{\beta}\,dy \tag{37}$$

for all cubes Q such that $2Q\subset B(x_0,6\sqrt{n}R)$. However, we will actually need an improved version of (37), as in the proof of Theorem 6.2. Let Q_0 be an open cube centered at x_0 and with side length $4R$. Then $2Q_0\subset B(x_0,6\sqrt{n}R)$. Choose $\varepsilon>0$ for which $\overline{\dim}_{\mathrm{A}}(G^c\cap\lambda B_0)<\varepsilon\beta+n$. Let us observe that $1<q/p$ and $\delta_{G^c}^{\beta}\in L^{q/p}(Q_0)$ by inequality (37) and the assumptions. Hence, by Proposition 6.3 there exists a constant $C>0$ such that the left hand side of (37) is dominated by $C\big(\fint_{2Q}\delta_{G^c}^{\varepsilon\beta}\big)^{1/\varepsilon}$ if $2Q\subset Q_0$.

With the help of the (q,p,β)-Hardy–Sobolev inequality for $C_0^\infty(G\cup B(x_0,6\sqrt{n}R))$ and the improved version of inequality (37), we can now proceed as follows. Fix a cube Q that is centered at $G^c\cap B_0$ and whose side length is bounded by $\frac{2}{5}R$; then $4Q\subset Q_0$ and $\delta_{G^c}(y)=\delta_{G^c\cap\lambda B_0}(y)$ whenever $y\in 4Q$. Hence,

$$\begin{aligned}\left(\int_Q \delta_{G^c}(y)^{(q/p)(n-p+\beta)-n}\,dy\right)^{p/q} &\le C\ell(Q)^{n-p}\fint_{2Q}\delta_{G^c}(y)^{\beta}\,dy\\ &\le \ell(Q)^{n-p}\left(\fint_{2Q}\delta_{G^c}(y)^{(q/p)\beta}\,dy\right)^{p/q}\\ &\le C\ell(Q)^{n-p}\left(\fint_{4Q}\delta_{G^c}(y)^{\varepsilon\beta}\,dy\right)^{1/\varepsilon}\\ &= C\ell(Q)^{n-p}\left(\fint_{4Q}\delta_{G^c\cap\lambda B_0}(y)^{\varepsilon\beta}\,dy\right)^{1/\varepsilon}\le C\ell(Q)^{n-p+\beta}\,.\end{aligned}$$

Since $\delta_{G^c}\le\delta_{G^c\cap B_0}$ we can again use Lemma 3.1(E,C) to conclude that

$$\overline{\dim}_{\mathrm{A}}(G^c\cap B_0)<\frac{q}{p}(n-p+\beta)\,. \qquad \square$$

The global dichotomy results of Theorem 4.6 can now be proved using similar arguments as in the local results of Propositions 8.1 and 8.2. We outline the main ideas:

Proof of Theorem 4.6. Let us first consider the case $\beta\ge 0$. It suffices to prove that inequality

$$\overline{\dim}_{\mathrm{A}}(G^c)<\frac{q}{p}(n-p+\beta) \tag{38}$$

holds if $\dim_{\mathrm{H}}(G^c) < n-p+\beta$ and $q(n-p+\beta)/p < n$. Fix $\omega \in G^c$ and $0 < r < d(G^c)$, and write $B_0 = \overline{B}(\omega, 3r)$. Since $\dim_{\mathrm{H}}(G^c \cap 2B_0) \le \dim_{\mathrm{H}}(G^c) < n-p+\beta$, we can repeat the proof of Proposition 8.1 to obtain that

$$\int_{B(\omega,r)} \delta_{G^c}(x)^{(q/p)(n-p+\beta)-n}\,dx \le Cr^{(q/p)(n-p+\beta)}\,,$$

where the constant C is independent of both ω and r. Thus, $(q/p)(n-p+\beta) \in \mathcal{A}(G^c)$ and inequality (38) follows from Lemma 3.1(C).

In the case $\beta < 0$, the claim for $q = p$ follows from the results in [23], and hence it suffices to prove inequality (38) while assuming that $p < q$ and $\underline{\dim}_{\mathrm{M}}(G^c) < n-p+\beta$. In particular, then $|G^c| = 0$. Arguing as in the proof of Proposition 8.2, we find that the (q,p,β)-Hardy–Sobolev inequality (23) actually holds for all $f \in C_0^\infty(\mathbb{R}^n)$. In particular, the assumptions of Theorem 6.2 are valid with $E = G^c$, and so inequality (38) follows. □

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