WEIGHTED POINTWISE HARDY INEQUALITIES

PEKKA KOSKELA AND JUHA LEHRBÄCK

ABSTRACT. We introduce the concept of a visual boundary of a domain $\Omega \subset \mathbb{R}^n$ and show that the weighted Hardy inequality $\int_\Omega |u|^p d_\Omega{}^{\beta-p} \leq C \int_\Omega |\nabla u|^p d_\Omega{}^\beta$, where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, holds for all $u \in C_0^\infty(\Omega)$ with exponents $\beta < \beta_0$ when the visual boundary of Ω is sufficiently large. Here $\beta_0 = \beta_0(p, n, \Omega)$ is explicit, essentially sharp, and may even be greater than $p-1$, which is the known bound for smooth domains. For instance, in the case of the usual von Koch snowflake domain the sharp bound is shown to be $\beta_0 = p - 2 + \lambda$, with $\lambda = \log 4/\log 3$. These results are based on new pointwise Hardy inequalities.

1. Introduction

The classical Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx \,, \tag{1}$$

where $1 < p < \infty$, $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, and $u \in C_0^\infty(\Omega)$, was first considered by G. H. Hardy [6], [7] in the one-dimensional, unweighted ($\beta = 0$) case with $\Omega = (0, \infty) \subset \mathbb{R}$. It was later proved by Hardy et al. (cf. [8, Section 9.8] and references therein) that if u is an absolutely continuous function and $u(0) = 0$, the weighted inequality (1) holds in $(0, \infty)$ with a constant $C = C(p, \beta) > 0$ whenever $\beta < p - 1$. Since these first results, it has been a question of considerable interest to find conditions which guarantee that the (p, β)-Hardy inequality (1) is valid in some more general settings. The situation we are interested in this paper is the one where Ω is a domain in $\mathbb{R}^n$, $n \geq 2$, and (1) holds for all functions $u \in C_0^\infty(\Omega)$ with a constant $C_\Omega = C_\Omega(p, \beta) > 0$. If this is the case, we say that the domain Ω admits the (p, β)-Hardy inequality.

The main purpose of this paper is to establish new sufficient conditions for domains to admit these Hardy inequalities. Before going into our results, let us briefly review some of the know conditions for (p, β)-Hardy inequalities in this setting. First of all, we state the following version of the well-known result due to Ancona [1] (in the case $p = 2$), Lewis [15], and Wannebo [25], concerning the unweighted case $\beta = 0$, i.e. the p-Hardy inequality.

Theorem 1.1 (Ancona, Lewis, Wannebo). *Let $\Omega \subset \mathbb{R}^n$ be a domain and suppose that there exist an exponent $0 \leq \lambda \leq n$ and a constant $C > 0$ such that*

$$\mathcal{H}^\lambda_\infty\big(\Omega^c \cap B(w, r)\big) \geq C r^\lambda \quad \textit{for all } w \in \partial\Omega \textit{ and } r > 0. \tag{2}$$

Then Ω admits the p-Hardy inequality whenever $p > n - \lambda$ (and $1 < p < \infty$).

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Here $\mathcal{H}^\lambda_\infty$ is the λ-dimensional Hausdorff content. To be precise, Wannebo [25] proved in fact more, namely that under condition (2) there exists some small positive $\beta_0 = \beta_0(p, n, \Omega)$ such that Ω admits the weighted (p, β)-Hardy inequality (1) for all $\beta < \beta_0$. In the original results corresponding to Theorem 1.1 the density condition (2) was actually given in terms of the local p-capacity of the complement of Ω, but such a condition is equivalent to (2); see Section 2.4 for a brief discussion on these conditions. We also remark that if $\Omega \subsetneq \mathbb{R}^n$, that is, Ω is a proper subdomain of $\mathbb{R}^n$, then Ω satisfies (2) with $\lambda = 0$, and hence Ω admits the p-Hardy inequality for every $p > n$.

For a general domain satisfying condition (2) there is no explicit expression for the number β_0 mentioned above. However, it is already due to Nečas [19] that if $\partial\Omega$ is sufficiently nice, we can take $\beta_0 = p - 1$:

Theorem 1.2 (Nečas)**.** *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p, β)-Hardy inequality for all $\beta < p - 1$.*

See also Kufner [14] for this and related results for domains with Hölder-boundary. The bound $\beta < p - 1$ for Lipschitz domains is the best possible, as easy calculations show that the (p, β)-Hardy inequality fails e.g. in the unit ball $B(0, 1) \subset \mathbb{R}^n$ for every $\beta \geq p - 1$.

Notice that a Lipschitz domain Ω satisfies condition (2) with $\lambda = n - 1$, and so the bound $\beta < p - 1$ can be written as $\beta < p - n + \lambda$. Moreover, the bound $p > n - \lambda$ from Theorem 1.1, where $\beta = 0$, also takes the form $\beta < p - n + \lambda$. Hence the number $\beta_0 = p - n + \lambda$ appears to be, at least in these cases, somehow critical for (p, β)-Hardy inequalities. One could then proceed to ask if condition (2), with an exponent $0 \leq \lambda \leq n$, would be sufficient to guarantee (p, β)-Hardy inequalities for all $\beta < p - n + \lambda$. For example, if Ω is the von Koch snowflake domain in the plane, then (2) holds with $\lambda = \dim \partial\Omega = \log 4/ \log 3$, and, indeed, direct calculations indicate that Ω should admit the (p, β)-Hardy inequality for $\beta < p - 2 + \lambda$.

It turns out, however, that this is not true in general; in this paper we give a construction which proves the following result.

Theorem 1.3. *For every $1 < \lambda < 2$ there exists a simply connected domain $\Omega = \Omega_\lambda \subset \mathbb{R}^2$ which satisfies condition (2) with the exponent λ, but fails to admit the $(p, p-1)$-Hardy inequality for any $1 < p < \infty$. In particular, for each $1 < p < \infty$, the (p, β)-Hardy inequality* does not *hold in Ω for every $\beta < p - 2 + \lambda$.*

See Example 7.3 for the proof of Theorem 1.3. Actually, in Example 7.4 we construct for any $1 \leq \sigma < \lambda < 2$ a domain $\Omega = \Omega_{\lambda,\sigma}$ such that Ω satisfies condition (2) with the given exponent λ, and admits the (p, β)-Hardy inequality for all $\beta < p - 2 + \lambda$ *but* for $\beta(\sigma) = p - 2 + \sigma$. We conclude that the thickness of the complement Ω^c, in the sense of (2) with an exponent λ, is not sufficient to guarantee (p, β)-Hardy inequalities for all $\beta < p - n + \lambda$.

Despite these negative results, the von Koch snowflake example leads one to ask if it is possible to obtain Hardy inequalities for all $\beta < p-n+\lambda$ under some additional conditions on the domain. For instance, it is well-known that snowflake type domains are John domains, which implies that all the boundary points are "visible" or "easily accessible" from the points inside the domain. This motivates the definition of the *visual boundary* $v_x(c)$*–*$\partial\Omega$ *near* $x \in \Omega$ (see Section 4). The next result, the main theorem of this paper, states that if the visual part of the boundary is large enough (in terms of Hausdorff content) near every $x \in \Omega$, then Ω admits the desired Hardy inequalities.

Theorem 1.4. *Let* $1 < p < \infty$ *and let* $\Omega \subset \mathbb{R}^n$ *be a domain. Assume that there exist* $0 \le \lambda \le n$, $c \ge 1$, *and* $C > 0$ *such that*

$$\mathcal{H}^\lambda_\infty\big(v_x(c)\text{–}\partial\Omega\big) \ge C d_\Omega(x)^\lambda \quad \textit{for every } x \in \Omega.$$

Then Ω *admits the* (p,β)*-Hardy inequality for all* $\beta < p - n + \lambda$*.*

To make the statement of Theorem 1.4 a bit more accessible, let us record the following rather immediate corollary to Theorem 1.4 for a well-known class of domains, *uniform domains*. See Section 2.5 for the definition of uniform domains and Proposition 4.3 for the proof of this corollary.

Corollary 1.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a uniform domain and let* $1 < p < \infty$*. Assume that there exist* $0 \le \lambda \le n$ *and* $C > 0$ *such that*

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap B\big(x, 2d_\Omega(x)\big)\big) \ge C d_\Omega(x)^\lambda \quad \textit{for every } x \in \Omega. \tag{3}$$

Then Ω *admits the* (p,β)*-Hardy inequality for all* $\beta < p - n + \lambda$*.*

We refer to estimates of the type (3) as inner boundary density conditions. As a direct consequence of Corollary 1.5 we obtain the fact that a von Koch snowflake type domain Ω, as in Example 7.1, with $\dim \partial\Omega = \lambda \in (1,2)$, really admits (p,β)-Hardy inequalities for all $1 < p < \infty$ and $\beta < \beta_0 = p - 2 + \lambda$; notice that here $\beta_0 > p - 1$ is strictly larger than the critical exponent for domains with a "smooth" boundary (Theorem 1.2). Furthermore, this β_0 is sharp, since Ω fails to admit the (p,β)-Hardy inequality whenever $\beta \ge \beta_0$; see Example 7.1.

We should however mention that a domain $\Omega \subset \mathbb{R}^n$ being uniform — or more generally John — is by no means necessary for Ω to admit Hardy inequalities; see the examples from Section 7. On the other hand, even though the visual boundary condition in Theorem 1.4 is closely related to John domains, the conclusion of Theorem 1.4 does not hold for all John domains satisfying the density condition (3); the domain constructed in Example 7.3 works as a counterexample. Nevertheless, we prove that each simply connected John domain in the plane admits (p,β)-Hardy inequalities for all $\beta < p-1$. This improves on the result of Nečas (Theorem 1.2), where Ω was assumed to be a bounded Lipschitz domain; each Lipschitz domain is in fact a John domain. More generally, if $\Omega \subset \mathbb{R}^n$, $n \ge 3$, is a John domain, and if in addition Ω is quasiconformally equivalent to the unit ball of $\mathbb{R}^n$, then Ω admits (p,β)-Hardy inequalities for all $\beta < p - 1$. See Section 6 for the proofs.

In spite of these results on John domains, we presume that, in the case $\beta < p - 1$, the visibility of the boundary plays no essential role. We state this as a conjecture.

Conjecture 1.6. *Let $1 < p < \infty$. If $\Omega \subset \mathbb{R}^n$ satisfies the inner boundary density condition (3) with the exponent $\lambda = n - 1$, then Ω admits the (p, β)-Hardy inequality whenever $\beta < p-1$. In particular, if Ω is a simply connected domain in the plane, then Ω admits the (p, β)-Hardy inequality for all $\beta < p - 1$.*

In order to prove the (p, β)-Hardy inequalities of Theorem 1.4, we in fact establish as a tool stronger inequalities, *pointwise (p, β)-Hardy inequalities*, which are also of their own independent interest. In the unweighted case $\beta = 0$, pointwise Hardy inequalities were introduced independently by Hajłasz [5] and Kinnunen and Martio [13]. Generalizing the approach of [5] we call the inequality

$$|u(x)| \le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| d_\Omega{}^{\beta/p}\big)(x), \tag{4}$$

where $u \in C_0^\infty(\Omega)$ and $1 < q < p$, a pointwise (p, β)-Hardy inequality; see Proposition 3.1 for the justification of this notion. In (4) we denote $M_{R,q} f = (M_R |f|^q)^{1/q}$, where $M_R f$ is the usual restricted Hardy-Littlewood maximal function of f. We say that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality if there exist *some* $1 < q < p$ and a constant $C > 0$ such that the inequality (4) holds for every $u \in C_0^\infty(\Omega)$ and all $x \in \Omega$ with these q and C.

It was proved in [5] and [13] that, under the assumptions of Theorem 1.1, the domain $\Omega \subset \mathbb{R}^n$ actually admits the pointwise p-Hardy inequality for all $p > n - \lambda$. Likewise, we prove that under the assumptions of Theorem 1.4 the domain $\Omega \subset \mathbb{R}^n$ actually admits the pointwise (p, β)-Hardy inequality for all $\beta < p - n + \lambda$; see Theorem 5.1. However, the pointwise (p, β)-Hardy inequality is not equivalent to the usual (p, β)-Hardy inequality, since there are domains which admit the latter for some p and β, but where the corresponding pointwise inequality fails, the easiest example being the punctured ball $B(0, 1) \setminus \{0\} \subset \mathbb{R}^n$ (cf. Example 7.2) which admits the p-Hardy inequality when $p \neq n$, but where the pointwise p-Hardy inequality only holds for $p > n$; see also Examples 7.3 and 7.4. Also, when $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$ is fixed, the set of β's for which Ω admits the pointwise (p, β)-Hardy inequality is always an interval (see Lemma 3.2), but this is not necessarily the case with the usual Hardy inequality, as can be seen for instance from Example 7.2, where $\Omega = B(0, 1) \setminus \{0\}$. This phenomenon has certainly been known to exist, but as it appears that there are no references to this in the literature, we formulate this as a proposition.

Proposition 1.7. *There exist domains in $\mathbb{R}^n$ which admit the (p, β_0)-Hardy inequality, for some $1 < p < \infty$ and $\beta_0 \in \mathbb{R}$, but where the (p, β)-Hardy inequality does not hold for every $\beta < \beta_0$.*

Let us now conclude this Introduction with a brief sketch of the outline of this paper. In Section 2 we go through some basic notation and definitions used in the rest of the paper. Section 3 is devoted to some preliminary results on pointwise Hardy inequalities and John domains. The exact definition of the visual boundary and the formulation of our main sufficient condition for Hardy inequalities, Condition 4.1, are given in Section 4, and then our main results are proved in Section 5. Next, in Section 6, we establish the aforementioned results on John domains, and finally, in Section 7 we give some examples which show that Condition 4.1 is essentially the weakest possible condition that guarantees a domain Ω to admit the (pointwise) (p,β)-Hardy inequality for all $1<p<\infty$ and $\beta<n-p+\lambda$. As noted before, these examples also shed some light to many other questions concerning usual and pointwise Hardy inequalities and their relations.

2. Basic definitions

2.1. **Notation.** Let A be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$, $n\geq 1$. Then ∂A denotes the boundary of A, $\overline{A}$ is the closure of A, and the n-dimensional Lebesgue measure of A is denoted $|A|$, provided that A is measurable. The characteristic function of A is χ_A, and $\operatorname{diam}(A)$ is the usual Euclidean diameter of A. Depending on the situation, $d(\cdot,\cdot)$ denotes either the Euclidean distance between two points, two sets, or a point and a set. We also use the notation $|x|$ for the Euclidean norm of $x\in\mathbb{R}^n$; then $|x-y|=d(x,y)$. An open ball with center $x\in\mathbb{R}^n$ and radius $r>0$ is denoted $B(x,r)$. When $L>0$ and $B=B(x,r)$ is a ball, we denote $LB=B(x,Lr)$. An open and connected set $\Omega\subset\mathbb{R}^n$ is called a *domain*. As in the Introduction, we denote $d_\Omega(x)=d(x,\partial\Omega)$ for $x\in\Omega$. For $A\subset\mathbb{R}^n$, $b\in\mathbb{R}^n$, and $\kappa\in\mathbb{R}$ (we allow also $\kappa\in\mathbb{C}$ if $n=2$) we denote $\kappa A+b=\{\kappa x+b : x\in A\}$, unless A is a ball (cf. above) or a cube (cf. 2.2).

Let $U\subset\mathbb{R}^n$ and let $f\colon U\to\mathbb{R}^m$ be a mapping. If $A\subset U$, $f|_A$ denotes the restriction of f to A. The *support* of f, $\operatorname{spt}(f)$, is the closure of the set where f is non-zero. If $A\subset\mathbb{R}^n$ is measurable with $0<|A|<\infty$, and if $f\in L^1(A)$, we denote

$$f_A=\fint_A f(x)\,dx=\frac{1}{|A|}\int_A f(x)\,dx\,.$$

All the integrals in this paper are taken with respect to the n-dimensional Lebesgue measure, if not stated otherwise.

A continuous mapping $\gamma\colon[a,b]\to\mathbb{R}^n$, $a,b\in\mathbb{R}$, $n\geq 2$, as well as the image $\gamma=\gamma([a,b])\subset\mathbb{R}^n$, is called a *curve*. The Euclidean length of a curve γ is denoted $l(\gamma)$. A curve γ is *rectifiable* if $l(\gamma)<\infty$. Every rectifiable curve γ can be parametrized by arc length, i.e. $\gamma=\gamma\colon[0,l]\to\mathbb{R}^n$ so that $l(\gamma|_{[0,t]})=t$ for all $t\in[0,l]$. We say that a curve $\gamma\colon[a,b]\to\mathbb{R}^n$ *joins x to y* (in $A\subset\mathbb{R}^n$), if $\gamma(a)=x$ and $\gamma(b)=y$ (and $\gamma\subset A$). When $x,y\in\mathbb{R}^n$, $[x,y]$ is the line segment with endpoints x and y.

We use the letter C to denote various positive constants, which may vary from expression to expression. If g and h are some quantities, we write $g\lesssim h$ if there exists a constant $C>0$ so that $g\leq Ch$. When F is a finite set, $\#F$ denotes the cardinality of F.

2.2. **Whitney decompositions.** Let $\Omega\subsetneq\mathbb{R}^n$, $n\geq 2$, be a proper subdomain. Then $\mathcal{W}=\mathcal{W}(\Omega)$ denotes a *Whitney decomposition* of Ω, i.e. a collection of closed cubes $Q\subset\Omega$ with pairwise disjoint interiors and having edges parallel to the coordinate axes, such that $\Omega=\bigcup_{Q\in\mathcal{W}}Q$. Also, the diameters of $Q\in\mathcal{W}$ are in the set $\{2^{-j}:j\in\mathbb{Z}\}$ and satisfy the condition

$$\operatorname{diam}(Q)\leq d(Q,\partial\Omega)\leq 4\operatorname{diam}(Q).$$

We refer to [23] for the existence and further properties of Whitney decompositions.

For $j \in \mathbb{Z}$ we define

$$\mathcal{W}_j = \{Q \in \mathcal{W} : \operatorname{diam}(Q) = 2^{-j}\}.$$

When Q is a cube, c_Q denotes the center point of Q, and if $L > 0$, then LQ is the cube with the center point c_Q and diameter $L \operatorname{diam}(Q)$.

2.3. **Maximal functions.** The classical *restricted Hardy-Littlewood maximal function* of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M_R f(x) = \sup_{0<r<R} \fint_{B(x,r)} |f(y)|\, dy\,,$$

where $0 < R \leq \infty$ may depend on x. In the case $R = \infty$ we denote $M_\infty f = Mf$. The well-known maximal theorem of Hardy, Littlewood and Wiener (see e.g. [23]) states that if $1 < p < \infty$, we have $||M_R f||_p \leq C(n,p)||f||_p$ for all $0 < R \leq \infty$. When $1 < q < \infty$, we define $M_{R,q} f = \big(M_R|f|^q\big)^{1/q}$. From the maximal theorem it follows that M_q is bounded on L^p for each $q < p < \infty$.

2.4. **Hausdorff content and capacity.** The λ-*Hausdorff content* of a set $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^\lambda_\infty(A) = \inf\left\{ \sum_{k=1}^\infty \operatorname{diam}(E_k)^\lambda : A \subset \bigcup_{k=1}^\infty E_k \right\},$$

and the *Hausdorff dimension* of A is then

$$\dim(A) = \inf\{\lambda > 0 : \mathcal{H}^\lambda_\infty(A) = 0\}.$$

Hausdorff contents are closely related to the concept of variational capacity. Although we do not need capacities in our results, we recall some basic definitions in order to establish this connection and to be able to comment on the original results related to Theorem 1.1.

When $\Omega \subset \mathbb{R}^n$ is a domain, the *(variational) p-capacity* of a compact set $E \subset \Omega$ (relative to Ω) is defined as

$$\operatorname{cap}_p(E,\Omega) = \inf\left\{ \int_\Omega |\nabla u|^p\, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E \right\}.$$

A closed set $E \subset \mathbb{R}^n$ is said to be *uniformly p-fat* if there exists a constant $C > 0$ such that

$$\operatorname{cap}_p\big(E \cap \overline{B}(x,r), B(x,2r)\big) \geq C \operatorname{cap}_p\big(\overline{B}(x,r), B(x,2r)\big)$$

for all $x \in E$ and $r > 0$. Note that for each ball $B(x,r) \subset \mathbb{R}^n$ we have $\operatorname{cap}_p(\overline{B}(x,r), B(x,2r)) = C(n,p)r^{n-p}$. For this and other basic properties of the p-capacity we refer to [11].

As mentioned in the Introduction, the original formulations of Theorem 1.1 were given in terms of capacity (cf. for example [15] or [5]). More precisely, the claim was that if the complement of Ω is uniformly p-fat, then Ω admits the p-Hardy inequality. But now, if the density condition (2) from Theorem 1.1 holds for some $\lambda > n - p$, i.e.

$$\mathcal{H}^\lambda_\infty\big(\Omega^c \cap B(w,r)\big) \geq C r^\lambda \quad \text{for all } w \in \partial\Omega \text{ and } r > 0,$$

then Ω^c is uniformly p-fat; see e.g. [11, Lemma 2.31] and notice that it clearly suffices to consider only boundary points. Conversely, positive p-capacity implies positive $(n-p)$-Hausdorff content with estimates, cf. [11, Theorem 2.27]. Since uniform fatness has a self-improving property [15, Theorem 1], we conclude that uniform p-fatness of Ω^c is in fact equivalent to the requirement that condition (2) holds with some exponent $\lambda > n-p$, and thus Theorem 1.1 is equivalent to the results of Ancona, Lewis, and Wannebo.

2.5. **John domains and uniform domains.** Let $\Omega \subset \mathbb{R}^n$ be a domain and let $c \geq 1$. We say that Ω is a *c-John domain* with center point x_0 if for every $x \in \Omega$ there exists a curve (called a John curve) $\gamma\colon [0,l] \to \Omega$, parametrized by arc length, such that $\gamma(0) = x$, $\gamma(l) = x_0$, and

$$d(\gamma(t), \partial\Omega) \geq \tfrac{1}{c} t \tag{5}$$

for each $t \in [0,l]$. Geometrically this means that each point in Ω can be joined to the central point by a "twisted cone", which is sometimes also called a "carrot". If Ω is a c-John domain with center point x_0, then $\Omega \subset B\big(x_0, c\, d(x_0, \partial\Omega)\big)$, so in particular Ω is bounded. Also, if Ω is a c-John domain, then for each $w \in \partial\Omega$ there is a curve $\gamma\colon [0,l] \to \Omega \cup \{w\}$ joining w to x_0 and satisfying (5). We say in this case, too, that γ joins w to x_0 *in* Ω. John domains were introduced in [21] and named in [16] after F. John who had considered a similar class of domains earlier (cf. [12]). There are several other ways to define John domains, see [20]. However, for bounded domains $\Omega \subset \mathbb{R}^n$ these definitions are equivalent, but with possibly different constants.

A domain $\Omega \subset \mathbb{R}^n$ is *uniform* if there is a constant $C \geq 1$ such that each pair of points $x, y \in \Omega$ can be joined by a curve $\gamma\colon [0,l] \to \Omega$, parametrized by arc length, so that $l \leq C d(x,y)$ and $d(z, \partial\Omega) \geq \frac{1}{C} \min\{d(z,x), d(z,y)\}$ for each $z \in \gamma$. Such a curve γ is called a "double cone" or a "cigar" arc. Every bounded uniform domain is also a c-John domain for some $c \geq 1$.

Let then Ω be a c-John domain with center point x_0, and let $\mathcal{W} = \mathcal{W}(\Omega)$ be a Whitney decomposition of Ω. When $w \in \partial\Omega$, we let $\mathcal{J}_c(w, x_0)$ denote the collection of all c-John curves joining w to x_0 in Ω. We then define

$$P(w) = \big\{Q \in \mathcal{W} : Q \cap \gamma \neq \emptyset \text{ for some } \gamma \in \mathcal{J}_c(w, x_0)\big\}.$$

When $E \subset \partial\Omega$, we also denote

$$P(E) = \bigcup_{w \in E} P(w).$$

The *(John-)shadow* $S(Q)$ of a cube $Q \in \mathcal{W}$ on the boundary $\partial\Omega$ is now defined by

$$S(Q) = \{w \in \partial\Omega : Q \in P(w)\}.$$

Then $S(Q)$ is a closed set for each $Q \in \mathcal{W}$. Indeed, if $w_j \in S(Q)$ and $w_j \to w$ as $j \to \infty$, we have for each $j \in \mathbb{N}$ a curve $\gamma_j \in \mathcal{J}_c(w_j, x_0)$ such that $\gamma_j \cap Q \neq \emptyset$. It follows from the Arzelà-Ascoli theorem that there exists a subsequence of (γ_j) converging uniformly to a curve γ. Since $l(\gamma) \leq \liminf_{j\to\infty} l(\gamma_j)$, it is easy to show that γ, after a possible reparametrization, is a c-John curve joining w to x_0 and intersecting Q.

Estimates for the sizes of these shadows will provide us with one key element in the proof of our main theorem.

3. Preliminary results

We begin by recording some basic properties of weighted pointwise Hardy inequalities. First, let us justify the notation of the pointwise (p,β)-Hardy inequality, i.e. that the pointwise inequality always implies the usual (p,β)-Hardy inequality. We give the proof, which uses some well-known arguments, for the sake of completeness.

Proposition 3.1. *Suppose that the pointwise (p,β)-Hardy inequality (4) holds for a function $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with a constant $C_1 > 0$. Then u satisfies the (p,β)-Hardy inequality (1) with a constant $C = C(C_1, p, n) > 0$.*

Proof. Denote $R = R(x) = 2d_\Omega(x)$. Divide the inequality (4) by $d_\Omega(x)^{1-\frac{\beta}{p}}$, integrate to power p over Ω, and use the fact that $M_{R,q}$ is bounded on L^p to obtain

$$\begin{aligned}\int_\Omega |u(x)|^p d_\Omega(x)^{\beta-p}\,dx &\le C_1^p \int_\Omega \Big(M_{R,q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x)\Big)^p\,dx\\ &\le C\int_\Omega \Big(|\nabla u(x)|^q d_\Omega(x)^{\beta(q/p)}\Big)^{p/q}\,dx\\ &= C\int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta\,dx\,.\end{aligned}$$

Here the constant $C > 0$ depends only on C_1, p, and the constant from the maximal function theorem, so that $C = C(C_1, n, p)$. $\square$

From the next lemma we obtain the fact that the pointwise (p,β_0)-Hardy inequality implies pointwise (p,β)-Hardy inequalities for all $\beta < \beta_0$.

Lemma 3.2. *Let $u \in C_0^\infty(\Omega)$ and let $1 < p < \infty$, $\beta_0 \in \mathbb{R}$. If u satisfies the pointwise (p,β_0)-Hardy inequality (4) at $x \in \Omega$ with constants $1 < q < p$ and $C_1 > 0$, then u satisfies the pointwise (p,β)-Hardy inequality at x for all $\beta < \beta_0$ with q and a constant $C = C(C_1, p, \beta_0, \beta) > 0$.*

Proof. Let $\beta < \beta_0$ and denote $\alpha = \beta_0 - \beta > 0$. If $0 < r < 2d_\Omega(x)$ and $y \in B(x,r)$, we have that $d_\Omega(y) \le 3d_\Omega(x)$. Thus we obtain from the pointwise (p,β)-Hardy inequality that

$$\begin{aligned}|u(x)| &\le C_1 d_\Omega(x)^{1-\frac{\beta_0}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta_0/p}\big)(x)\\ &\le C d_\Omega(x)^{1-\frac{\beta_0}{p}} d_\Omega(x)^{\frac{\alpha}{p}}\\ &\qquad\qquad \cdot\left(\sup_{0<r<2d_\Omega(x)} \fint_{B(x,r)} |\nabla u(y)|^q d_\Omega(y)^{\beta_0\frac{q}{p}-\alpha\frac{q}{p}}\,dy\right)^{1/q}\\ &\le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x),\end{aligned}$$

where $C = C(C_1, p, \beta_0, \beta) > 0$. $\square$

The next lemma allows us to modify Whitney decompositions, if needed, in such a way that if Q is a Whitney cube and $x \in Q$, then we may assume that Q is contained in a sufficiently small ball around x.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $x \in \Omega$. Then there exists a Whitney decomposition $\mathcal{W}$ of Ω and a cube $Q \in \mathcal{W}$ such that $x \in Q \subset B\big(x, \frac{2}{3} d_\Omega(x)\big)$.*

Proof. Let $\mathcal{W}'$ be any Whitney decomposition of Ω and take $Q' \in \mathcal{W}$ such that $x \in Q'$. If $Q' \subset B(x, \frac{2}{3} d_\Omega(x))$, we are done. Otherwise, divide Q' into 2^n congruent subcubes Q_i, $i = 1, \dots, 2^n$ with $\operatorname{diam}(Q_i) = \operatorname{diam}(Q')/2$ for all $i = 1, \dots, 2^n$. Then

$$\operatorname{diam}(Q_i) = \operatorname{diam}(Q')/2 \le d(Q', \partial\Omega)/2 \le d(Q_i, \partial\Omega)/2 \le d(Q_i, \partial\Omega)$$

for all $i = 1, \dots, 2^n$. On the other hand, as $x \in Q'$ but Q' is not a subset of $B(x, \frac{2}{3} d_\Omega(x))$, we have that $\operatorname{diam}(Q') \ge \frac{2}{3} d_\Omega(x)$, and thus

$$d(Q_i, \partial\Omega) \le d(Q_i, x) + d_\Omega(x) \le \tfrac{1}{2} \operatorname{diam}(Q') + \tfrac{3}{2} \operatorname{diam}(Q') = 4 \operatorname{diam}(Q_i)$$

for all $i = 1, \dots, 2^n$. Hence, if we replace the cube Q' in the Whitney decomposition $\mathcal{W}'$ by cubes $Q_1, \dots, Q_{2^n}$, we obtain a new Whitney decomposition $\mathcal{W}$ for Ω.

Finally, we may assume that $x \in Q_1$. Since $\operatorname{diam}(Q_1) = \operatorname{diam}(Q')/2 \le d_\Omega(x)/2$, it follows that $Q_1 \subset B\big(x, \frac{2}{3} d_\Omega(x)\big)$. $\square$

We end this section with some simple results on John domains that we need in the proof of our main theorem. First of all, the diameter of the shadow of a Whitney cube is bounded, up to a constant, by the diameter of the cube itself.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a c-John domain, and let $\mathcal{W}$ be a Whitney decomposition of Ω. Then there exists a constant $C = C(c) > 0$ such that*

$$\operatorname{diam}(S(Q)) \le C \operatorname{diam}(Q)$$

for each $Q \in \mathcal{W}$.

Proof. If $S(Q) = \emptyset$ there is nothing to prove, so we may assume that $S(Q) \neq \emptyset$. Let $w \in S(Q)$. Then there exists, by definition, a c-John curve γ joining w to x_0 in Ω so that $\gamma(t_Q) \in Q$ for some $t_Q \in [0, l(\gamma)]$. It follows that

$$\begin{aligned} d(w, Q) &\le d(w, \gamma(t_Q)) \le l\big(\gamma|_{[0,t_Q]}\big) = t_Q \\ &\le c\, d(\gamma(t_Q), \partial\Omega) \le 5c \operatorname{diam}(Q), \end{aligned} \tag{6}$$

and hence, by the triangle inequality, $\operatorname{diam}(S(Q)) \le (10c + 1) \operatorname{diam}(Q)$. $\square$

Next we show that the shadows of the Whitney cubes of a given size have bounded overlap, and hence we obtain a bound for the sum of the measures of these shadows as well.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a c-John domain, and let $\mathcal{W}$ be a Whitney decomposition of Ω. Then there exists a constant $C = C(n, c) > 0$ such that (i) for each $j \in \mathbb{Z}$ and each $w \in \partial\Omega$,*

$$\#\{Q \in \mathcal{W}_j : w \in S(Q)\} \le C.$$

(ii) if μ is a Borel measure on $\partial\Omega$, we have for every measurable subset $E \subset \partial\Omega$ and each $j \in \mathbb{Z}$ that

$$\sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E) \le C\mu(E).$$

Proof. (i) When $w \in S(Q)$ (i.e. $Q \in P(w)$) we have, by (6), that

$$Q \subset B\big(w, (5c+1)\operatorname{diam}(Q)\big).$$

Now, let us fix $w \in \partial\Omega$ and $j \in \mathbb{Z}$. Also, denote

$$a_j = \#\{Q \in \mathcal{W}_j : w \in S(Q)\} = \#\{Q \in \mathcal{W}_j : Q \in P(w)\},$$

and let $d_j = 2^{-j}$. Since the cubes $Q \in \mathcal{W}_j$ are essentially disjoint, we obtain that

$$\begin{aligned} a_j\, d_j{}^n = C(n) \sum_{Q \in \mathcal{W}_j \cap P(w)} |Q| &\le C(n)\, \big|B\big(w, (5c+1)d_j\big)\big| \\ &\le C(n,c) d_j{}^n\,. \end{aligned}$$

Thus $a_j \le C(n,c)$.

(ii) Let μ be a Borel measure on $\partial\Omega$ and let $E \subset \partial\Omega$ be a μ-measurable set. By the first part of the lemma, $\#\{Q \in \mathcal{W}_j : w \in S(Q)\}$ is uniformly bounded on E by a constant $C = C(n,c) > 0$, independent of j. Since $S(Q)$ is closed, it is μ-measurable, and hence

$$\sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E) = \int_E \sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(w)\, d\mu(w) \le C\mu(E).$$

This proves the lemma. □

4. Visual boundary and the main condition

Let us now present our sufficient condition for Hardy inequalities. When $\Omega \subset \mathbb{R}^n$ is a domain, $x \in \Omega$, and $c \ge 1$ is a constant, we define a subdomain $\Omega_x(c)$ by

$$\Omega_x(c) = \bigcup\{U \subset \Omega : U \text{ is a } c\text{-John domain with center point } x\}.$$

Then clearly $\emptyset \ne \Omega_x(c) \subset \Omega$ and $\Omega_x(c)$ is also a c-John domain with center point x. We say that the set

$$v_x(c)\text{–}\partial\Omega = \partial\Omega \cap \partial\Omega_x(c)$$

is the *c-visual boundary of Ω near x*. In our main theorem, as well as in the corresponding pointwise result, we assume that the visual boundary of Ω near each point $x \in \Omega$ is uniformly large, in the sense of the following density condition.

Condition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. We require that there exist constants $c \ge 1$, $C_0 > 0$, and $0 \le \lambda \le n$ such that for each $x \in \Omega$*

$$\mathcal{H}^\lambda_\infty\big(v_x(c)\text{–}\partial\Omega\big) \ge C_0 d_\Omega(x)^\lambda\,. \tag{7}$$

Notice that a domain $\Omega \subset \mathbb{R}^n$ satisfies Condition 4.1 if and only if for each $x \in \Omega$ there exists *some* c-John domain $U_x \subset \Omega$ with center point x so that

$$\mathcal{H}^{\lambda}_{\infty}(\partial U_x \cap \partial\Omega) \geq C_0 d_{\Omega}(x)^{\lambda} .$$

To make the formulations of our results a bit less technical, we will make use of the above fact in the following lemma, which allows us, under Condition 4.1, to consider only a part of the visual boundary $v_x(c)$–$\partial\Omega$, lying inside $B\big(x, 2d_{\Omega}(x)\big)$.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying Condition 4.1 with constants $c \geq 1$, $C_0 > 0$, and $0 \leq \lambda \leq n$. Then, for each $x \in \Omega$, there exists a c-John domain $U_x \subset \Omega \cap B\big(x, 2d_{\Omega}(x)\big)$, with center point x, such that*

$$\mathcal{H}^{\lambda}_{\infty}(\partial U_x \cap \partial\Omega) \geq C_1 d_{\Omega}(x)^{\lambda},$$

where the constant $C_1 = C_1(C_0, c, \lambda) > 0$ is independent of x.

Proof. Let $x \in \Omega$. If $c \leq 2$, then $\Omega_x(c) \subset B(x, 2d_{\Omega}(x))$, and there is nothing to prove. Otherwise, we choose $w \in \partial\Omega$ such that $d(x,w) = d_{\Omega}(x)$ and denote $x' = w + (x-w)/(c-1)$. Then, by easy calculations, $\Omega_{x'}(c) \subset B\big(x, 2d_{\Omega}(x)\big)$, and $U_x = \Omega_{x'}(c) \cup B(x, d_{\Omega}(x))$ is a c-John domain with center point x. In particular, $v_{x'}(c)$–$\partial\Omega \subset \partial U_x \cap \partial\Omega$, so using Condition 4.1 for x' and the fact that $d_{\Omega}(x') = d_{\Omega}(x)/(c-1)$, we conclude

$$\mathcal{H}^{\lambda}_{\infty}(\partial U_x \cap \partial\Omega) \geq \mathcal{H}^{\lambda}_{\infty}(v_{x'}(c)\text{–}\partial\Omega) \geq C_0 d_{\Omega}(x')^{\lambda} = C_1 d_{\Omega}(x)^{\lambda},$$

where $C_1 = C_0(c-1)^{-\lambda} > 0$ is independent of x. □

If $\Omega \subset \mathbb{R}^n$ is a uniform domain, then it is sufficient to assume merely that the usual boundary $\partial\Omega$ satisfies an inner boundary density condition, since then we obtain, using the uniformity, that Ω satisfies Condition 4.1. Let us state this as a proposition:

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and let $1 < p < \infty$. Assume that there exist $C > 0$ and $0 \leq \lambda \leq n$ such that*

$$\mathcal{H}^{\lambda}_{\infty}\big(\partial\Omega \cap B(x, 2d_{\Omega}(x))\big) \geq C d_{\Omega}(x)^{\lambda}$$

for every $x \in \Omega$. Then Ω satisfies Condition 4.1.

Proof. Let C_U be the constant from the uniformity condition for Ω. Let $x_0 \in \Omega$ and let $B_0 = B\big(x_0, 2d_{\Omega}(x_0)\big)$. When $x \in B_0 \cap \Omega$ we can find, by modifying the double cone arc joining x to x_0, a c-John arc γ_x joining x to x_0 in Ω, with a constant $c = c(C_U, n) > 0$; cf. [20] and references therein. Thus $\Omega \cap B_0 \subset \Omega_{x_0}(c)$, and so $\partial\Omega \cap B_0 \subset v_x(c)$–$\partial\Omega$. This, together with the density assumption of the proposition, implies that Condition 4.1 holds in Ω. □

Corollary 1.5 follows now immediately from Proposition 4.3 and Theorem 1.4. In fact, a uniform domain Ω satisfying the density condition of Proposition 4.3 admits the pointwise (p, β)-Hardy inequality as well for $1 < p < \infty$ and $\beta < p - n + \lambda$; see Theorem 5.1.

5. The proof of the Main Theorem

In this section, we give the proof of our main result, Theorem 1.4. In fact, we prove the next theorem, which is the corresponding result for pointwise Hardy inequalities.

Theorem 5.1. *Let $1 < p < \infty$. Assume that a domain $\Omega \subset \mathbb{R}^n$ satisfies Condition 4.1 with an exponent λ, and let $\beta < p - n + \lambda$. Then Ω admits the pointwise (p, β)-Hardy inequality, i.e. there exist $1 < q < p$ and $C > 0$ such that*

$$|u(x)| \le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x)$$

whenever $u \in C_0^\infty(\Omega)$ and $x \in \Omega$.

By Proposition 3.1, Theorem 1.4 is a direct consequence of Theorem 5.1. The proof of Theorem 5.1 relies on the following lemma.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying Condition 4.1 with an exponent λ. Then, if $1 < p < \infty$ and $\beta < p - n + \lambda$, there exists some $1 \le q_0 < p$ with the following property: For every $q_0 < q < p$ there is a constant $C > 0$ such that the inequality*

$$\big|u_{\frac{2}{3}B_x}\big| \le C d_\Omega(x)^{1-\frac{\beta}{p}} \left(\fint_{2B_x} |\nabla u(y)|^q \, d_\Omega(y)^{\frac{\beta}{p}q} \, dy \right)^{1/q},$$

where $B_x = B\big(x, d_\Omega(x)\big)$, holds for every $x \in \Omega$ and every $u \in C_0^\infty(\Omega)$. Here q_0 depends only on n, p, β, and λ, while C may depend in addition on q and the constants from Condition 4.1.

Proof. Let $x \in \Omega$ and let $\Omega_x = \Omega_x(c)$ be the c-John domain with center point x from the definition of the visual boundary in Section 4. Actually, by Lemma 4.2, we may assume that $\Omega_x \subset 2B_x = B\big(x, 2d_\Omega(x)\big)$. By Lemma 3.3, there exists a Whitney decomposition $\mathcal{W} = \mathcal{W}(\Omega_x)$ for Ω_x, and a cube $Q_0 \in \mathcal{W}$ such that $x \in Q_0 \subset \frac{2}{3}B_x$. Take $j_1 \in \mathbb{Z}$ so that $\operatorname{diam}(Q_0) = 2^{-j_1}$, and denote $j_0 = j_1 - 3$. Then it is easy to see that $\operatorname{diam}(Q) \le 2^{-j_0}$ for every $Q \in \mathcal{W}$, and in particular $2^{-j_0} \le C \operatorname{diam}(Q_0)$.

Now let $1 < p < \infty$ and $\beta < p - n + \lambda$, and define

$$q_0 = \max\Big(1, p\,\frac{n-\lambda}{p-\beta}\Big).$$

Since $0 \le n - \lambda < p - \beta$, we have that $1 \le q_0 < p$. Let $q_0 < q < p$ and denote $\beta' = \frac{q}{p}\beta$. Then $q/p > (n-\lambda)/(p-\beta)$, and thus we obtain

$$\lambda + q - \beta' - n = \lambda + \tfrac{q}{p}(p - \beta) - n > \lambda + (n - \lambda) - n = 0.$$

We will first show that there exists a constant $C_1 > 0$, depending only on the given data, such that

$$|u_{Q_0}| \le C_1 \operatorname{diam}(Q_0)^{1-\frac{\beta}{p}} \left(\fint_{2B_x} |\nabla u(y)|^q \, d_\Omega(y)^{\frac{\beta}{p}q} \, dy \right)^{1/q}. \tag{8}$$

To this end, denote $E = \partial\Omega \cap \partial\Omega_x$, so that E is a part of the c-visual boundary of Ω near x satisfying

$$\mathcal{H}^\lambda_\infty(E) \ge C d_\Omega(x)^\lambda. \tag{9}$$

When $w \in E$, there exists a c-John curve γ_w joining w to x in Ω_x. We apply a chaining argument involving the Poincaré inequality on cubes, similar to the one in [22, Lemma 8], for the cubes $Q \in \mathcal{W}$ intersecting γ_w, and obtain that

$$|u_{Q_0}| = |u_{Q_0} - u(w)| \le C \sum_{Q \in P(w)} \operatorname{diam}(Q) \fint_Q |\nabla u(y)| \, dy \,,$$

where the constant $C > 0$ is independent of w. A simple use of Hölder's inequality leads us to

$$|u_{Q_0}| \le C \sum_{Q \in P(w)} \operatorname{diam}(Q)^{1-\frac{\beta}{p}} \left(\fint_Q |\nabla u(y)|^q \, d_\Omega(y)^{\frac{\beta}{p} q} \, dy \right)^{1/q} . \tag{10}$$

Note that here we have to use different sides of the inequality

$$\operatorname{diam}(Q) \le d_\Omega(y) \le 5 \operatorname{diam}(Q) \qquad \text{for all } y \in Q \in \mathcal{W}$$

depending whether $\beta \ge 0$ or $\beta < 0$.

From now on, let us denote $g(y) = |\nabla u(y)| \, d_\Omega(y)^{\beta/p}$. We apply Frostman's lemma (see e.g. [17, Theorem 8.8]) and choose a Radon measure μ such that μ is supported on E, $\mu(B(x,r)) \le r^\lambda$ for all $x \in \mathbb{R}^n$ and $r > 0$, and $\mu(E) \ge C\mathcal{H}^\lambda_\infty(E)$. Integration of (10) over E with respect to the measure μ yields

$$|u_{Q_0}| \, \mu(E) \le C \int_E \sum_{Q \in P(w)} \operatorname{diam}(Q)^{1-\frac{\beta}{p}} \left(\fint_Q g(y)^q \, dy \right)^{1/q} d\mu(w). \tag{11}$$

We then interchange the order of summation and integration in (11) and use Hölder's inequality for sums to obtain

$$\begin{aligned} |u_{Q_0}| &\le C\mu(E)^{-1} \sum_{Q \in P(E)} \mu(S(Q)) \operatorname{diam}(Q)^{1-\frac{\beta}{p}-\frac{n}{q}} \left(\int_Q g(y)^q \, dy \right)^{1/q} \\ &\le C\mu(E)^{-1} \Bigg(\sum_{Q \in P(E)} \mu(S(Q))^{\frac{q}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \Bigg)^{\frac{q-1}{q}} \\ &\qquad \cdot \Bigg(\sum_{Q \in P(E)} \int_Q g(y)^q \, dy \Bigg)^{1/q} . \end{aligned} \tag{12}$$

In the next step we estimate the sum in (12):

$$\begin{aligned} &\sum_{Q \in P(E)} \mu(S(Q))^{\frac{q}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \\ &\qquad \le \sum_{j=j_0}^{\infty} \max_{Q \in \mathcal{W}_j} \left(\mu(S(Q))^{\frac{1}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \right) \sum_{Q \in \mathcal{W}_j} \mu(S(Q)) \,, \end{aligned} \tag{13}$$

where by Lemma 3.5(ii)

$$\sum_{Q \in \mathcal{W}_j} \mu(S(Q)) \le C\mu(E). \tag{14}$$

For the cubes $Q \in \mathcal{W}_j$ we have by definition that $\operatorname{diam}(Q) = 2^{-j}$, and further, by the properties of μ and Lemma 3.4, we obtain

$$\mu(S(Q)) \le C \operatorname{diam}(S(Q))^\lambda \le C \operatorname{diam}(Q)^\lambda \le C 2^{-j\lambda}.$$

Recall that by the choice of q and β' we have $\lambda + q - \beta' - n > 0$, so that

$$\begin{aligned}\sum_{j=j_0}^{\infty} \max_{Q\in\mathcal{W}_j} \Big(\mu(S(Q))^{\frac{1}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}}\Big) &\le \sum_{j=j_0}^{\infty} C 2^{-j\,\frac{\lambda+q-\beta'-n}{q-1}} \\ &\le C\, 2^{-j_0\,\frac{\lambda+q-\beta'-n}{q-1}}.\end{aligned}$$

Combining this with equations (12), (13), and (14) yields

$$\begin{aligned}|u_{Q_0}| &\le C\mu(E)^{-1+\frac{q-1}{q}} \Big(2^{-j_0\,\frac{\lambda+q-\beta'-n}{q-1}}\Big)^{\frac{q-1}{q}} \bigg(\sum_{Q\in P(E)} \int_Q g(y)^q\,dy\bigg)^{1/q} \\ &\le C\mu(E)^{-\frac{1}{q}} \operatorname{diam}(Q_0)^{\frac{\lambda+q-\beta'-n}{q}} \bigg(\int_{2B_x} g(y)^q\,dy\bigg)^{1/q}.\end{aligned} \tag{15}$$

Finally, the properties of the Frostman measure μ and (9) imply that

$$\operatorname{diam}(Q_0)^\lambda \le C\mathcal{H}^\lambda_\infty(E) \le C\mu(E),$$

and hence we obtain from (15) that estimate (8) holds, i.e.

$$|u_{Q_0}| \le C_1 \operatorname{diam}(Q_0)^{1-\frac{\beta}{p}} \bigg(\fint_{2B_x} |\nabla u(y)|^q\, d_\Omega(y)^{\beta'}\,dy\bigg)^{1/q}.$$

To finish the proof, we estimate the difference $\big|u_{Q_0} - u_{\frac{2}{3}B_x}\big|$, using the Poincaré inequality and the facts that $|\frac{2}{3}B_x| \le C|Q_0|$, and that $d_\Omega(x)^{\beta/p} \le C d_\Omega(y)^{\beta/p}$ for every $y \in \frac{2}{3}B_x$, as follows:

$$\begin{aligned}\big|u_{Q_0} - u_{\frac{2}{3}B_x}\big| &\le C\big|\tfrac{2}{3}B_x\big|^{-1} \int_{\frac{2}{3}B_x} \big|u(y) - u_{\frac{2}{3}B_x}\big|\,dy \\ &\le C d_\Omega(x) \fint_{\frac{2}{3}B_x} |\nabla u(y)|\,dy \\ &\le C d_\Omega(x)^{1-\frac{\beta}{p}} \fint_{\frac{2}{3}B_x} |\nabla u(y)| d_\Omega(y)^{\frac{\beta}{p}}\,dy \\ &\le C d_\Omega(x)^{1-\frac{\beta}{p}} \bigg(\fint_{2B_x} |\nabla u(y)|^q\, d_\Omega(y)^{\frac{\beta}{p}q}\,dy\bigg)^{1/q};\end{aligned} \tag{16}$$

the last inequality follows from Hölder's inequality and the fact that $|2B_x| \le C|\frac{2}{3}B_x|$. The lemma follows from (8) and (16), since $\operatorname{diam}(Q_0) \le d_\Omega(x) \le 5\operatorname{diam}(Q_0)$ and $\big|u_{\frac{2}{3}B_x}\big| \le |u_{Q_0}| + \big|u_{Q_0} - u_{\frac{2}{3}B_x}\big|$. □

We are now ready to prove the pointwise (p,β)-Hardy inequality for the domains in question.

Proof of Theorem 5.1. Let $x \in \Omega$ and let $u \in C_0^\infty(\Omega)$. By Lemma 5.2, there exist some $1 < q < p$ and a constant $C > 0$, independent of x and u, such that

$$\big|u_{\frac{2}{3}B_x}\big| \le C d_\Omega(x)^{1-\frac{\beta}{p}}\, M_{2d_\Omega(x),q}\, g(x),$$

where we denote as before $g(x) = |\nabla u(x)|\, d_\Omega(x)^{\beta/p}$.

On the other hand, using the well-known inequalities [4, Lemma 7.16] and [9, Lemma (a)], we obtain that

$$\big|u(x) - u_{\frac{2}{3}B_x}\big| \le C d_\Omega(x) M_{\frac{2}{3}d_\Omega(x)} |\nabla u(x)|,$$

where $C = C(n) > 0$. Therefore we have for *each* $1 < q < \infty$ that

$$\begin{aligned}\big|u(x) - u_{\frac{2}{3}B_x}\big| &\le C d_\Omega(x)^{1-\frac{\beta}{p}} \sup_{r<\frac{2}{3}d_\Omega(x)} \fint_{B(x,r)} |\nabla u(y)| d_\Omega(y)^{\beta/p}\, dy \\ &\le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\, g(x),\end{aligned}$$

where the second inequality follows from Hölder's inequality. Hence

$$|u(x)| \le \big|u(x) - u_{\frac{2}{3}B_x}\big| + \big|u_{\frac{2}{3}B_x}\big| \le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\, g(x),$$

where the constant $C > 0$ depends only on the given data. This proves the theorem. □

Condition 4.1 is not very meaningful when $\lambda = 0$; in fact, this kind of a condition is satisfied whenever $\Omega \subsetneq \mathbb{R}^n$ is a proper subdomain, since then $\partial\Omega \neq \emptyset$, and thus $v_x(1)$–$\partial\Omega \neq \emptyset$ for each $x \in \Omega$. Hence we obtain that

$$\mathcal{H}^0_\infty\big(v_x(1)\text{–}\partial\Omega\big) = 1 = d_\Omega(x)^0$$

for every $x \in \Omega$. As Lemma 5.2 holds also in this case, we obtain the following corollary which generalizes the result known for $\beta = 0$.

Corollary 5.3. *Let $1 < p < \infty$. Then each subdomain $\Omega \subsetneq \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality for every $\beta < p - n$.*

We note that the bound $\beta < p - n$ in Corollary 5.3 is sharp, since the $(p, p - n)$-Hardy inequality fails e.g. in $\mathbb{R}^n \setminus \{0\}$ and in $B(0, 1) \setminus \{0\} \subset \mathbb{R}^n$; see Example 7.2.

6. John domains quasiconformally equivalent to the unit ball

Recall that a homeomorphism $f\colon \Omega \to \Omega'$ between domains $\Omega, \Omega' \subset \mathbb{R}^n$, $n \ge 2$, is called a $(K\text{-})$*quasiconformal (qc) mapping* if f belongs to the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and there is a constant $K \ge 1$ such that

$$||Df(x)||^n \le K\, J_f(x) \quad \text{for a.e. } x \in \Omega.$$

Here $||\cdot||$ denotes the operator norm and J_f is the Jacobian determinant of f. Domains Ω and Ω' are said to be *quasiconformally equivalent* if there exists a qc mapping $f\colon \Omega \to \Omega'$. We refer to [24] for the basic theory of qc mappings.

John domains which are quasiconformally equivalent to the unit ball have some special properties among all John domains (see [10] and references therein). In the proof of the next theorem we need to use one of those properties together with general results on qc mappings and John domains.

Theorem 6.1. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a John domain. Assume in addition that Ω is quasiconformally equivalent to the unit ball $B(0, 1) \subset \mathbb{R}^n$. Then Ω admits the pointwise (p, β)-Hardy inequality for each $\beta < p - 1$.*

Proof. Denote $B = B(0,1) \subset \mathbb{R}^n$ and let $f\colon B \to \Omega$ be a K-qc mapping. Fix a point $y \in \Omega$ and take $x \in B$ so that $f(x) = y$. Since Ω is a John domain, f extends continuously to $\partial\Omega$ (cf. [24, Corollary 17.14] and [20, 2.17]). It follows from [24, Theorem 18.1] that there is a constant $\alpha = \alpha(n,K) > 0$ such that $f\big(B\big(x, \alpha\, d(x,\partial B)\big)\big) \subset B\big(y, \frac{1}{2}\, d_\Omega(y)\big)$. By [2, Corollary 6.4], we have that

$$\mathcal{H}^{n-1}_\infty\big(f(S_x)\big) \geq C\, d_\Omega(y)^{n-1},$$

where S_x is the radial projection of the ball $B\big(x, \alpha\, d(x,\partial B)\big)$ on ∂B and $C = C(n,K,\alpha) > 0$. Note that in [2] they have $\alpha = \frac{1}{2}$, but the results hold for any fixed $0 < \alpha < 1$ as well.

In order to prove the theorem it is now enough to show that for each $w \in f(S_x) \subset \partial\Omega$ there is a John curve γ joining w to y, with a John constant independent of w and y, since then Ω satisfies the visual boundary Condition 4.1 with $\lambda = n-1$, and Theorem 5.1 gives the claim. To this end, let $w \in f(S_x)$ and let $w' \in S_x$ be a preimage of w. Choose $x' \in B\big(x, \alpha\, d(x,\partial B)\big)$ so that $x' \in [0, w']$. We now define a curve $\gamma_1\colon [0,1] \to \Omega$ by $\gamma_1(t) = f\big(w' + t(x' - w')\big)$ for all $t \in [0,1]$. Note that γ_1 need not to be rectifiable. However, by [10, Theorem 3.1], there exists a constant $b = b(n,K,\Omega) \geq 1$ such that if $z = w' + t(x' - w') \in [w', x']$ for $t \in [0,1]$, we have

$$\operatorname{diam}\big(\gamma_1([0,t])\big) = \operatorname{diam}\big(f[w',z]\big) \leq b\, d\big(f(z),\partial\Omega\big) = b\, d\big(\gamma_1(t),\partial\Omega\big).$$

Now, by [16, Lemma 2.7] (see also [20, Section 2]), there exists a constant $c = c(b,n) \geq 1$ and a c-John curve γ_2 joining w to $f(x')$ in Ω.

If $f(x') = y$, we take $\gamma = \gamma_2$ and the proof is complete. Otherwise we define our curve γ in parts:

$$\gamma(t) = \begin{cases} \gamma_2(t)\,, & 0 \leq t \leq l(\gamma_2) \\ f(x') + \big(t - l(\gamma_2)\big)\dfrac{y - f(x')}{|y - f(x')|}\,, & l(\gamma_2) < t \leq l(\gamma_2) + |y - f(x')|\,, \end{cases}$$

and it easily follows that γ is a c_1-John curve joining w to y in Ω, with a constant $c_1 = c_1(n,K,\Omega) > 0$. □

The following planar result is an immediate consequence of Theorem 6.1, since by the Riemann mapping theorem each simply connected proper subdomain $\Omega \subsetneq \mathbb{R}^2$ is conformally, and thus especially quasiconformally equivalent to the unit ball $B(0,1) \subset \mathbb{R}^2$.

Corollary 6.2. *Let Ω be a simply connected John domain in the plane and let $1 < p < \infty$. Then Ω admits the pointwise (p,β)-Hardy inequality for each $\beta < p - 1$.*

7. Examples

In this section, we give various planar examples which prove the essential sharpness of our theorems; higher dimensional examples can be constructed along same lines. The first brief example, however, shows that, at least in the case of $\beta < p - 1$, Condition 4.1 is not very restrictive. Also, we record for von Koch -type snowflake domains the Hardy inequalities mentioned in the Introduction.

Example 7.1. (a) It is not necessary for a domain to be John, or even bounded, in order to satisfy Condition 4.1: Let $\Omega \subset \mathbb{R}^n$ be a strip,

$$\Omega = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_n < 1\}.$$

Then Ω is unbounded, but it satisfies Condition 4.1 with $\lambda = n - 1$, and thus Ω admits the pointwise (p, β)-Hardy inequality for all $\beta < p - 1$.

(b) Let $\Omega \subset \mathbb{R}^n$ be a "room-and-corridor"-type domain which are widely used in the study of the Poincaré inequalities (see e.g. [22, Section 10] and references therein). Then Ω is not necessarily a John domain, but it satisfies clearly Condition 4.1 with $\lambda = n-1$ and admits the pointwise (p, β)-Hardy inequality for all $\beta < p - 1$.

(c) Let $1 < \lambda < 2$ and let $\Omega \subset \mathbb{R}^2$ be a λ-snowflake domain, i.e. a "triangle" whose edges are copies of the von Koch -type snowflake curve K_λ with $\dim(K_\lambda) = \lambda$. Then Ω is a uniform domain, and by the self-similarity of the snowflake curve (cf. [3]) it is clear that the density condition of Proposition 4.3 is satisfied. Hence Condition 4.1 holds in Ω with the exponent λ, and we conclude that Ω admits the pointwise (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < \beta_0 = p + \lambda - 2$. Furthermore, by considering functions $u_j \in C_0^\infty(\Omega)$ such that $u_j(x) = 1$ if $d_\Omega(x) \geq 2^{-j}$, and $|\nabla u| \lesssim 2^j$ if $d_\Omega(x) \leq 2^{-j}$, it is easy to see that the (p, β)-Hardy inequality fails whenever $\beta \geq \beta_0$.

Next we record some properties of the punctured ball $B(0,1) \setminus \{0\}$ and the punctured space $\mathbb{R}^n \setminus \{0\}$ concerning Hardy inequalities. These also prove Proposition 1.7.

Example 7.2. Let $1 < p < \infty$. Then $B(0,1) \setminus \{0\}$ admits the (p, β)-Hardy inequality if and only if $\beta < p - n$ or $p - n < \beta < p - 1$, and $\mathbb{R}^n \setminus \{0\}$ admits the (p, β)-Hardy inequality if and only if $\beta \neq p - n$.

Denote $\Omega_1 = B(0,1) \setminus \{0\}$ and $\Omega_2 = \mathbb{R}^n \setminus \{0\}$. As was noted by Lewis (see the remark after Theorem 2 in [15]), Ω_1 admits the p-Hardy inequality if and only if $1 < p < n$ or $p > n$. This can be seen by using one-dimensional Hardy inequalities on rays starting from the origin, and integrating with respect to polar coordinates. Exactly the same idea gives the above weighted inequalities in the cases $\beta > n-p$ as well, for both Ω_1 (when $p-n < \beta < p-1$) and Ω_2; the case $\beta < n - p$ is trivial by Corollary 5.3.

To see that the $(p, p - n)$-Hardy inequality fails in both domains, it suffices to consider functions $u_k \in C_0^\infty(\Omega_j)$, $j = 1, 2$, such that $\operatorname{spt}(u_k) \in B(0, 1/2) \setminus B(0, 2^{-(k+1)})$, $u(x) = 1$ when $2^{-k} < |x| < 1/4$, $|\nabla u(x)| \lesssim 2^k$ when $2^{-(k+1)} < |x| < 2^{-k}$, and $|\nabla u(x)| \lesssim C$ when $1/4 < |x| < 1/2$. Then, easy calculations show that $\int_{\Omega_i} |u_k|^p {d_\Omega}^{(p-n)-p} \xrightarrow{k \to \infty} \infty$, while $\int_{\Omega_i} |\nabla u_k|^p {d_\Omega}^{p-n}$ remains uniformly bounded.

When $\beta \geq p-1$, we use the following functions $u_k \in C_0^\infty$ to show that the (p, β)-Hardy inequality fails in Ω_1: $\operatorname{spt}(u_k) \in B(0, 1-2^{-(k+1)}) \setminus B(0, 2^{-(k+1)})$, $u(x) = 1$ when $2^{-k} < |x| < 1 - 2^{-k}$, and $|\nabla u(x)| \lesssim 2^k$ when $2^{-(k+1)} < |x| < 2^{-k}$ or $1 - 2^{-k} < |x| < 1 - 2^{-(k+1)}$. Then, again by straight-forward calculations,

$$\frac{\int_{\Omega_1} |u_k|^p {d_\Omega}^{\beta - p}}{\int_{\Omega_1} |\nabla u_k|^p {d_\Omega}^{\beta}} \xrightarrow{k \to \infty} \infty.$$

This shows that the (p,β)-Hardy inequality fails in Ω_1 for every $\beta \geq p-1$.

Let us then turn to the bit more involved examples, which, among other things, prove Theorem 1.3. In these examples we show that a density condition of the type

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap B\big(x, 2d_\Omega(x)\big)\big) \geq C d_\Omega(x)^\lambda \quad \text{for all } x \in \Omega, \tag{17}$$

for some $C > 0$, is not sufficient to guarantee (p,β)-Hardy inequalities in Ω for all $\beta < p - n + \lambda$. Therefore some additional accessibility condition similar to our visible boundary condition 4.1 is really needed.

We use both complex and vector notation in the following constructions, so, for instance, i denotes always the imaginary unit, and when $x \in \mathbb{R}^2 = \mathbb{C}$, we write $x = (x_1, x_2) = x_1 + ix_2$.

Example 7.3. Let $1 < p < \infty$ and $1 < \lambda < 2$. We construct a simply connected John domain $\Omega_\lambda \subset \mathbb{R}^2$ which satisfies the condition (17) with the exponent λ, but fails to admit the (p,β)-Hardy inequality for $\beta = p-1 < p-2+\lambda$. Furthermore, by Lemma 3.2, the pointwise (p,β)-Hardy inequality fails in Ω_λ for every $\beta \geq p-1$ as well. Nevertheless, it turns out that Ω_λ admits the usual (p,β)-Hardy inequality also when $p-1 < \beta < p-2+\lambda$.

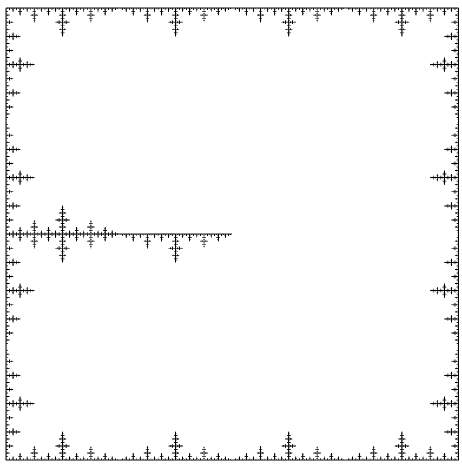

FIGURE 1. The domain Ω_λ of Example 7.3 for $\lambda = 1.45$

We begin by constructing a self-similar fractal called "the antenna set" in the (complex) plane. Let $0 < \alpha < \frac{1}{2}$ and let $F^\alpha = \{f_1, f_2, f_3, f_4\}$ be the iterated function system of similitudes

$$\begin{aligned} f_1(x) &= \tfrac{1}{2}x\,, & f_3(x) &= \alpha i x + \tfrac{1}{2}\,,\\ f_2(x) &= \tfrac{1}{2}x + \tfrac{1}{2}\,, & f_4(x) &= -\alpha i x + \tfrac{1}{2} + \alpha i\,. \end{aligned}$$

Then there exists a unique compact set $K = K^\alpha \subset \mathbb{R}^2$ which is invariant under F^α, i.e. $K = \bigcup_{j=1}^4 f_j(K)$. This K is the antenna set. It is easy to check that K satisfies the open set condition, and hence the Hausdorff dimension of K is $\lambda = \lambda(\alpha)$, where $1 < \lambda < 2$ is the solution of the equation $2 \cdot 2^{-\lambda} + 2\alpha^\lambda = 1$, and furthermore, we have that $0 < \mathcal{H}^\lambda(K) < \infty$. See e.g. [3, Chapter 9] for detailed information about iterated function systems, self-similar sets, and the open set condition. We now choose $0 < \alpha < \frac{1}{2}$ so that $\dim(K^\alpha) = \lambda$ for the fixed λ.

Take $\kappa = \frac{1}{4}$. Consider the unit square $[0,1]^2$ and replace each of the edges by four copies of κK, that is, K dilated by the factor $\frac{1}{4}$, oriented so that the "antennas" are inside the unit square. We call this domain Ω^1_λ. Notice that Ω^1_λ satisfies Condition 4.1 for λ, even though it is not a uniform domain. Next we remove from Ω^1_λ the sets $\kappa K + \frac{i}{2}$, $-\kappa K + \frac{i}{2} + \kappa$, and finally the set $A = -\kappa K + \frac{i}{2} + 2\kappa$. We have then constructed our domain $\Omega = \Omega_\lambda$ (see Fig. 1) which can quite easily be seen to be a simply connected John domain.

Let then $x \in \Omega$. Pick a point $w_x \in \partial\Omega$ so that $d(x, w_x) = d_\Omega(x)$. Then $B\big(w_x, d_\Omega(x)\big) \subset B\big(x, 2d_\Omega(x)\big)$, and from the self-similarity of the antenna set we obtain

$$\mathcal{H}^\lambda_\infty\big(B(w_x, d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda\,,$$

where $C > 0$ is a constant independent of x. Hence Ω satisfies the condition (17), but it does *not* satisfy Condition 4.1 for any $\lambda > 1$.

Since Ω is a simply connected John domain, we know by Corollary 6.2 that Ω admits the (p,β)-Hardy inequality (even pointwise) whenever $1 < p < \infty$ and $\beta < p - 1$. Next we show that Ω fails to admit the $(p, p-1)$-Hardy inequality for every $1 < p < \infty$.

The failure happens above the "one-sided antenna" A. To see this, choose an open square $S \subset \Omega$ so that one edge of S is a subset of A and $d_\Omega(x) = d(x, A)$ for every $x \in S$. It is then enough to show that the $(p, p-1)$-Hardy inequality fails in the upper half plane $H_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ for functions in $C_0^\infty([0,3]^2)$, since then the case with $[0,3]^2$ and ∂H_+ replaced by S and A follows with a composition of a dilatation and a transformation.

We choose a sequence of functions $u_j \in C_0^\infty([0,3]^2)$ with the following properties: $\operatorname{spt}(u_j) \subset [0,3] \times [2^{-(j+1)}, 3]$, $u_j(x) = 1$ for all $x \in [1,2] \times [2^{-j}, 2]$, $|\nabla u_j(x)| \leq 2^{j+2}$ for all $x \in [0,3] \times [2^{-(j+1)}, 2^{-j}]$, and $|\nabla u_j(x)| \leq 2$ for all other $x \in \operatorname{spt}(|\nabla u_j|)$. Then, for $1 < p < \infty$ we have

$$\begin{aligned}\int_{H_+} |u_j(x)|^p\, d(x, \partial H_+)^{(p-1)-p}\, dx &\geq \sum_{k=0}^{j} \int_{[1,2]\times[2^{-k}, 2^{-k+1}]} d(x, \partial H_+)^{-1}\, dx \\ &\geq \sum_{k=0}^{j} 2^{-k}\, 2^{k-1} = \tfrac{1}{2}\,(j+1) \xrightarrow{j\to\infty} \infty\,,\end{aligned}$$

but

$$\begin{aligned}&\int_{H_+} |\nabla u_j(x)|^p\, d(x, \partial H_+)^{p-1}\, dx \\ &\quad\leq \int_{[0,3]\times[2,3]} 2^p\, 3^{p-1}\, dx + \sum_{k=0}^{j} \int_{[0,3]\times[2^{-k}, 2^{-k+1}]} 2^p\, 2^{(-k+1)(p-1)}\, dx \\ &\qquad\quad + \int_{[0,3]\times[2^{-(j+1)}, 2^{-j}]} 2^{(j+2)p}\, 2^{-j(p-1)}\, dx \\ &\quad\leq C_1(p) + C_2(p) \sum_{k=0}^{j} 2^{-kp} + C_3(p) = C(p) < \infty\,.\end{aligned}$$

Thus the functions in $C_0^\infty([0,3]^2)$ do not satisfy the $(p,p-1)$-Hardy inequality with a universal constant in the upper half plane, and so we conclude that our domain Ω does not admit the $(p,p-1)$-Hardy inequality. This also means that Ω does not admit the pointwise $(p,p-1)$-Hardy inequality, and hence, by Lemma 3.2, the pointwise (p,β)-Hardy inequality fails in Ω for each $\beta \geq p-1$.

Next we show that Ω admits the usual (p,β)-Hardy inequality also when $p-1 < \beta < p-2+\lambda$: Denote $S_b = \big(\frac14,\frac12\big) \times \big(\frac12,\frac34\big)$, so that S_b is a square above the antenna A, and let $\Omega_g = \Omega \setminus S_b$. Fix $1 < p < \infty$ and $p-1 < \beta < p-2+\lambda$. Let $u \in C_0^\infty(\Omega)$ and denote $\tilde u(s,t) = u\big(\frac14 + s, \frac12 + t\big)$, $\tilde{d_\Omega}(s,t) = d_\Omega\big(\frac14 + s, \frac12 + t\big)$. With an application of Fubini's theorem and integration by parts, we then calculate (recall that we denote $\kappa = \frac14$)

$$
\begin{aligned}
\int_{S_b} |u|^p \, {d_\Omega}^{\beta-p} &\lesssim \int_0^\kappa \int_0^\kappa |\tilde u(s,t)|^p \, t^{\beta-p} \, dt \, ds \\
&\lesssim \int_0^\kappa \Big[|\tilde u(s,\kappa)|^p \, \kappa^{\beta-p+1} + \int_0^\kappa |\tilde u(s,t)|^{p-1} |\nabla \tilde u(s,t)| \, t^{\beta-p+1} \, dt \Big] ds \\
&\lesssim \int_0^\kappa v_s(\kappa) \, ds + \int_{S_b} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} \,,
\end{aligned}
\tag{18}
$$

where we have denoted $v_s(t) = |\tilde u(s,t)|^p \, \tilde{d_\Omega}(s,t)^{\beta-p+1}$. Notice that in (18) we need to use the fact $\beta \neq p-1$. Now

$$
|v_s'(t)| \lesssim |\tilde u(s,t)|^{p-1} |\nabla \tilde u(s,t)| \, \tilde{d_\Omega}(s,t)^{\beta-p+1} + |\tilde u(s,t)|^p \, \tilde{d_\Omega}(s,t)^{\beta-p}
$$

since $|\nabla d_\Omega| \leq 1$, and thus, by changing the integration to the square $\big(\frac14,\frac12\big) \times \big(\frac34, 1\big) \cap \Omega \subset \Omega_g$ above S_b, we obtain

$$
\begin{aligned}
\int_0^\kappa v_s(\kappa) \, ds &\lesssim \int_0^\kappa \int_\kappa^{2\kappa} |v_s'(t)| \, dt \, ds \\
&\lesssim \int_{\Omega_g} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p}.
\end{aligned}
\tag{19}
$$

The pointwise (p,β)-Hardy inequality (4) holds for all $x \in \Omega_g$ with a constant independent of x and u, since these points satisfy the visual boundary condition (7) with the exponent λ. The use of this fact and the maximal function theorem yields, together with (18), (19), and Hölder's inequality, that

$$
\begin{aligned}
\int_\Omega |u|^p \, {d_\Omega}^{\beta-p} &= \int_{S_b} |u|^p \, {d_\Omega}^{\beta-p} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p} \\
&\lesssim \int_{\Omega_g} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p} \\
&\qquad + \int_{S_b} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p} \\
&\lesssim \int_\Omega |\nabla u|^p \, {d_\Omega}^{\beta} + \int_\Omega |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} \\
&\lesssim \int_\Omega |\nabla u|^p \, {d_\Omega}^{\beta} + \Big(\int_\Omega |u|^p \, {d_\Omega}^{\beta-p} \Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\nabla u|^p \, {d_\Omega}^{\beta} \Big)^{\frac1p}.
\end{aligned}
\tag{20}
$$

It is obvious that all the constants in the above calculations depend only on p, β, and Ω. We obtain the (p,β)-Hardy inequality from (20), since if $a, b, C_1 > 0$ and $a \le C_1(b + a^{1-1/p}\, b^{1/p})$, it follows from Young's inequality that $a \le Cb$.

Finally, it is clear (cf. Example 7.1(c)) that Ω does not admit the (p,β)-Hardy inequality when $\beta \ge p - 2 + \lambda$.

Remark. By the truncation technique of Maz'ja [18], the (p,β)-Hardy inequality could also be proven by showing that

$$\int_{\{x\in\Omega : |u(x)|\ge 1\}} d_\Omega{}^{\beta-p} \le C \int_\Omega |\nabla u|^p \, d_\Omega{}^\beta$$

for each $u \in C_0^\infty(\Omega)$.

Example 7.4. Let $1 < p < \infty$ and $1 < \sigma < \lambda < 2$. We construct a simply connected John domain $\Omega_{\lambda,\sigma} \subset \mathbb{R}^2$ which satisfies the density condition (17) with the exponent λ, admits the (p,β)-Hardy inequality for all $\beta < p-2+\sigma$ and $p-2+\sigma < \beta < p-2+\lambda$, but fails to admit the $(p, p-2+\sigma)$-Hardy inequality.

The idea of this construction is to modify the domain Ω_λ of Example 7.3 so that, instead of a "straight antenna" A, the boundary $\partial\Omega_{\lambda,\sigma}$ would contain a more complicated set A_σ, a "snowflake antenna" (see Fig. 2). This A_σ is constructed as follows:

Let F_σ be the standard von Koch snowflake curve of dimension σ, obtained as the invariant set of four similitudes $\varphi_1, \dots, \varphi_4$ with contraction ratio $\frac{1}{4} < \rho = \rho(\sigma) < \frac{1}{2}$, ordered so that $\varphi_1(0) = 0$, $\varphi_j(1) = \varphi_{j+1}(0)$ for $j = 1, 2, 3$, and $\varphi_4(1) = 1$. Denote $z_0 = \varphi_2(1)$, so that z_0 is the "top" of F_σ, and let $z_j = \varphi_j(z_0)$ for $j \in \{1, 2, 3, 4\}$. Denote $\delta = d(z_1, z_2) > 0$ and $K' = K \cup (-K + 1)$, where K is the basic antenna set of dimension λ. Then choose $0 < \tau < \delta$, let $K_0 = i\tau K' + z_0$, and define

$$K_{j_1,\dots,j_k} = \varphi_{j_1} \circ \dots \circ \varphi_{j_k}(K_0).$$

We then have, for example, that

$$d(K_1, K_2) \ge \delta - 2\rho \operatorname{diam}(K_0) > 0\,. \tag{21}$$

Let A'_σ be the union of F_σ and all the images of K_0 under iterations of $\varphi_1, \dots, \varphi_4$:

$$A'_\sigma = F_\sigma \cup \bigcup_{k=1}^{\infty} \bigcup_{j_1,\dots,j_k=1}^{4} K_{j_1,\dots,j_k}\,.$$

Now, we construct the domain $\Omega_{\lambda,\sigma}$ in the same way as Ω_λ in Example 7.3, except that in the last stage we remove, instead of A, the set $A_\sigma = -\kappa A'_\sigma + \frac{i}{2} + 2\kappa$. Using (21), the definition of A'_σ, and properties of F_σ and K, it is then straight-forward to verify that $\Omega_{\lambda,\sigma}$ is a simply connected John domain. Also, it is rather easy to see that $\Omega_{\lambda,\sigma}$ satisfies the visible boundary condition 4.1 with the exponent σ, and hence, by Theorem 5.1, $\Omega_{\lambda,\sigma}$ admits (pointwise) (p,β)-Hardy inequalities for all $\beta < p - 2 + \sigma$.

Let then $w \in \partial\Omega_{\lambda,\sigma}$ and let $0 < r < 1$. If $w \notin A_\sigma$, we obtain as before that $\mathcal{H}^\lambda_\infty\big(B(w,r) \cap \partial\Omega\big) \ge Cr^\lambda$. If $w \in A_\sigma$, we choose $k \in \mathbb{N}$ so that $\rho^k < r \le \rho^{k-1}$; recall that ρ is the contraction ratio of the similitudes in

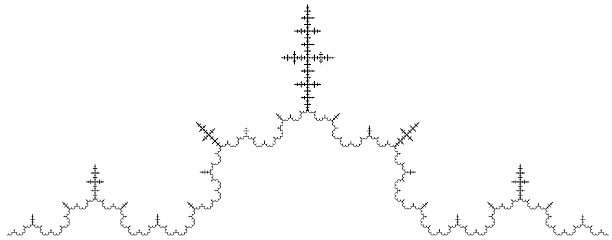

FIGURE 2. The set A'_σ in the construction of Example 7.4 for $\sigma = 1.15$ (and $\lambda = 1.45$)

the construction of the snowflake curve F_σ. Then it follows that there exist $j_1, \ldots, j_k \in \{1,2,3,4\}$ such that $-\kappa K_{j_1,\ldots,j_k} + \frac{i}{2} + 2\kappa \subset B(w,r)$, and thus

$$\mathcal{H}^\lambda_\infty\big(B(w,r)\cap\partial\Omega\big) \geq \kappa^\lambda\,\mathcal{H}^\lambda_\infty\big(K_{j_1,\ldots,j_k}\big) = \kappa^\lambda(\rho^k)^\lambda\mathcal{H}^\lambda_\infty\big(K_0\big) \geq Cr^\lambda,$$

where $C = C(\lambda,\sigma) > 0$. This shows that $\Omega_{\lambda,\sigma}$ satisfies the density condition (17) with the exponent λ.

However, $\Omega_{\lambda,\sigma}$ does not admit the $(p, p-2+\sigma)$-Hardy inequality. This can be seen similarly as above for Ω_λ, by considering a suitable subdomain $S_\sigma \subset \Omega_{\lambda,\sigma}$ above A_σ, chosen so that $\dim(\partial S_\sigma \cap \partial\Omega_{\lambda,\sigma}) = \sigma$, $d(x,\partial\Omega_{\lambda,\sigma}) = d(x, A_\sigma)$ for all $x \in S_\sigma$, and $|S^k_\sigma| \gtrsim (4\rho^2)^k$ for all k greater than some $j_0 \in \mathbb{N}$, where

$$S^k_\sigma = \big\{x \in S_\sigma : \rho^{k+1} \leq d(x,\partial\Omega_{\lambda,\sigma}) \leq \rho^k\big\}.$$

We can then choose functions $u_j \in C_0^\infty(S_\sigma)$ in such a way that $|\nabla u_j| \lesssim \rho^{-j}$ in S^j_σ, $|\nabla u_j| \lesssim 1$ elsewhere in $\operatorname{spt}(|\nabla u_j|)$, and $\int_{S^k_\sigma} |u_j|^p \gtrsim |S^k_\sigma| \gtrsim (4\rho^2)^k$ for all j greater than j_0 and all $k \in \{j_0, \ldots, j-1\}$. Then it follows with easy calculations and the use of the fact $\sigma = (\log 4)/(-\log\rho)$ that

$$\int_{\Omega_{\lambda,\sigma}} |u_j(x)|^p\, d(x,\partial\Omega_{\lambda,\sigma})^{(p-2+\sigma)-p}\,dx \gtrsim (j-j_0) \xrightarrow{j\to\infty} \infty\,,$$

but

$$\int_{\Omega_{\lambda,\sigma}} |\nabla u_j(x)|^p\, d(x,\Omega_{\lambda,\sigma})^{p-2+\sigma}\,dx \leq C(p,\sigma) < \infty\,.$$

Hence the $(p, p-2+\sigma)$-Hardy inequality fails in $\Omega_{\lambda,\sigma}$. Still, it can be shown that $\Omega_{\lambda,\sigma}$ admits the (p,β)-Hardy inequality also when $p-2+\sigma < \beta < p-2+\lambda$; the calculations are similar to those in Example 7.3, and we omit the details.

Examples 7.3 and 7.4 show that, for planar domains, the density condition (17) with $1 < \lambda < 2$ is not sufficient to guarantee (p,β)-Hardy inequalities for all $\beta < p-2+\lambda$. For instance, the $(p, p-1)$-Hardy inequality fails in the domain Ω_λ, since the dense part of the boundary is completely on the "wrong side" of the boundary for points above A. Next we give yet another example in which a density condition even stronger than (17) is satisfied, but another kind of a phenomenon prevents the $(p, p-1)$-Hardy inequality.

Example 7.5. Let $1 < p < \infty$ and $1 < \lambda < 2$. We construct a simply connected domain $\Omega = \Omega_\lambda \subset \mathbb{R}^2$ which satisfies the condition (17) with the exponent λ, but fails to admit the $(p, p-1)$-Hardy inequality. Contrary to the previous examples, we have in fact for each $x \in \Omega$ that

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap \partial D(x)\big) \geq C d_\Omega(x)^\lambda, \tag{22}$$

where $D(x)$ is the x-component of the set $\Omega \cap B\big(x, 2d_\Omega(x)\big)$. Hence this example shows that $\Omega_x(c)$ being a *c-John* domain is essential in the definition of the visual boundary in Section 4.

Let K be the basic antenna set of dimension λ, as defined in Example 7.3. We now take the square $S = [-1, 1] \times [0, 2]$ and replace the edges, with the exception of the part $\big[-\frac{1}{2}, \frac{1}{2}\big] \times \{0\}$, by copies of $\frac{1}{2}K$, oriented so that the antennas are inside S.

Now choose an increasing sequence (n_j), $n_j \in \mathbb{N}$, and $n_1 \geq 2$, so that $2^{-jp} n_j \to \infty$ as $j \to \infty$. Let $K_j^R = i2^{-n_j}K$, $K_j^L = i2^{-n_j}(-K+1)$, and $K_j = K_j^R \cup K_j^L$ for $j \in \mathbb{N}$. Then define

$$T_j = \bigcup_{k=0}^{2^{n_j-1}-1} \big(K_j + ik2^{-n_j}\big),$$

so that T_j is a "stick", made of copies of K_j, with $\operatorname{diam}(T_j) = \frac{1}{2}$. Furthermore, T_j^R and T_j^L are defined analogically, with K_j replaced by K_j^R and K_j^L, respectively.

To complete the construction of Ω, we remove from the already modified square S the following union of sticks:

$$\begin{aligned}\bigcup_{j\in\mathbb{N}} \Bigg[&\Big(\big(T_j^L \cup 2^{-n_j}K\big) + 2^{-(j+1)}\Big) \\ &\cup \bigcup_{k=1}^{2^{-(j+1)+n_j}-1} \Big(\big(T_j \cup 2^{-n_j}K\big) + 2^{-(j+1)} + k2^{-n_j}\Big) \cup \Big(T_j^R + 2^{-j}\Big)\Bigg] \\ &\cup \Big(\tfrac{i}{2}(-K+1) + \tfrac{1}{2}\Big),\end{aligned}$$

as well as the reflection of the above set with respect to the line $i\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2$. Finally, we have to remove also the line segment $[0, i2^{-1}]$ in order to obtain a simply connected domain Ω. It is quite obvious that Ω satisfies Condition 4.1 with exponent 1, and hence Ω admits (p, β)-Hardy inequalities for all $\beta < p-1$. Next we show that Ω does not admit the $(p, p-1)$-Hardy inequality.

Choose functions $u_j \in C_0^\infty(\Omega)$ with the following properties for each $j \geq 2$: $\operatorname{spt}(u_j) \subset [-2^{-j}, 2^{-j}] \times [2^{-1} + 2^{-n_j}, 1]$, $u_j = 1$ in $[-2^{-(j+1)}, 2^{-(j+1)}] \times [2^{-1} + 2^{-n_j+1}, 1-2^{-j}]$, $|\nabla u_j| \lesssim 2^{n_j}$ in $[-2^{-j}, 2^{-j}] \times [2^{-1} + 2^{-n_j}, 2^{-1} + 2^{-n_j+1}]$, and $|\nabla u_j| \lesssim 2^j$ elsewhere in $\operatorname{spt}(|\nabla u_j|)$. Then elementary calculations, similar to those in Example 7.3, give

$$\int_\Omega |u_j|^p \, {d_\Omega}^{-1} \gtrsim 2^{-j} n_j$$

and

$$\int_\Omega |\nabla u_j|^p \, {d_\Omega}^{p-1} \lesssim 2^{-j} \, 2^{jp} \, .$$

From these estimates we see that the $(p, p-1)$-Hardy inequality fails in Ω, since $2^{-jp}\, n_j \to \infty$ as $j \to \infty$.

It is left to show that the modified density condition (22) holds for each $x \in \Omega$. By the construction of Ω and the self-similarity of the antenna set, it is clear that this condition holds with some fixed constant $C > 0$ for all $x \in \Omega$ satisfying $d(x, \partial S) \leq \frac{1}{2}$. Hence we only need to consider points above the sticks, and, in fact, it is enough to consider points $x = it$ for $\frac{1}{2} < t < 1$, since other points can be treated similarly.

When $2^{-j+1} \leq t - \frac{1}{2} \leq 2^{-j+2}$ and $x = it$, the ball $B\bigl(x, 2d_\Omega(x)\bigr)$ intersects roughly $(2^{-j+n_j})^2$ copies of the scaled antenna K_j ; let $\mathcal{K}_j$ denote the union of these copies. Let $\mathcal{K}_j \subset \bigcup_{k\in\mathbb{N}} A_k$ so that $\sum_k \operatorname{diam}(A_k)^\lambda \leq 2\mathcal{H}^\lambda_\infty(\mathcal{K}_j)$. If $\operatorname{diam}(A_k) \leq 2^{-n_j}$ for all $k \in \mathbb{N}$, we obtain that

$$\begin{aligned}\sum_k \operatorname{diam}(A_k)^\lambda &\gtrsim (2^{-j+n_j})^2 \mathcal{H}^\lambda_\infty(K_j)\\ &\gtrsim (2^{-j+n_j})^2\, (2^{-n_j})^\lambda \mathcal{H}^\lambda_\infty(K) \gtrsim (2^{-j})^\lambda,\end{aligned}$$

and thus

$$\mathcal{H}^\lambda_\infty\bigl(\partial\Omega \cap \partial D(x)\bigr) \geq C d_\Omega(x)^\lambda.$$

On the other hand, if $\operatorname{diam}(A_{k_0}) = \delta > 2^{-n_j}$ for some $k_0 \in \mathbb{N}$, we deduce, using the facts that the set $\mathcal{K}_j$ is made of similar sticks with constant distances and that $\{A_k\}$ is an "almost optimal" covering of $\mathcal{K}_j$ with respect to the λ-Hausdorff content, that there exists another "almost optimal" covering $\{\tilde{A}_k\}$ consisting of sets satisfying $\frac{1}{2}\,\delta \leq \operatorname{diam}(\tilde{A}_k) \leq 2\delta$. To cover the set $\mathcal{K}_j$, we need to have at least an amount of the order $(2^{-j})^2/\delta^2$ of such sets, and hence we obtain with simple calculations that (22) holds also in this case. This proves that the density condition (22) holds for all points in Ω, and hence such a condition is not sufficient to guarantee the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$.

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland
E-mail address: pkoskela@maths.jyu.fi, juhaleh@maths.jyu.fi