HARDY INEQUALITIES AND THICKNESS CONDITIONS

JUHA LEHRBÄCK

1. Introduction

The study of the different forms of Hardy inequalities has gathered some considerable interest in the past few decades. These inequalities have natural applications for instance in the theory of partial differential equations and in spectral theory, but they also lead to many interesting questions and perhaps surprising connections between different areas of mathematical analysis. For instance, Hardy inequalities are closely related to the quasiadditivity properties of capacities, see e.g. the work of Hiroaki Aikawa in [1].

In a domain (an open and connected set) $\Omega \subset \mathbb{R}^n$, the (p, β)-Hardy inequality, for $1 < p < \infty$ and $\beta \in \mathbb{R}$, reads as

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \le C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx. \tag{1}$$

Here $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$ denotes the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$, and $u \in C_0^\infty(\Omega)$, i.e., u is a smooth test-function with a compact support in Ω. We say that $\Omega \subset \mathbb{R}^n$ *admits* the (p, β)-Hardy inequality if there exists a constant $C > 0$ so that (1) holds for all $u \in C_0^\infty(\Omega)$ with this constant. In the unweighted case $\beta = 0$ we simply speak of the p-Hardy inequality.

In this talk, I will concentrate on the close connection between the validity of these Hardy inequalities and their variants in a domain and the size and geometry of the complement of that domain. I will first introduce different ways how to measure the size of the complement, for instance in terms of capacities or Hausdorff type contents, and then present some recent developments concerning sufficient and necessary conditions for different types of Hardy inequalities in terms of thickness and other conditions.

2. Thickness conditions

Let us begin with a definition of fatness given in terms of a capacity density condition:

Definition 2.1. (i) Let $\Omega \subset \mathbb{R}^n$ be a domain and let $E \subset \Omega$ be a compact subset. Then the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega), \ u \ge 1 \text{ on } E \right\}.$$

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(ii) A closed set $E \subset \mathbb{R}^n$ is said to be *uniformly p-fat* if there exists a constant $C > 0$ such that

$$\operatorname{cap}_p\big(E \cap \overline{B}(x,r), B(x,2r)\big) \geq C \operatorname{cap}_p\big(\overline{B}(x,r), B(x,2r)\big)$$

for every $x \in E$ and all $r > 0$.

For the basic properties of the p-capacity we refer to [8]; for instance, it is well-known that $\operatorname{cap}_p(\overline{B}(x,r), B(x,2r)) = C(n,p)r^{n-p}$ for each ball $B(x,r) \subset \mathbb{R}^n$.

It is easy to see (with the help of Hölder's inequality) that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $p' > p$, then E is also uniformly p'-fat. In the other direction we have the following highly non-trivial 'self-improvement' result due to John Lewis [20, Thm. 1]:

Theorem 2.2. *Let $1 < p < \infty$ and assume that $E \subset \mathbb{R}^n$ is uniformly p-fat. Then there exists $1 < q < p$ such that E is also uniformly q-fat.*

A more geometric interpretation of fatness can be achieved by means of Hausdorff contents.

Definition 2.3. The *λ-Hausdorff content* of a set $E \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\infty(E) = \inf\Big\{\sum_{i=1}^\infty r_i^\lambda : E \subset \bigcup_{i=1}^\infty B(z_i, r_i)\Big\}.$$

The *Hausdorff dimension* of $E \subset \mathbb{R}^n$ is the number

$$\dim_{\mathcal{H}}(E) = \inf\big\{\lambda > 0 : \mathcal{H}^\lambda_\infty(E) = 0\big\}.$$

We have the following relation between uniform fatness and 'thickness' in terms of Hausdorff contents:

Proposition 2.4. *Let $1 < p < \infty$. Then a closed set $E \subset \mathbb{R}^n$ is uniformly p-fat if and only if there exists some exponent $\lambda > n - p$ and a constant $C > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(E \cap B(w,r)\big) \geq Cr^\lambda \quad \textit{for every } w \in E \textit{ and all } r > 0.$$

For the idea of the proof, see e.g. the discussion in [11]. It is worth a mention that Theorem 2.2 is needed in the proof of the necessity part of Proposition 2.4.

On the other hand, if we are only interested in domains and their complements, we can use the following equivalence from [15]:

Theorem 2.5. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Then the complement Ω^c is uniformly p-fat if and only if the following 'inner' Hausdorff content density condition holds for the boundary $\partial\Omega$ with an exponent $\lambda > n - p$:*

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \quad \textit{for every } x \in \Omega. \tag{2}$$

Note however that the validity of condition (2) with $\lambda > n - p$ does not guarantee the uniform p-fatness of the boundary $\partial\Omega$ (!); an example of this is given by domains with outer cusps.

A similar equivalence as in Proposition 2.4 can be obtained also with the help of Minkowski contents:

Definition 2.6. When $E \subset \mathbb{R}^n$ and $r > 0$, we denote

$$\mathcal{M}_r^\lambda(E) = \inf\left\{ N r^\lambda : E \subset \bigcup_{i=1}^N B(z_i, r),\ z_i \in E \right\}.$$

Then the λ-*Minkowski content* of $E \subset \mathbb{R}^n$ is

$$\mathcal{M}_\infty^\lambda(E) = \inf_{r>0} \mathcal{M}_r^\lambda(E).$$

Lemma 2.7. *Let $E \subset \mathbb{R}^n$ be a closed set. Assume that there exist $0 < \lambda \le n$ and $C_0 > 0$ such that*

$$\mathcal{M}_\infty^\lambda\big(B(w,r) \cap E\big) \ge C_0\, r^\lambda$$

for every $w \in E$ and all $r > 0$. Then, for every $0 < \lambda' < \lambda$, there exists a constant $C = C(C_0, \lambda, \lambda', n) > 0$ such that

$$\mathcal{H}_\infty^{\lambda'}\big(B(w,r) \cap E\big) \ge C\, r^{\lambda'}$$

for every $w \in E$ and all $r > 0$.

As a corollary from Proposition 2.4 and Lemma 2.7 we obtain

Corollary 2.8. *Let $1 < p < \infty$. Then a closed set $E \subset \mathbb{R}^n$ is uniformly p-fat if and only if there exists some exponent $\lambda > n - p$ and a constant $C > 0$ so that the Minkowski thickness condition (2.7) holds for every $w \in E$ and all $r > 0$.*

3. Hardy inequalities

3.1. **One-dimensional inequalities.** The origins of Hardy inequalities trace back to the early 20th century. In the famous 1925 paper [5], G.H. Hardy proved that the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p\, dx, \tag{3}$$

where $1 < p < \infty$, holds whenever $f \ge 0$ is measurable and, moreover, that the constant on the right-hand side is the best possible. An excellent account on the interesting — and not that straigt-forward — progress leading to the discovery of inequality (3) can be found in [14]. Writing $u(x) = \int_0^x f(t)\, dt$ we see that equation (3) corresponds exactly to the unweighted p-Hardy inequality (1) for the domain $(0, \infty) \subset \mathbb{R}$.

The proof of (3) is rather simple, the only tools needed are integration by parts and Hölder's inequality. Exactly the same method can be applied in order to prove one-dimensional weighted Hardy inequalities (cf. [6, §330]), which can be formulated as follows (essentially [13, Thm. 5.2]):

Theorem 3.1. *Let $1 < p < \infty$ and $\beta \ne p - 1$. If $u\colon (0,\infty) \to \mathbb{R}$ is absolutely continuous with $\lim_{x\to 0} u(x) = 0 = \lim_{x\to\infty} u(x)$, then u satisfies the inequality*

$$\int_0^\infty |u(x)|^p\, x^{\beta - p}\, dx \le \left(\frac{p}{|p-1-\beta|} \right)^p \int_0^\infty |u'(x)|^p\, x^\beta\, dx, \tag{4}$$

where the constant on the right-hand side is the best possible.

3.2. **Higher dimensional inequalities.** Hardy inequalities were introduced to higher dimensional Euclidean spaces $\mathbb{R}^n$, $n \geq 2$, by J. Nečas, who also proved the following basic theorem in the 1960's:

Theorem 3.2 (Nečas [21]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$.*

Recall that a domain Ω is Lipschitz if the boundary can be represented locally as graphs of Lipschitz-continuous functions. The proof of Theorem 3.2 is based on this fact and the analog of the one-dimensional inequality (4) for bounded intervals, in which it only holds for $\beta < p - 1$. However, it is nowadays well-understood that the 'smoothness' of the boundary is not that important, but it is indeed the 'thickness' or 'fatness' of the complement (or the boundary) that arises as a natural sufficient condition for Hardy inequalities. The following theorem, dating to the late 1980's, has been of fundamental importance.

Theorem 3.3 (Ancona [2] ($p = 2$), Lewis [20], Wannebo [22]). *Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then Ω admits the p-Hardy inequality*

Note that the exact formulations in [2], [20], and [22] are a bit different to that of Theorem 3.3, but the content is exactly the same. Wannebo actually showed even more, namely that the uniform p-fatness of Ω^c suffices for (p, β)-Hardy inequalities for all $\beta \leq \beta_0$, where $\beta_0 > 0$ is some small (positive) number. However, no explicit expression for β_0 was given.

On the other hand, Lewis showed in [20] that the converse to Theorem 3.3 holds in $\mathbb{R}^n$ *only* in the case $p = n$, that is, the n-Hardy inequality implies uniform n-fatness for the complement, but the p-Hardy inequality may actually hold for some $1 < p < n$ when the complement is *thin* enough.

Recently, I was able to establish the following result, which includes both Theorem 3.2 and Theorem 3.3 as special cases, and gives a sharp bound for β for which the (p, β)-Hardy inequality holds under the inner boundary density condition (2).

Theorem 3.4. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that* (2) *holds in Ω with an exponent $0 \leq \lambda \leq n - 1$. Then Ω admits the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$.*

Indeed, it is easy to see that a bounded Lipschitz domain Ω satisfies (2) with $\lambda = n - 1$, and so Theorem 3.4 gives Hardy inequalities for all $\beta < p - n + (n - 1) = p - 1$; and since uniform p-fatness of Ω^c implies (2) with some $\lambda > n - p$, the value $\beta = 0$ is admissible in the Theorem for domains with uniformly p-fat complements.

The requirement $\lambda \leq n - 1$ (and thus $\beta < p - 1$) is essential in Theorem 3.4, as examples from my earlier work [11] with Pekka Koskela show that the conclusion of Theorem 3.4 need not hold for $\beta \geq p - 1$ even if (2) holds with an exponent $\lambda > n - 1$.

3.3. **Pointwise Hardy inequalities.** A new chapter in the development of Hardy inequalities was opened in the late 1990's, when P. Hajłasz [4] and J. Kinnunen and O. Martio [9] noticed, independently, that the uniform p-fatness of Ω^c implies the following *pointwise* variant of the p-Hardy inequality.

Proposition 3.5 (Hajłasz, Kinnunen-Martio). *Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exist a constant $C > 0$ such that the inequality*

$$|u(x)| \leq C d_\Omega(x) \big(M_{2d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p} \tag{5}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

If the conclusion of Theorem 3.5 holds in a domain Ω we say that Ω admits the pointwise p-Hardy inequality. In (5) M_R is the usual restricted Hardy-Littlewood maximal operator, defined by

$$M_R f(x) = \sup_{0<r\leq R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

It is very easy to show, using the maximal theorem, that if $p < q$ and the pointwise p-Hardy inequality (5) holds for a function $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with a constant $C_1 > 0$, then u satisfies the usual q-Hardy inequality, with a constant $C = C(C_1, p, q, n) > 0$. However, it is *a priori* not at all obvious, if the pointwise p-Hardy inequality (5) suffices for the usual p-Hardy inequality as well. We will return to this question in Section 4.

On the other hand, there are many domains which admit the p-Hardy inequality, but where the corresponding pointwise inequality fails. For example, the punctured unit ball $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$, admits the pointwise p-Hardy inequality only in the trivial case $p > n$, but the usual p-Hardy inequality holds in this domain also when $1 < p < n$; see [11] for more details. Notice that this same domain gives an example of the fact that uniform p-fatness of the complement is not necessary for a domain to admit the p-Hardy inequality, as the complement of $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ is not uniformly p-fat for any $p \leq n$.

In the weighted case, a pointwise analogue of the (p,β)-Hardy inequality was introduced in the paper [11] with P. Koskela: We say that Ω admits the pointwise (p,β)-Hardy inequality, if there exist $C > 0$ and $1 < q < p$ such that

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{2d_\Omega(x)} \big(|\nabla u|^q d_\Omega{}^{\frac{\beta}{p} q} \big)(x) \Big)^{1/q} \quad \text{for all } x \in \Omega \tag{6}$$

whenever $u \in C_0^\infty(\Omega)$; note the small (but slightly inconvenient) difference between inequality (6) in the case $\beta = 0$ and the pointwise p-Hardy inequality (5). Using the maximal theorem and the fact that we have $1 < q < p$ in (6) it is easy to see that inequality (6) implies the usual (p,β)-Hardy inequality (1).

The first general sufficient condition for weighted pointwise Hardy inequalities was given in [11] in the following form:

Theorem 3.6. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that there exist $0 \leq \lambda \leq n$, $c \geq 1$, and $C > 0$ so that*

$$\mathcal{H}^{\lambda}_{\infty}\big(v_x(c)\text{–}\partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{7}$$

Then Ω admits the pointwise (p, β)-Hardy inequality whenever $\beta < p - n + \lambda$.

Here $v_x(c)$–$\partial\Omega$, the *c-visible boundary near $x \in \Omega$*, consists of the points $w \in \partial\Omega$ which are accessible from x by a c-John curve γ in Ω, i.e. a curve which, the end-point $w \in \partial\Omega$ excluded, stays relatively far away from the boundary. Besides Lipschitz-domains, where (7) holds with $\lambda = n - 1$, this condition is satisfied for instance in snowflake-type domains in $\mathbb{R}^n$ with $n-1 < \lambda < n$. Thus, contrary to Theorem 3.4, values $\beta > p - 1$ can be reached in Theorem 3.6 — and thus also in the corresponding usual (p, β)-Hardy inequalities — provided that $\lambda > n - 1$.

For $\beta \leq 0$ the accessibility part of the assumption in Theorem 3.6 is actually not needed, as the following result from [19] shows:

Proposition 3.7. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition* (2) *holds with an exponent $0 \leq \lambda \leq n$. Then, if $\beta \leq 0$ and $\beta < p-n+\lambda$, Ω admits the pointwise (p, β)-Hardy inequality.*

Proposition 3.7 is actually used, together with a 'shift'-property of usual Hardy inequalities (cf. [16]), in the proof of Theorem 3.4.

Contrary to Theorem 3.6 and Proposition 3.7, Theorem 3.4 only treats usual Hardy inequalities, not pointwise. Hence there appears a gap concerning our knowledge on pointwise Hardy inequalities: For $0 < \beta < p - 1$ we do not know if the inner boundary density condition (2) with an exponent $\lambda > n - p + \beta$ suffices for Ω to admit the pointwise (p, β)-Hardy inequality; for $\beta \leq 0$ this is sufficient but for $\beta \geq p - 1$ it is not.

4. Pointwise Hardy inequalities and uniformly fat sets

In the recent work [10] with Riikka Korte and Heli Tuominen we prove the following result; related (but weaker) results can be found in [15].

Theorem 4.1. *Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality* (5) *if and only if the complement Ω^c is uniformly p-fat.*

This equivalence between uniform fatness and pointwise Hardy inequalities has some interesting consequences. For instance, by Theorem 2.2, uniform p-fatness enjoys a self-improvement property. Now, it is immediate from Theorem 4.1 that pointwise p-Hardy inequalities, for $1 < p < \infty$, possess this same property. In particular, we obtain an answer to the question that was mentioned in Section 3.3:

Corollary 4.2. *Let $1 < p < \infty$. If a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then Ω admits the integral p-Hardy inequality as well.*

There are (at least) two distinct ways to prove this. Either one uses the fact that the pointwise p-Hardy inequality implies a pointwise q-Hardy inequality for some $q < p$, and then applies the maximal theorem to obtain the usual p-Hardy, or then one can deduce from the pointwise p-Hardy inequality that Ω^c is uniformly p-fat, and then use a clever integration trick due to A. Wannebo [22] which yields the usual p-Hardy inequality. This latter approach uses only 'elementary' tools, whereas the (known) proofs for the self-improvement use rather sophisticated tools from non-linear potential theory.

To be a bit more precise, we prove in [10] that the following equivalences hold:

Theorem 4.3. *Let $1 \le p < \infty$. Then, for a domain $\Omega \subsetneq \mathbb{R}^n$, the following assertions are quantitatively equivalent:*

(a) *The complement Ω^c is uniformly p-fat.*

(b) *For all $B = B(w, r)$, with $w \in \Omega^c$ and $r > 0$, and for every $u \in C_0^\infty(\Omega)$*

$$\int_B |u|^p \, dx \le C r^p \int_B |\nabla u|^p \, dx.$$

(c) *For all $x \in \Omega$ and every $u \in C_0^\infty(\Omega)$*

$$|u_{B_x}|^p \le C d_\Omega(x)^p \fint_{2B_x} |\nabla u|^p \, dx,$$

where $B_x = B(x, d_\Omega(x))$.

(d) *The domain Ω admits the pointwise p-Hardy inequality* (5).

Here we use the notation

$$u_B = \fint_B u \, dx = |B|^{-1} \int_B u \, dx.$$

Part (a) $\Longrightarrow$ (b) of the proof of Theorem 4.3 follows from a Lemma due to V.G. Maz'ja:

Lemma 4.4. *There is a constant $C > 0$ such that for each $u \in C_0^\infty(\Omega)$ and for all balls $B \subset \mathbb{R}^n$ we have*

$$\fint_B |u|^p \, dx \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u = 0\}, B)} \int_B |\nabla u|^p \, dx. \tag{8}$$

Parts (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (d) are then simple consequences of basic inequalities from Sobolev theory.

The proof of (d) $\Longrightarrow$ (a), which is the only previously unknown part of Theorem 4.3 in the Euclidean case, is also based on the use of 'elementary' tools; in particular, the Poincaré inequality and a basic '$5r$'-covering lemma are needed.

5. A word on metric spaces

Analysis in metric spaces is a rapidly developing area with plenty of applications in various branches of mathematics. The basic context here is a metric measure space $X = (X, d, \mu)$, where μ is a Borel regular outer measure satisfying $0 < \mu(B) < \infty$ for all balls $B \subset X$.

It is usually assumed that (i) the measure μ is *doubling*, i.e., there exists a constant $C_d \geq 1$ such that

$$\mu(2B) \leq C_d\, \mu(B)$$

for all balls B of X; and that (ii) the space X supports a *(weak)* $(1,p)$*-Poincaré inequality*. This means that there exist constants $C_p > 0$ and $\tau \geq 1$ such that for all balls $B \subset X$, all continuous functions u, and for all *upper gradients* g_u of u, we have the inequality

$$\fint_B |u - u_B|\, d\mu \leq C_p r \Big(\fint_{\tau B} g_u^p \, d\mu \Big)^{1/p}. \tag{9}$$

Here a Borel function $g \geq 0$ is said to be an upper gradient of a function u (on an open set $\Omega \subset X$), if for all curves γ joining points x and y (in Ω) we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

Under the above conditions (i) and (ii) the majority of the tools needed to develop meaningful first-order calculus are available in the space X. Examples of metric spaces satisfying both (i) and (ii) include (weighted) Euclidean spaces, compact Riemannian manifolds, Carnot groups, and metric graphs. See for instance the book [7] by Juha Heinonen and the references therein for more information on analysis on metric spaces based on upper gradients and Poincaré inequalities.

As the main tools used in the proofs of our recent results on Hardy inequalities are Poincaré inequalities, basic covering theorems, the maximal theorem, and the self-improvement of uniform p-fatness, and all of these are available under conditions (i) and (ii) (the last one due to J. Björn, P. MacManus, and N. Shanmugalingam [3]), all of our results hold in these metric spaces as well; however, instead of smooth test functions we consider Lipchitz functions with compact supports or the so-called *Newtonian* functions, which offer a substitute for Sobolev functions in this setting. See [10] and [19] for the exact metric space versions of the above results.

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland

E-mail address: `juha.lehrback@jyu.fi`