Thickness: past and present

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PART I:

Conditions for thickness

Capacity

When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E,\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E \right\}.$$

Extention of $\operatorname{cap}_p(\cdot,\Omega)$ for arbitrary sets can be done via the standard procedure (Choquet).

We say that $\operatorname{cap}_p(E) = 0$, if $\operatorname{cap}_p(E',\Omega) = 0$ for all compact $E' \subset E$ and all open $\Omega \supset E'$.

Variational capacity was used by Gehring and Serrin in the early 1960's, and it has close connections to modulus of path families and potentials of Riesz and Bessel. However, these will not be discussed in this talk.

Hausdorff content and measure

The λ-dimensional Hausdorff δ-content of $A \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} r_i^\lambda : A \subset \bigcup_{i=1}^{\infty} B(z_i, r_i), r_i < \delta \right\}.$$

We may in addition assume that $z_i \in A$.

The λ-dimensional Hausdorff measure is

$$\mathcal{H}^\lambda(A) = \lim_{\delta \to 0} \mathcal{H}^\lambda_\delta(A).$$

Note that for each $0 < \delta \leq \infty$ we have $\mathcal{H}^\lambda_\infty(A) \leq \mathcal{H}^\lambda_\delta(A) \leq \mathcal{H}^\lambda(A)$, but still

$$\mathcal{H}^\lambda_\infty(A) = 0 \iff \mathcal{H}^\lambda(A) = 0.$$

Hausdorff dimension

The Hausdorff dimension of A is

$$\dim_{\mathcal{H}}(A) = \inf\{\lambda > 0 : \mathcal{H}^\lambda(A) = 0\}$$
$$= \inf\{\lambda > 0 : \mathcal{H}^\lambda_\infty(A) = 0\}$$

For us, the following difference between measure and content is important: If $\lambda < \dim_{\mathcal{H}}(A)$, then

$$\mathcal{H}^\lambda(A) = \infty \qquad (\to \text{ useless })$$

but *always*

$$\mathcal{H}^\lambda_\infty(A) \le \operatorname{diam}(A)^\lambda \qquad (\to \text{ useful })$$

(For $\lambda = \dim_{\mathcal{H}}(A)$ the *measure* $\mathcal{H}^\lambda(A)$ is usually better, though)

Exceptional sets

The first interest to results concerning the relation between capacities and Hausdorff measures/contents came from the attempts to understand small (or exeptional) sets. Some references include Frostman (1935), Wallin (1963/5), Reshetnyak (1969), Meyers (1970) and Havin–Maz'ya (1972).

The following basic result is well-known, see e.g. Adams–Hedberg or Heinonen–Kilpeläinen–Martio:

Theorem

Let $E \subset \mathbb{R}^n$.
(a) If $\operatorname{cap}_p(E) = 0$, *then* $\dim_{\mathcal{H}}(E) \leq n - p$.
(b) If $\mathcal{H}^{n-p}(E) < \infty$, *then* $\operatorname{cap}_p(E) = 0$.

Thickness

We say that a (closed) set $E \subset \mathbb{R}^n$ is λ-thick at $w \in E$, if there exists $C > 0$ so that

$$\mathcal{H}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \geq Cr^\lambda \quad \text{for all } r > 0.$$

E is λ-thick, if it is λ-thick at every $w \in E$ with the same constant.

Then actually

$$\mathcal{H}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \approx r^\lambda$$

for every $w \in E$ and all $r > 0$.

Quantitative estimates

The basic theorem can be improved to give quantitative estimates, see for instance Reshetnyak (1969), Martio (1978/79):

Theorem

Let $E \subset \mathbb{R}^n$.
(a) If E *is* λ*-thick at* $w \in E$ *and* $\lambda > n - p$*, then*

$$\operatorname{cap}_p\left(E \cap \overline{B}(w,r), B(w,2r)\right) \geq Cr^{n-p}$$

for all $r > 0$.
(b) If $w \in E$ *and*

$$\operatorname{cap}_p\left(E \cap \overline{B}(w,r), B(w,2r)\right) \geq Cr^{n-p}$$

for all $r > 0$*, then* E *is* $(n-p)$*-thick at* w.

Notice the difference in (a) and (b).

Metric spaces

For simplicity, we mainly consider $\mathbb{R}^n$ in this talk, but in fact most of the considerations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \leq C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the 'dimension' of X is at most $s = \log_2 C_d$)
- X supports a (weak) p-Poincaré inequality:

$$\fint_B |u - u_B| \, d\mu \leq C_P r \Big(\fint_{\tau B} g_u^p \, d\mu \Big)^{1/p}$$

whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u: For all (or p-almost all) curves γ joining $x, y \in X$

$$|u(x) - u(y)| \leq \int_\gamma g_u \, ds.$$

Quantitative estimates in metric spaces

Quantitative estimates of the previous theorem were generalized to metric spaces by Heinonen–Koskela (1998) and Costea (2009) (partially with some additional conditions).

Similar estimates appear in the recent work Korte–L–Tuominen (preprint 2009), where instead of the usual Hausdorff content we consider the following *Hausdorff content of co-dimension q* of a set $E \subset X$:

$$\widetilde{\mathcal{H}}^q_R(E) = \inf\Big\{ \sum r_i^{-q} \mu\big(B(x_i, r_i)\big) : E \subset \bigcup B(x_i, r_i),\ r_i \le R \Big\}.$$

Then

$$\widetilde{\mathcal{H}}^q_{r/2}\big(E \cap \overline{B}(w, r)\big) \ge C \mu\big(\overline{B}(w, r)\big) r^{-q} \quad \text{for all } r > 0 \tag{1}$$

with some $1 \le q < p$ leads to the p-capacity estimate

$$\operatorname{cap}_p \big(E \cap \overline{B}(w, r), B(w, 2r)\big) \ge C r^{-p} \mu\big(\overline{B}(w, r)\big) \quad \text{for all } r > 0,$$

and conversely, the above p-capacity estimate gives (1) for p-co-content.

Uniform fatness

A closed set $E \subset \mathbb{R}^n$ is *uniformly p-fat* if

$$\operatorname{cap}_p \left(E \cap \overline{B}(x,r), B(x,2r)\right) \geq C \operatorname{cap}_p \left(\overline{B}(x,r), B(x,2r)\right)$$

for every $x \in E$ and all $r > 0$.
Actually, then

$$\operatorname{cap}_p \left(E \cap \overline{B}(x,r), B(x,2r)\right) \approx r^{n-p}$$

for each $x \in E$ and all $r > 0$.

Recall the famous Wiener criterion for the regularity of a boundary point $x_0 \in \partial\Omega$ for the Dirichlet problem for the p-Laplacian:

$$\int_0^1 \left(\frac{\operatorname{cap}_p \left(E \cap \overline{B}(x_0,r), B(x_0,2r)\right)}{\operatorname{cap}_p \left(\overline{B}(x_0,r), B(x_0,2r)\right)} \right)^{1/(p-1)} \frac{dr}{r} = \infty.$$

Uniform fatness is (of course) stronger than this and implies even Hölder continuity up to the boundary for the solutions
(see Heinonen–Kilpeläinen–Martio).

Uniform fatness: self-improvement

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $q > p$, then E is also uniformly q-fat.

smaller $p \leftrightarrow$ *fatter set*

On the other hand, we have a deep result by J. Lewis:

Theorem (Lewis 1988)

If $E \subset \mathbb{R}^n$ is uniformly p-fat for $1 < p < \infty$, then there exists some $1 < q < p$ such that E is uniformly q-fat.

Mikkonen (1996) proved this in weighted $\mathbb{R}^n$ and Björn, MacManus and Shanmugalingam (2001) in metric spaces.

Equivalence: Uniform fatness and thickness

Using this self-improvement and the previous quantitative estimates we obtain for $1 < p < \infty$:

$E \subset \mathbb{R}^n$ is λ-thick for some $\lambda > n - p$

$\implies$ E is uniformly p-fat

$\implies$ E is uniformly q-fat for some $1 < q < p$

$\implies$ E is $(n - q)$-thick (and $n - q > n - p$).

This can be written as

Corollary

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat if and only if E is λ-thick for some $\lambda > n - p$.

Minkowski content

Let us define a Minkowski-type content of a compact set $A \subset \mathbb{R}^n$: first set

$$\mathcal{M}_r^\lambda(A) = \inf\left\{ N r^\lambda : A \subset \bigcup_{i=1}^{N} B(z_i, r) \right\}$$

(we may again assume $z_i \in A$) and then define

$$\mathcal{M}_\infty^\lambda(A) = \inf_{r>0} \mathcal{M}_r^\lambda(A).$$

Sidenote: the (lower) Minkowski dimension of A is

$$\underline{\dim}_{\mathcal{M}}(A) = \inf\{\lambda > 0 : \mathcal{M}_\infty^\lambda(A) = 0\}.$$

Note that for each compact set $A \subset \mathbb{R}^n$

$$\mathcal{H}_\infty^\lambda(A) \leq \mathcal{M}_\infty^\lambda(A).$$

From Minkowski to Hausdorff

Although Minkowski content can in general be much larger than Hausdorff content, a *uniform* estimate for $\mathcal{M}^{\lambda_0}_\infty$ yields a similar estimate for $\mathcal{H}^{\lambda}_\infty$:

Lemma (L. AASFM 2009)

Let $E \subset \mathbb{R}^n$ be a closed set. Assume that there exist $0 < \lambda_0 \le n$ and $C_0 > 0$ such that

$$\mathcal{M}^{\lambda_0}_\infty\big(\overline{B}(w,r) \cap E\big) \ge C_0\, r^{\lambda_0} \qquad \text{for all } w \in E,\ r > 0.$$

Then, for every $0 < \lambda < \lambda_0$, there exists a constant $C > 0$ such that

$$\mathcal{H}^{\lambda}_\infty\big(\overline{B}(w,r) \cap E\big) \ge C\, r^{\lambda} \qquad \text{for all } w \in E,\ r > 0.$$

Idea of the proof: Fix $\lambda < \lambda_0$ and use the λ_0-Minkowski estimate repetedly to construct a Cantor type subset $C \subset E$, and then show that C is indeed λ-thick.

Equivalence: Minkowski content

As the direction $\mathcal{H}^\lambda_\infty \to \mathcal{M}^\lambda_\infty$ is trivial, we have a further equivalent condition for uniform fatness:

Corollary

Let $1 < p < \infty$. Then the following are equivalent for a closed set $E \subset \mathbb{R}^n$:
(a) E is uniformly p-fat
(b) E is λ-thick for some $\lambda > n - p$, i.e.

$$\mathcal{H}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \geq r^\lambda \qquad \text{for all } w \in E,\ r > 0.$$

(c) E satisfies a uniform Minkowski-content estimate for some $\lambda > n - p$:

$$\mathcal{M}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \geq r^\lambda \qquad \text{for all } w \in E,\ r > 0.$$

PART II:

Thickness, fatness, and Hardy inequalities

Hardy inequalities and uniform fatness

Let us now consider the following p-Hardy inequality in $\mathbb{R}^n$:

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

Theorem (Ancona 1986 ($p = 2$), Lewis 1988, Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality, that is, there exist $C > 0$ so that (2) holds for all $u \in C_0^\infty(\Omega)$ with the same constant C.

However, uniform p-fatness of the complement is *necessary* for the p-Hardy inequality in $\mathbb{R}^n$ only when $p = n$ (Ancona $n = 2$, Lewis).

For instance, $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ admits p-Hardy when $1 < p < n$ or $p > n$, but the complement is uniformly p-fat only when $p > n$.

Pointwise p-Hardy inequality

It is quite straight-forward to obtain the following stronger(?) pointwise inequalities from uniform p-fatness of the complement:

Theorem (Hajłasz 1999, Kinnunen-Martio 1997)

Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exists a constant $C > 0$ such that the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

Here $M_R f$ is the usual restricted Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, defined by $M_R f(x) = \sup_{r \leq R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$

Pointwise implies integral

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$$|u(x)|^{p} \leq C d_\Omega(x)^{p} \left(M_{2d_\Omega(x)}\left(|\nabla u|^q\right)(x)\right)^{p/q}$$

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$$|u(x)|^p d_\Omega(x)^{-p} \quad \leq C \quad \left(M_{2d_\Omega(x)}\left(|\nabla u|^q\right)(x)\right)^{p/q}$$

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$$\int_\Omega |u(x)|^p d_\Omega(x)^{-p}\,dx \le C \int_\Omega \big(M_{2d_\Omega(x)}(|\nabla u|^q)(x)\big)^{p/q}\,dx$$
$$\le C \int_\Omega \big(|\nabla u|^q\big)^{p/q}\,dx$$

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$$\le C \int_\Omega |\nabla u|^p\,dx$$

A boundary Poincaré inequality

In the proof of

$$\Omega^c \text{ uniformly } p\text{-fat} \;\Rightarrow\; \text{pointwise } p\text{-Hardy for } \Omega$$

the following Sobolev-type estimate due to Maz'ya plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\frac{1}{|B|}\int_B |u|^p\,dx \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)}\int_B |\nabla u|^p\,dx. \tag{3}$$

Now, if Ω^c is uniformly p-fat and $u \in C_0^\infty(\Omega)$, it follows from (3) that

$$\int_B |u|^p\,dx \le Cr^p \int_B |\nabla u|^p\,dx$$

for each ball centered at $\partial\Omega$ (a "boundary Poincaré inequality").

This, combined with standard estimates (or a chaining argument) for the maximal functions yields the pointwise p-Hardy inequality.

p-Hardy from p-fatness?

Pointwise Hardy inequalities show that the proof of

$$\Omega^c \text{ uniformly } p\text{-fat} \;\Rightarrow\; p\text{-Hardy for } \Omega$$

can not be completely trivial; indeed, a deep result (self-improvement of p-fatness, Lewis) or clever integration tricks (Wannebo) are being used.

Recall:
uniform p-fatness of Ω^c $\Rightarrow$ pointwise p-Hardy for Ω
$\Rightarrow$ usual q-Hardy for all $q > p$.

How to obtain p-Hardy?

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Use the self-improvement of p-fatness first.

But are there alternative (more 'direct') proofs?

Equivalence: Pointwise Hardy and uniform fatness

In Korte–L–Tuominen (2009) we prove that if Ω admits the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x)\big(M_{2d_\Omega(x)}\big(|\nabla u|^p\big)(x)\big)^{1/p},$$

then the complement Ω^c *has to be* uniformly p-fat, so we obtain an equivalence between these two conditions.

(In particular, pointwise p-Hardy inequalities self-improve!)

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This equivalence means that in a proof of

$$\Omega^c \text{ uniformly } p\text{-fat} \;\Rightarrow\; \Omega \text{ admits } p\text{-Hardy}$$

we have to justify why we can "integrate the above maximal function inequality with exponent 1" to obtain

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx.$$

Hence such a proof should not be 'too easy'.

From pointwise Hardy to fatness

How to prove [pointwise p-Hardy $\Rightarrow$ uniform p-fatness of Ω^c] ?
Main ideas:

From pointwise Hardy to fatness

How to prove [pointwise p-Hardy $\Rightarrow$ uniform p-fatness of Ω^c] ?
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \leq v \leq 1$ and $v \geq 1$ in $B \cap \Omega^c$.
- If $\fint_B v \geq C$, we are done by Poincaré:

$$1 \leq C \fint_B v \leq CR \Big(\fint_{2B} |\nabla v|^p \Big)^{1/p} \;\Rightarrow\; \int_{2B} |\nabla v|^p \geq CR^{n-p}$$

- Otherwise $u = 1 - v$ must have values $\geq C_1$ in a large set $E \subset \frac{1}{4}B$; $|E| \geq C_2|B|$. Moreover, $u = 0$ on $\Omega^c \cap B$.
- $\Rightarrow$ we may use the pw p-Hardy on points $x \in E$; let r_x be the corresponding "almost" best radii ($0 < r_x < 2d_\Omega(x) < R/2$).
- "$5r$"-covering thm. $\Rightarrow$ we find $x_i \in E$ s.t. $B_i = B(x_i, r_i)$ are pairwise disjoint but $E \subset \bigcup 5B_i$.

From pointwise Hardy to fatness...cont'd

- Thus $R^n \leq C|E| \leq C \sum {r_i}^n$
- On the other hand

$$C_1^p \leq |u(x_i)|^p \leq C d_\Omega(x_i)^p M_{2d_\Omega(x)} |\nabla u|^p(x) \leq C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

$$\Rightarrow \; {r_i}^n \leq C R^p \int_{B_i} |\nabla u|^p$$

- Combining the above inequalities with the facts that $|\nabla u| = |\nabla v|$ in B and B_i's are pairwise disjoint, we get

$$R^n \leq C R^p \sum_{i=1}^{\infty} \int_{B_i} |\nabla u|^p \leq C R^p \int_{2B} |\nabla v|^p$$

- Hence $\mathrm{cap}_p(\Omega^c \cap \overline{B}, 2B) \geq C R^{n-p}$, and so Ω^c is unif. p-fat.

Equivalence: boundary Poincaré

Since the validity of the boundary p-Poincaré inequality

$$\int_B |u|^p \, dx \leq Cr^p \int_B |\nabla u|^p \, dx \tag{4}$$

for all $u \in C_0^\infty(\Omega)$ and all balls centered at $\partial\Omega$ follows from the uniform p-fatness of Ω^c and implies the pointwise p-Hardy for Ω, we may conclude that the validity of (4) is in fact equivalent with the other two "p"-properties.

So once more we have a new characterization of thickness.

(And also (4) self-improves; recall here Keith–Zhong (2008))

Uniformly perfect sets and Hardy inequalities

A set $E \subset \mathbb{R}^n$ is *uniformly perfect*, if $\#E \geq 2$ and there exists $c \geq 1$ such that for all $x \in E, \ r > 0$

$$E \cap B(x, cr) \setminus B(x, r) \neq \emptyset$$

(if $E \setminus B(x, cr) \neq \emptyset$).

For unbounded sets, uniform perfecness is equivalent to uniform n-fatness (Järvi–Vuorinen 1996, Sugawa 2003) and the n-Hardy inequality Korte–Shanmugalingam 2009).

Hence, by the previous equivalence results we have:

$E \subset \mathbb{R}^n$ is uniformly perfect and unbounded
$\Leftrightarrow$ E is uniformly n-fat
$\Leftrightarrow$ E is λ-thick for some $\lambda > 0$
$\Leftrightarrow$ E is Minkowski λ-thick for some $\lambda > 0$
$\Leftrightarrow$ E^c admits pointwise n-Hardy $\Leftrightarrow$ E^c admits n-Hardy $\Leftrightarrow$ . . .

PART III:

Boundary conditions
and
weighted Hardy inequalities

Pointwise Hardy implies inner boundary density

If a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then it is easy to see that the following *inner boundary density condition* for $\partial\Omega$ holds for $\lambda = n - p$ (L. PAMS 2008):

there exists a constant $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

Idea: Let $\overline{B}(x, 2d_\Omega(x)) \cap \partial\Omega \subset \bigcup_{i=1}^{N} B(z_i, r_i)$ and use the pointwise p-Hardy for the test function

$$\varphi(y) = \min_{1 \leq i \leq N} \big\{1,\ r_i^{-1} d(y, B(z_i, r_i))\big\} \cdot \chi_\Omega(y) \cdot (\text{cut-off})$$

Pointwise Hardy from inner boundary density

Conversely, if $\Omega \subset \mathbb{R}^n$ and for some $\lambda > n - p$

$$\mathcal{H}^\lambda_\infty\big(\overline{B}(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \quad \text{for every } x \in \Omega, \tag{5}$$

then Ω admits the pointwise p-Hardy inequality (L. PAMS 2008), (KLT 2009).

Using the self-improvement of pointwise Hardy inequalities, we have from the previous results that

$\Omega \subset \mathbb{R}^n$ admits pointwise p-Hardy
$\Longrightarrow$ Ω admits pointwise q-Hardy for some $1 \leq q < p$
$\Longrightarrow$ $\partial\Omega$ satifies (5) with $\lambda = n - q > n - p$
$\Longrightarrow$ $\Omega \subset \mathbb{R}^n$ admits pointwise p-Hardy.

Question: Would it be possible obtain
Ω admits pointwise p-Hardy $\Longrightarrow$ (5) holds with some $\lambda > n - p$
directly? ($\rightarrow$ new proof for self-improvement?)

Inner boundary density and thickness

From the previous slide we obtain also the following interesting characterization:
Ω^c is uniformly p-fat $\Leftrightarrow$ $\partial\Omega$ satisfies inner density condition with some $\lambda > n - p$.

We have compared thickness conditions with uniform p-fatness
($\leftrightarrow$ λ-thickness for $\lambda > n - p$)
and pointwise p-Hardy inequalities
($\leftrightarrow$ inner λ-thickness of $\partial\Omega$ for $\lambda > n - p$)

But since $1 < p < \infty$, the relevant values of λ have been $0 \leq \lambda \leq n - 1$.

However, (Hausdorff) thickness conditions make sense for each $0 \leq \lambda \leq n$.

Inner boundary density and thickness

Let us take another look at the λ-thickness conditions for $0 \leq \lambda \leq n$:

$$(1) \qquad \mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(w,r)\cap\partial\Omega\big) \geq Cr^{\lambda} \text{ for every } r>0,\ w\in\partial\Omega$$

$$(2) \qquad \mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(x,2d_{\Omega}(x))\cap\partial\Omega\big) \geq Cd_{\Omega}(x)^{\lambda} \text{ for every } x\in\Omega$$

$$(3) \qquad \mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(w,r)\cap\Omega^{c}\big) \geq Cr^{\lambda} \text{ for every } r>0,\ w\in\Omega^{c}\ \ (\partial\Omega)$$

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(2) $$\mathcal{H}^\lambda_\infty\big(\overline{B}(x,2d_\Omega(x))\cap\partial\Omega\big) \ge Cd_\Omega(x)^\lambda \text{ for every } x\in\Omega$$

(3) $$\mathcal{H}^\lambda_\infty\big(\overline{B}(w,r)\cap\Omega^c\big) \ge Cr^\lambda \text{ for every } r>0,\ w\in\Omega^c \quad (\partial\Omega)$$

Then
$(1) \Rightarrow (2)$ (trivial) but $(1) \not\Leftarrow (2)$ (cusp in $n \ge 3$)

Inner boundary density and thickness

Let us take another look at the λ-thickness conditions for $0 \le \lambda \le n$:

(1) $\quad \mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(w,r) \cap \partial\Omega\big) \ge Cr^{\lambda}$ for every $r > 0$, $w \in \partial\Omega$

(2) $\quad \mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \ge Cd_{\Omega}(x)^{\lambda}$ for every $x \in \Omega$

(3) $\quad \mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(w,r) \cap \Omega^c\big) \ge Cr^{\lambda}$ for every $r > 0$, $w \in \Omega^c$ $(\partial\Omega)$

Then
(1) $\Rightarrow$ (2) (trivial) but (1) $\not\Leftarrow$ (2) (cusp in $n \ge 3$)

(2) $\Rightarrow$ (3) (L. PAMS 2008) but (2) $\not\Leftarrow$ (3) (ball, $\lambda > n-1$).

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(2) $\Rightarrow$ (3) (L. PAMS 2008) but (2) $\not\Leftarrow$ (3) (ball, $\lambda > n-1$).

However, if we fix $\mu < n-1$, then
(2) with some $\lambda > \mu \iff$ (3) with some $\lambda > \mu$.

Weighted Hardy inequalities

Let us now add a weight $d_\Omega(x)^\beta$, $\beta \in \mathbb{R}$, to the both sides of the p-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x) \quad ^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \qquad dx$$

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This is the (p, β)-*Hardy inequality* for $u \in C_0^\infty(\Omega)$.

The following results hold for weighted Hardy inequalities:

Theorem (Nečas 1962)

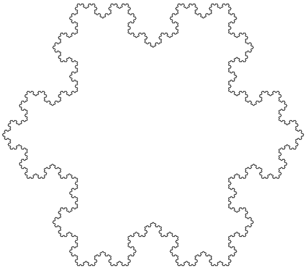
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$ (sharp).

Theorem (Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then there exists some $\beta_0 > 0$ so that Ω admits the (p, β)-Hardy inequality for all $\beta < \beta_0$.

Ball and snowflake

Consider domains $B = B(0,1) \subset \mathbb{R}^2$ and snowflake domain $\Omega \subset \mathbb{R}^2$. Both B and Ω have 2-thick complements, but ∂B satisfies only inner 1-density condition whereas $\partial\Omega$ satisfies inner density condition for $\lambda = \log 4/\log 3$.



p-Hardy inequalities do not 'see' this difference, but *weighted* Hardy inequalities do: For a fixed $1 < p < \infty$,
B admits (p, β)-Hardy iff $\beta < p - 1$ ($= p - n + (n-1)$),
whereas Ω (should) admit (p, β)-Hardy iff $\beta < p - 2 + \lambda$.

This observation of P. Koskela was the starting point for all my research on Hardy inequalities.

Weighted pointwise Hardy inequalities

We also have the following pointwise version of the weighted (p, β)-Hardy inequality:

$$|u(x)| \leq C d_\Omega(x)^{1} \quad \Big(M_{2 d_\Omega(x)} \big(|\nabla u|^{q} \qquad \big)(x) \Big)^{1/q}, \tag{6}$$

where we assume that $1 < q < p$ (self-improvement?).

Weighted pointwise Hardy inequalities

We also have the following pointwise version of the weighted (p, β)-Hardy inequality:

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{2d_\Omega(x)} \big(|\nabla u|^q d_\Omega^{\frac{\beta}{p}q} \big)(x) \Big)^{1/q}, \qquad (6)$$

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where we assume that $1 < q < p$ (self-improvement?).

We say that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality if there exist some $1 < q < p$ and a constant $C > 0$ so that (6) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with these q and C.

As in the unweighted case, the pointwise (p, β)-Hardy inequality implies the usual weighted (p, β)-Hardy inequality.

Accessibility

Theorem (Koskela-L. JLMS 2009)

Let $1 < p < \infty$ *and let* $\Omega \subset \mathbb{R}^n$ *be a domain. Assume that there exist* $0 \leq \lambda \leq n$, $c \geq 1$, *and* $C > 0$ *so that*

$$\mathcal{H}^{\lambda}_{\infty}\big(v_x(c) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \qquad \text{for every } x \in \Omega. \tag{7}$$

Then Ω *admits the pointwise* (p, β)*-Hardy inequality whenever* $\beta < p - n + \lambda$.

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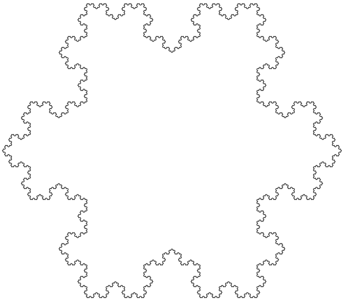
Then Ω *admits the pointwise* (p, β)*-Hardy inequality whenever* $\beta < p - n + \lambda$.

A point $w \in \partial\Omega$ is in the set $v_x(c)\text{–}\partial\Omega$, if w is *accessible* from x by a *c-John curve*, that is, there exists a curve $\gamma = \gamma_{w,x} \colon [0, l] \to \Omega$, parametrized by arc length, with $\gamma(0) = w$, $\gamma(l) = x$, and satisfying $d(\gamma(t), \partial\Omega) \geq t/c$ for every $t \in [0, l]$.
(Thus (7) is a stronger version of the inner boundary density condition introduced earlier)

Examples

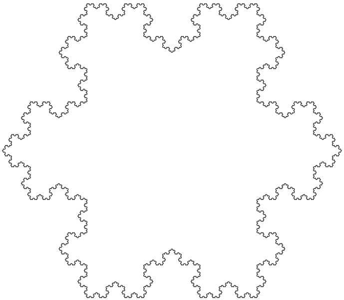


Here the boundary is λ-thick $(1 < \lambda < 2)$ and well accessible

$\Rightarrow (p, \beta)$-Hardy for all

$$\beta < \underbrace{p - n + \lambda}_{>p-1}$$

Examples



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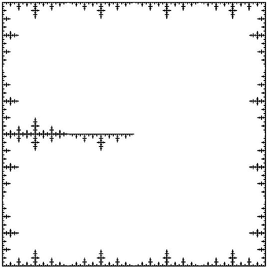


Here the boundary is λ-thick $(1 < \lambda < 2)$, but above the antenna in the middle the *accessible* part of the boundary is only 1-dimensional, and indeed the (p, β)-Hardy does not hold when

$$\beta = p - 1 < p - n + \lambda$$

Removing accessibility

The accessibility part of the previous theorem can actually be dropped (at least) whenever $\beta \leq 0$:

Theorem (L. 2010)

Let $1 < p < \infty$, *let* $\Omega \subset \mathbb{R}^n$ *be a domain, and assume that the inner boundary density condition holds with an exponent* $0 \leq \lambda \leq n$. *Then, if* $\beta \leq 0$ *and* $\beta < p - n + \lambda$, Ω *admits the pointwise* (p, β)*-Hardy inequality.*

This, together with a 'shift'-property of usual Hardy inequalities (L. ACV 2008) leads to the following result:

Theorem (L. 2010)

Let $1 < p < \infty$, *let* $\Omega \subset \mathbb{R}^n$ *be a domain, and assume that the inner boundary density condition holds with an exponent* $0 \leq \lambda \leq n - 1$. *Then* Ω *admits the* (p, β)*-Hardy inequality for all* $\beta < p - n + \lambda$.

In other words

We can rewrite the previous theorem as

Theorem (L. 2010)

Assume that Ω^c is uniformly q-fat for all $q > s \geq 1$. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - s$.

This is a nice generalization of both the Ancona–Lewis–Wannebo -theorem and Nečas' theorem.

Conclusion and a gap

In conclusion, if $1 < p < \infty$, $\beta \in \mathbb{R}$, and $\partial\Omega \subset \mathbb{R}^n$ is inner λ-thick for $\lambda > n - p + \beta$, then Ω admits

- (p, β)-Hardy if $\beta < p - 1$;
- pointwise (p, β)-Hardy if $\beta \leq 0$;
- pointwise (p, β)-Hardy if $\partial\Omega$ is in addition accessible.

On the other hand, inner λ-thickness for $\lambda > n - p + \beta$ *does not suffice* for (p, β)-Hardy if $\beta \geq p - 1$.

Above we have a gap: Does λ-thickness for $\lambda > n - p + \beta$ suffice for *pointwise* (p, β)-Hardy if $0 < \beta < p - 1$?

Equivalence: weighted pointwise for $\beta \leq 0$

Just like in the unweighted case, inner boundary fatness for some $\lambda > n - p + \beta$ is also necessary for the pointwise (p, β)-Hardy, (L. AASFM 2009). Thus, for $\beta \leq 0$ we obtain yet another equivalent condition for thickness (and uniform fatness):

For $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, and $\beta \leq 0$,
$\partial\Omega$ is inner λ-thick for some $\lambda > n - p + \beta$
$\Leftrightarrow$ Ω admits pointwise (p, β)-Hardy inequality.

Actually, for $\beta < 0$ we first obtain a Minkowski-type thickness condition for the boundary, which then yields a similar condition for Hausdorff-content.

What is necessary for Hardy?

If the complement of Ω contains a part of dimension μ, then Ω can not admit the $(p, p-n+\mu)$-Hardy.
(Koskela–Zhong 2003, $\beta = 0$ i.e. $p = n - \mu$), (L. MM 2008)

More precisely:

Theorem

Let $1 < p < \infty$, $\beta \neq p$, and assume that a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality. Then for a given ball $B \subset \mathbb{R}^n$ either

$$\dim_{\mathcal{H}}(4B \cap \Omega^c) > n - p + \beta$$

or

$$\dim_{\mathcal{A}}(B \cap \Omega^c) < n - p + \beta.$$

Here $\dim_{\mathcal{A}}$ is a concept of dimension, introduced by Aikawa, s.t. $\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{A}}(E)$ for all $E \subset \mathbb{R}^n$.

Aikawa–Assouad

When $E \subset \mathbb{R}^n$ is a closed set with an empty interior, we let $G(E)$ denote the set of $s > 0$ for which there exists $C_s > 0$ s.t.

$$\int_{B(x,r)} d(y,E)^{s-n}\,dy \leq C_s r^s$$

for every $x \in E$ and all $r > 0$.

Then $\dim_{\mathcal{A}}(E) = \inf G(E)$.
The above condition can also be expressed as a *uniform cube-counting condition* for Whitney cubes of E^c.

We then have $\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{A}}(E) \leq n$, but if E is sufficiently regular, e.g. a compact submanifold of $\mathbb{R}^n$ or a nice self-similar fractal, then $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{A}}(E)$.

It has recently turned out that $\dim_{\mathcal{A}}$ agrees with the *Assouad dimension*.

Thick and thin

If Ω^c (or $\partial\Omega$) contains a part E with $\dim_{\mathcal{A}}(E) = \mu$, then $(p, p - n + \mu)$-Hardy fails.

If $\partial\Omega$ contains in addition an inner λ-thick part, where $\lambda > \mu$, the (p, β)-Hardy may hold for some $\beta > p - n + \mu$
(L. MM 2008).

Then, for exponents $\beta > p - n + \mu$, the μ-dimensional boundary parts are 'too small' and we may neglect them, if the λ-thick part is visible.

Thick and thin: an example

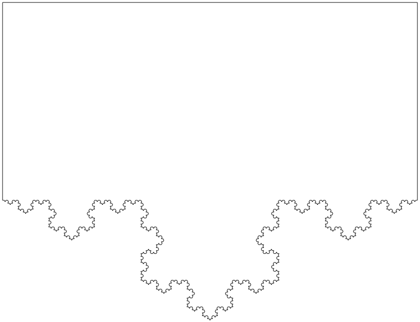
Here $\Omega \subset \mathbb{R}^2$ admits (p, β)-Hardy when

$\beta < p - 1$

or

$p - 1 < \beta < p - 2 + \lambda$,

where $\lambda > 1$ is the dimension of the snowflake curve.

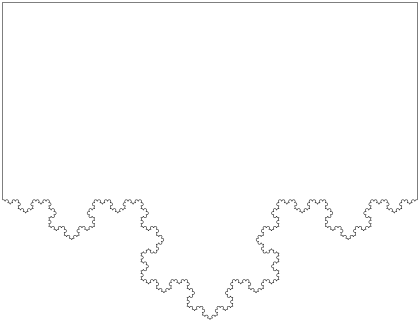


$\beta = p - 1 = p - 2 + 1$ is not possible since the boundary contains a 1-dimensional part.

Exponents $p - 1 < \beta < p - 2 + \lambda$ are ok;
1-dimensional parts are too small, and the thick part is visible.

Thickness is not enough

Existence of a thick boundary part is not sufficient: Let us add small 'rooms' on top of the previous Ω, and make the 'doors' smaller and smaller compared to the rooms.



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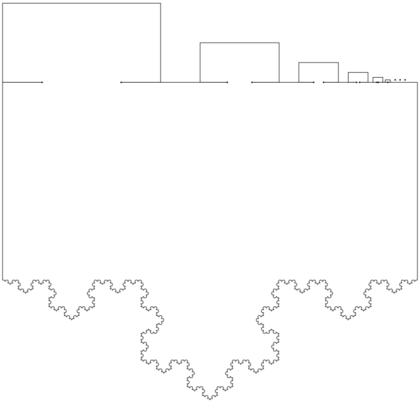
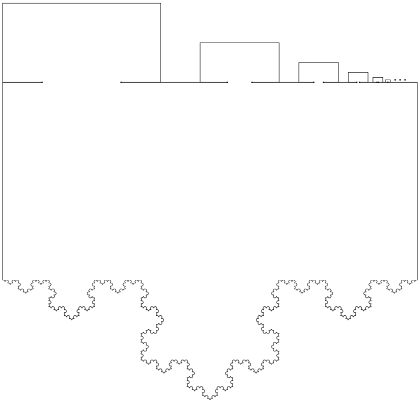


Thickness is not enough

Existence of a thick boundary part is not sufficient: Let us add small 'rooms' on top of the previous Ω, and make the 'doors' smaller and smaller compared to the rooms. Then the (p, β)-Hardy does not hold for any $\beta \geq p - 1$.



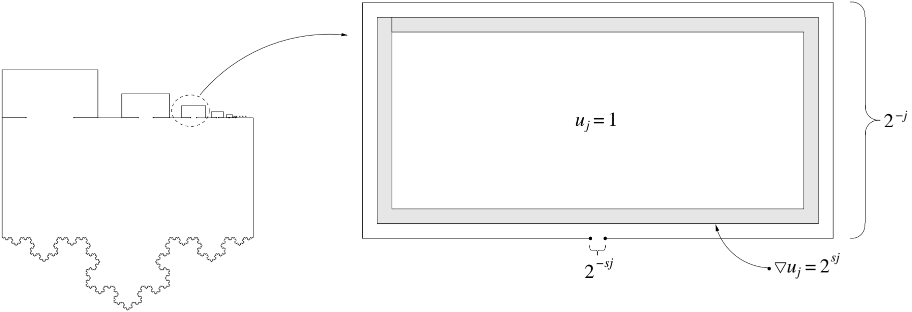
(without the rooms the inequality did hold for $p - 1 < \beta < p - 2 + \lambda$.)

The thick part is not visible!

Counterexamplefunctions

Let us show that the (p, β)-Hardy fails in Ω when $\beta > p - 1$.

Let the length of the j:th room be 2^{-j} and the width of the door be 2^{-sj}, where $s > 1$. Define functions $u_j \colon \Omega \to \mathbb{R}$ as in the figure.



Calculation

Then

$$\int |u_j|^p d_\Omega^{\beta-p} \gtrsim 2^{-2j} 2^{-j(\beta-p)} = 2^{-j(2-p+\beta)}$$

and

$$\int |\nabla u_j|^p d_\Omega^{\beta} \lesssim 2^{-sj} 2^{-j} 2^{sjp} 2^{-js\beta} = 2^{-js(1-p+\beta)},$$

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Then

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and so

$$\frac{\int |u_j|^p {d_\Omega}^{\beta-p}}{\int |\nabla u_j|^p {d_\Omega}^{\beta}} \gtrsim \frac{2^{-j(2-p+\beta)}}{2^{-js(1-p+\beta)} 2^{-j}} = 2^{-j(1-p+\beta)(1-s)} \xrightarrow{j\to\infty} \infty,$$

as $(1-p+\beta)(1-s) < 0$, when $s > 1$ and $\beta > p-1$.

Calculation

Then

$$\int |u_j|^p d_\Omega^{\beta-p} \gtrsim 2^{-2j} 2^{-j(\beta-p)} = 2^{-j(2-p+\beta)}$$

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as $(1-p+\beta)(1-s) < 0$, when $s > 1$ and $\beta > p-1$.

Thus the (p,β)-Hardy inequality fails, and we stop here.

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