Hardy inequalities, uniform fatness, and boundary density

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1. Introduction: Hardy inequalities

The original p-Hardy inequality

G.H. Hardy published in 1925 the inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p\, dx,$$

where $1 < p < \infty$ and $f \ge 0$ is measurable.

Taking $u(x) = \int_0^x f(t)\, dt$, the above *p-Hardy inequality* can be written as

$$\int_0^\infty \frac{|u(x)|^p}{x^p}\, dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |u'(x)|^p\, dx,$$

where $1 < p < \infty$ and u is absolutely continuous with $u(0) = 0$.

Hardy inequalities in $\mathbb{R}^n$

The 1-dimensional p-Hardy inequality

$$\int_0^\infty \frac{|u(x)|^p}{x^p}\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\,dx$$

can be generalized to higher dimensions in many ways; we consider the following:

$$\int_\Omega \frac{|u(x)|^p}{d_\Omega(x)^p}\,dx \le C \int_\Omega |\nabla u(x)|^p\,dx,$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

The p-Hardy inequality is not always valid

If the p-Hardy inequality

$$\int_\Omega \frac{|u(x)|^p}{d_\Omega(x)^p}\,dx \le C \int_\Omega |\nabla u(x)|^p\,dx$$

holds for all $u \in C_0^\infty(\Omega)$ with a constant $C > 0$, we say that the domain $\Omega \subset \mathbb{R}^n$ *admits* the p-Hardy inequality.
(In this talk, we are not interested in the optimality of the constant C)

Not all domains admit a p-Hardy inequality. For instance, it is easy to calculate that $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ does not admit the n-Hardy inequality.

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Not all domains admit a p-Hardy inequality. For instance, it is easy to calculate that $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ does not admit the n-Hardy inequality. (For $1 < p < n$ and $p > n$ this domain actually admits the p-Hardy inequality.)

Our main interest is in finding (e.g. geometric) conditions which guarantee the validity of the p-Hardy inequality on a domain Ω.

Metric spaces

For simplicity, we mainly consider $\mathbb{R}^n$ in this talk, but in fact most of the considerations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \leq C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the 'dimension' of X is at most $s = \log_2 C_d$)
- X supports a (weak) $(1, p)$-Poincaré inequality:

$$\fint_B |u - u_B| \, d\mu \leq C_P r \Big(\fint_{\lambda B} g_u^p \, d\mu \Big)^{1/p}$$

whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u: For all (or p-almost all) curves γ joining $x, y \in X$ we have $|u(x) - u(y)| \leq \int_\gamma g_u \, ds$.

We use here and in the following the notation

$$u_B = \fint_B u \, d\mu = \mu(B)^{-1} \int_B u \, d\mu.$$

Sufficient conditions for Hardy inequalities

Theorem (Nečas 1962)

Let $1 < p < \infty$ and assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then Ω admits the p-Hardy inequality

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The "smoothness" of the boundary is not relevant:

Theorem (Ancona 1986 ($p = 2$), Lewis 1988, Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

For instance, if $\Omega \subset \mathbb{R}^n$ is bounded Lipschitz, then Ω^c is uniformly p-fat for all $1 < p < \infty$.

Capacity and fatness

When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E,\Omega) = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E\right\}.$$

A closed set $E \subset \mathbb{R}^n$ is *uniformly p-fat* if

$$\operatorname{cap}_p\big(E \cap \overline{B}(x,r), B(x,2r)\big) \geq C\,\operatorname{cap}_p\left(\overline{B}(x,r), B(x,2r)\right)$$

for every $x \in E$ and all $r > 0$.
Actually, then

$$\operatorname{cap}_p\big(E \cap \overline{B}(x,r), B(x,2r)\big) \approx r^{n-p}$$

for each $x \in E$ and all $r > 0$.

Uniform fatness: self-improvement

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $q > p$, then E is also uniformly q-fat.

smaller $p \leftrightarrow$ *fatter set*

On the other hand, we have a deep result by J. Lewis:

Theorem (Lewis 1988)

If $E \subset \mathbb{R}^n$ is uniformly p-fat for $1 < p < \infty$, then there exists some $1 < q < p$ such that E is uniformly q-fat.

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(Björn, MacManus and Shanmugalingam (2001) proved the same in metric spaces.)

2. Pointwise Hardy inequalities

Hardy inequalities and uniform fatness

Recall the p-Hardy inequality

$$\int_\Omega \frac{|u(x)|^p}{d_\Omega(x)^p}\,dx \le C \int_\Omega |\nabla u(x)|^p\,dx,$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$;

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Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

However, uniform p-fatness of the complement is *necessary* for the p-Hardy inequality in $\mathbb{R}^n$ only when $p = n$ (Ancona $n = 2$, Lewis).

For instance, $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ admits p-Hardy when $1 < p < n$ or $p > n$, but the complement is uniformly p-fat only for $p > n$.

Pointwise p-Hardy inequality

It is quite straight-forward to obtain the following stronger(?) pointwise inequalities from uniform p-fatness of the complement:

Theorem (Hajłasz 1999, Kinnunen-Martio 1997)

Let $1 < p < \infty$ and assume that Ω^c is uniformly p-fat. Then there exists a constant $C > 0$ such that the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2 d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

Here $M_R f$ is the usual restricted Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, defined as $M_R f(x) = \sup_{r \leq R} ⨍_{B(x,r)} |f(y)|\, dy$.

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$$|u(x)|^{p'} d_\Omega(x)^{-p'} \quad \le C \quad \left(M_{2d_\Omega(x)}\left(|\nabla u|^p\right)(x)\right)^{p'/p}$$

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A boundary Poincaré inequality

In the proof of $\left[\ \Omega^c \text{ uniformly } p\text{-fat } \Rightarrow \text{ pointwise } p\text{-Hardy for } \Omega\ \right]$ the following Sobolev-type estimate due to Maz'ya plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\fint_B |u|^p\,dx \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)} \int_B |\nabla u|^p\,dx. \tag{1}$$

Now, if Ω^c is uniformly p-fat and $u \in C_0^\infty(\Omega)$, it follows from Hölder's inequality and (1) that the following 'boundary Poincaré inequality'

$$|u_B| \le \left(\fint_B |u|^p\right)^{1/p} \le C\left(r^{p-n}\int_B |\nabla u|^p\right)^{1/p} = Cr\left(\fint_B |\nabla u|^p\right)^{1/p}$$

holds for each ball $B = B(w,r)$ with $w \in \partial\Omega$.

Proof of the pointwise inequality:

The previous estimate, standard estimates (or a chaining argument) for the maximal function, and the usual Poincaré inequality now yield the pointwise p-Hardy inequality:
Let $x \in \Omega$, pick $w \in \partial\Omega$ such that $d(x,w) = d_\Omega(x)$, and write $B_x = B(x, d_\Omega(x))$, $B_w = B(w, d_\Omega(x)) \subset 2B_x$. Then

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Wannebo uses an inequality similar to the 'boundary Poincaré inequality' and a clever integration trick (this is not trivial, but still 'elementary').

But can we prove that the pointwise p-Hardy inequality implies the usual p-Hardy inequality?

Equivalence: Pointwise Hardy and uniform fatness

With Riikka Korte and Heli Tuominen (Math. Ann, 2011) we show that if Ω admits the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p},$$

then the complement Ω^c *has to be* uniformly p-fat, and so we have an equivalence between these two conditions.

(In particular, pointwise p-Hardy inequalities self-improve!)

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(In particular, pointwise p-Hardy inequalities self-improve!)

This equivalence means that in a proof of

$$\Omega^c \text{ uniformly } p\text{-fat} \;\Rightarrow\; \Omega \text{ admits } p\text{-Hardy}$$

we have to justify the "integration of the pointwise maximal function inequality with exponent 1" to obtain

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx.$$

Hence such a proof should not be 'too easy'.

From pointwise Hardy to fatness

So how to prove [pointwise p-Hardy $\Rightarrow$ uniform p-fatness of Ω^c] ?
Main ideas:

From pointwise Hardy to fatness

So how to prove [pointwise p-Hardy $\Rightarrow$ uniform p-fatness of Ω^c] ?
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$ and $v \in C_0^\infty(2B)$ be a capacity test function for $\Omega^c \cap B$, i.e. $0 \leq v \leq 1$ and $v = 1$ in $\Omega^c \cap B$.
- If $\fint_B v \geq c$ (where $0 < c < 1$ is a fixed small number), we are done by Poincaré (for $v \in C_0^\infty(2B)$):

$$c \leq \fint_B v \leq R\Big(\fint_{2B} |\nabla v|^p\Big)^{1/p} \;\Rightarrow\; \int_{2B} |\nabla v|^p \geq CR^{n-p}$$

- Otherwise $u = 1 - v$ must have values $\geq C_1 = C_1(c)$ in a large set $E \subset \frac{1}{4}B$; $|E| \geq C_2|B|$. Moreover, $u = 0$ on $\Omega^c \cap B$.
- We may use the pointwise p-Hardy on points $x \in E$; let r_x be the corresponding (almost) best radii ($0 < r_x < 2d_\Omega(x) < R/2$).
- By the $5r$-covering lemma we find $x_i \in E$ s.t. $B_i = B(x_i, r_i)$ are pairwise disjoint but $E \subset \bigcup 5B_i$.

From pointwise Hardy to fatness...cont'd

- Thus $R^n \le C|E| \le C \sum {r_i}^n$
- On the other hand

$$C_1^p \le |u(x_i)|^p \le C d_\Omega(x_i)^p M_{2d_\Omega(x)} |\nabla u|^p(x) \le C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

$$\Rightarrow \quad {r_i}^n \le C R^p \int_{B_i} |\nabla u|^p$$

- Combining the above inequalities with the facts that $|\nabla u| = |\nabla v|$ in B and that B_i's are pairwise disjoint, we get

$$R^n \le C R^p \sum_{i=1}^{\infty} \int_{B_i} |\nabla u|^p \le C R^p \int_{2B} |\nabla v|^p$$

- Hence $\mathrm{cap}_p(\Omega^c \cap \overline{B}, 2B) \ge C R^{n-p}$, and so Ω^c is uniformly p-fat.

Conclusion

We thus have for $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ that
Ω^c uniformly p-fat $\iff$ Ω admits pointwise p-Hardy,

and the proof is based on the use of 'elementary tools'; more precisely, sophisticated machinery from potential theory is not needed.

By Wannebo's integration trick,
Ω^c uniformly p-fat $\implies$ Ω admits usual p-Hardy,

and so we have an 'elementary' proof for the fact that
Ω admits pointwise p-Hardy $\implies$ Ω admits usual p-Hardy.

3. Weighted Hardy inequalities and boundary conditions

Weighted Hardy inequalities

Let us add a weight $d_\Omega(x)^\beta$, $\beta \in \mathbb{R}$, to the both sides of the p-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x) \;\; ^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \qquad\qquad dx$$

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$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta \, dx$$

This is the *(weighted)* (p,β)*-Hardy inequality* for $u \in C_0^\infty(\Omega)$.
The following have been known for weighted Hardy inequalities:

Theorem (Nečas 1962)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p,β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$ (sharp).

Theorem (Wannebo 1990)

Assume that Ω^c is uniformly p-fat. Then there exists some $\beta_0 > 0$ so that Ω admits the (p,β)-Hardy inequality for all $\beta < \beta_0$.

A precise statement under uniform fatness

We have the following recent result:

Theorem (L. PAMS, to appear)

Assume that Ω^c is uniformly q-fat for all $q > s \geq 1$. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - s$.

For instance, if $\Omega \subset \mathbb{R}^2$ is simply connected, then Ω admits the (p, β)-Hardy whenever $\beta < p - 1$.

The idea of the proof is quite simple: By the assumption, Ω^c is uniformly $(p - \beta)$-fat, and so Ω admits the $(p - \beta)$-Hardy inequality. Assume $\beta > 0$. Then, given $u \in C_0^\infty(\Omega)$, we can use the $(p - \beta)$-Hardy inequality for the test function $v = |u|^{\beta/(p-\beta)}$, and the (p, β)-inequality for u follows with a simple calculation using Hölder's inequality.

For $\beta < 0$, the claim follows from *weighted pointwise* Hardy inequalities (see below).

Hausdorff content and thickness

With the help of Hausdorff contents, even more can be said about weighted Hardy inequalities.

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Recall that the λ-dimensional Hausdorff δ-content of $A \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\delta(A) = \inf \Bigl\{ \sum_{i=1}^{\infty} r_i^\lambda : A \subset \bigcup_{i=1}^{\infty} B(z_i, r_i), r_i < \delta \Bigr\}.$$

(We may in addition assume that $z_i \in A$.)

We say that a (closed) set $E \subset \mathbb{R}^n$ is λ-thick, if there exists $C > 0$ so that

$$\mathcal{H}^\lambda_\infty\bigl(E \cap \overline{B}(w, r)\bigr) \geq C r^\lambda \quad \text{for all } r > 0, \ w \in E.$$

It is known that

E is uniformly p-fat $\implies$ E is $(n-p)$-thick

and

E is λ-thick, $\lambda > n - p \implies E$ is uniformly p-fat.

Equivalence: Uniform fatness and thickness

Thus, using the self-improvement of uniform fatness, we obtain for $1 < p < \infty$:

$E \subset \mathbb{R}^n$ is λ-thick for some $\lambda > n - p$

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$\implies$ E is uniformly p-fat
$\implies$ E is uniformly q-fat for some $1 < q < p$
$\implies$ E is $(n - q)$-thick (and $n - q > n - p$).

This can be written as

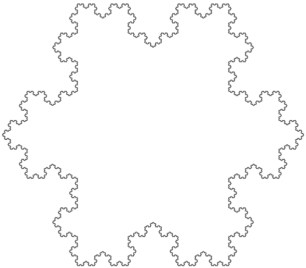
Corollary

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat if and only if E is λ-thick for some $\lambda > n - p$.

(Actually, even more is true: the above conditions are equivalent to a uniform thickness condition for the λ-dimensional Minkowski content, with $\lambda > n - p$.)

Ball and snowflake

Consider domains $B = B(0, 1) \subset \mathbb{R}^2$ and a 'snowflake' domain $\Omega \subset \mathbb{R}^2$. Both B and Ω have 2-thick complements, but ∂B is only 1-thick, whereas $\partial\Omega$ is thick up to $\lambda = \log 4/ \log 3$.



p-Hardy inequalities do not 'see' this difference, but *weighted* Hardy inequalities do: For a fixed $1 < p < \infty$,
B admits (p, β)-Hardy iff $\beta < p - 1 \quad (= p - n + (n - 1))$,
whereas Ω (should) admit (p, β)-Hardy iff $\beta < p - 2 + \lambda$.

This observation by P. Koskela was the starting point for all my research on Hardy inequalities.

Inner density

We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies *inner boundary density condition* for $0 \leq \lambda \leq n$ if there exists a constant $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(\partial\Omega \cap \overline{B}(x, 2d_{\Omega}(x))\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

We have the following characterization:

Theorem (L. PAMS 2008)

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$. Then Ω^c is uniformly p-fat if and only if $\partial\Omega$ satisfies inner boundary density for some $\lambda > n - p$.

Note that since $1 < p < \infty$, the relevant values of λ are $0 \leq \lambda \leq n - 1$. Thus, for these λ, inner boundary density is equivalent to the λ-thickness of the complement.

Weighted pointwise Hardy inequalities

We also have the following pointwise version of the weighted (p,β)-Hardy inequality:

$$|u(x)| \leq C d_\Omega(x)^{1} \quad \Big(M_{2d_\Omega(x)}\big(|\nabla u|^q \qquad\big)(x)\Big)^{1/q}, \tag{2}$$

where we assume that $1 < q < p$ (self-improvement?).

Weighted pointwise Hardy inequalities

We also have the following pointwise version of the weighted (p,β)-Hardy inequality:

$$|u(x)| \le C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{2d_\Omega(x)} \big(|\nabla u|^q d_\Omega{}^{\frac{\beta}{p} q} \big)(x) \Big)^{1/q}, \qquad (2)$$

where we assume that $1 < q < p$ (self-improvement?).

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where we assume that $1 < q < p$ (self-improvement?).

We say that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality if there exist some $1 < q < p$ and a constant $C > 0$ so that (2) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with these q and C.

As in the unweighted case, the pointwise (p, β)-Hardy inequality implies the usual weighted (p, β)-Hardy inequality (thanks to the built-in 'self-improvement').

Accessibility

Theorem (Koskela-L. JLMS, 2009)

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that there exist $0 \leq \lambda \leq n$, $c \geq 1$, and $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}(\partial^{\mathrm{vis}}_{x,c}\Omega) \geq C d_{\Omega}(x)^{\lambda} \qquad \text{for every } x \in \Omega. \tag{3}$$

Then Ω admits the pointwise (p, β)-Hardy inequality whenever $\beta < p - n + \lambda$.

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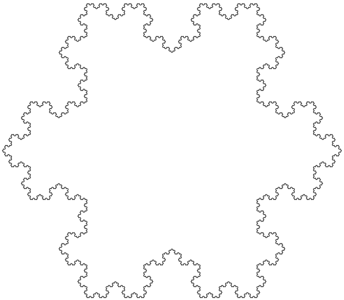
Then Ω *admits the pointwise* (p,β)*-Hardy inequality whenever* $\beta < p - n + \lambda$.

A point $w \in \partial\Omega$ is in the set $\partial^{\text{vis}}_{x,c}\Omega$, if w is *accessible* from x by a c*-John curve*, that is, there exists a curve $\gamma = \gamma_{w,x}\colon [0,l] \to \Omega$, parametrized by arc length, with $\gamma(0) = w$, $\gamma(l) = x$, and satisfying $d(\gamma(t), \partial\Omega) \geq t/c$ for every $t \in [0,l]$.
(Thus (3) is a stronger version of the inner boundary density condition introduced earlier)

Examples



Here the boundary is λ-thick $(1 < \lambda < 2)$ and well accessible

$\Rightarrow$ (p, β)-Hardy for all

$$\beta < \underbrace{p - 2 + \lambda}_{> p-1}$$

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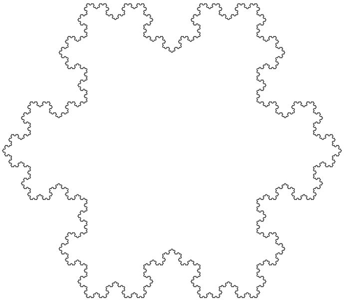

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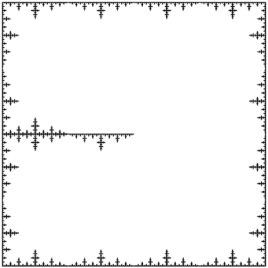

Here the boundary is λ-thick $(1 < \lambda < 2)$, but above the antenna in the middle the *accessible* part of the boundary is only 1-dimensional, and indeed the (p, β)-Hardy does not hold when

$$\beta = p - 1 < p - 2 + \lambda$$

Removing accessibility

The accessibility part of the previous theorem can be dropped for $\beta \le 0$:

Theorem (L. PAMS, to appear)

Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition holds with an exponent $0 \le \lambda \le n$. Then, if $\beta \le 0$ and $\beta < p - n + \lambda$, Ω admits the pointwise (p, β)-Hardy inequality.

This, together with a 'shift'-property of usual Hardy inequalities (L. ACV 2008) leads to the following result:

Theorem (L. PAMS, to appear)

Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition holds with an exponent $0 \le \lambda \le n - 1$. Then Ω admits the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$.

(Note that here $\beta < p - 1$.)

Conclusion and a gap

In conclusion, if $1 < p < \infty$, $\beta \in \mathbb{R}$, and $\partial\Omega \subset \mathbb{R}^n$ is inner λ-thick for $\lambda > n - p + \beta$, then Ω admits

- (p, β)-Hardy if $\beta < p - 1$;
- pointwise (p, β)-Hardy if $\beta \leq 0$;
- pointwise (p, β)-Hardy if $\partial\Omega$ is in addition accessible.

On the other hand, inner λ-thickness for $\lambda > n - p + \beta$ *does not suffice* for (p, β)-Hardy if $\beta \geq p - 1$ (by the 'antenna' example).

Thus one gap remains: Does inner λ-thickness for $\lambda > n - p + \beta$ suffice for *pointwise* (p, β)-Hardy if $0 < \beta < p - 1$?

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Thus one gap remains: Does inner λ-thickness for $\lambda > n - p + \beta$ suffice for *pointwise* (p, β)-Hardy if $0 < \beta < p - 1$?

This I do not know (at the moment).

4. A bit more on thickness conditions

Minkowski content

Let us define a Minkowski-type content of a compact set $A \subset \mathbb{R}^n$: first set

$$\mathcal{M}_r^\lambda(A) = \inf\left\{ N r^\lambda : A \subset \bigcup_{i=1}^{N} B(z_i, r) \right\}$$

(we may again assume $z_i \in A$) and then define

$$\mathcal{M}_\infty^\lambda(A) = \inf_{r>0} \mathcal{M}_r^\lambda(A).$$

Sidenote: the (lower) Minkowski dimension of A is

$$\underline{\dim}_{\mathcal{M}}(A) = \inf\{\lambda > 0 : \mathcal{M}_\infty^\lambda(A) = 0\}.$$

Note that for each compact set $A \subset \mathbb{R}^n$

$$\mathcal{H}_\infty^\lambda(A) \leq \mathcal{M}_\infty^\lambda(A).$$

From Minkowski to Hausdorff

The Minkowski content can in general be much larger than the Hausdorff content, but a *uniform* estimate for $\mathcal{M}^{\lambda}_{\infty}$ yields a similar estimate for $\mathcal{H}^{\lambda'}_{\infty}$:

Lemma (L. AASFM, 2009)

Let $E \subset \mathbb{R}^n$ be a closed set. Assume that there exist $0 < \lambda \leq n$ and $C_0 > 0$ such that

$$\mathcal{M}^{\lambda}_{\infty}\big(\overline{B}(w,r) \cap E\big) \geq C_0\, r^{\lambda} \qquad \text{for all } w \in E,\ \ r > 0.$$

Then, for every $0 < \lambda' < \lambda$, there exists a constant $C = C_{\lambda'} > 0$ such that

$$\mathcal{H}^{\lambda'}_{\infty}\big(\overline{B}(w,r) \cap E\big) \geq C\, r^{\lambda'} \qquad \text{for all } w \in E,\ \ r > 0.$$

Idea of the proof: Fix $\lambda' < \lambda$ and use the λ-Minkowski estimate repeatedly to construct a Cantor type subset $C \subset E$, and then show that C is indeed λ'-thick.

Equivalence: Minkowski content

It is trivial that $\mathcal{H}^\lambda_\infty(E) \le \mathcal{M}^\lambda_\infty(E)$, and so we have a further characterization for uniform fatness:

Corollary

Let $1 < p < \infty$. Then the following are equivalent for a closed set $E \subset \mathbb{R}^n$:
(a) E is uniformly p-fat
(b) E is λ-thick for some $\lambda > n - p$, i.e.

$$\mathcal{H}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \ge Cr^\lambda \quad \text{for all } w \in E,\ r > 0.$$

(c) E satisfies a uniform Minkowski content estimate for some $\lambda > n - p$:

$$\mathcal{M}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \ge Cr^\lambda \quad \text{for all } w \in E,\ r > 0.$$

(Note that λ in (b) and (c) may be different.)

One last thing

The condition $\mathcal{M}^{\lambda}_{\infty}\big(E \cap \overline{B}(w,R)\big) \geq CR^{\lambda}$ for all $w \in E$, $0 < R < \operatorname{diam}(E)$, is equivalent to the condition:
(LA) for all $0 < r < R < \operatorname{diam}(E)$, $w \in E$, at least $C(r/R)^{-\lambda}$ balls of radius r are needed to cover $E \cap B(w,R)$.

This is in a sense 'dual' to the following condition:
(UA) for all $0 < r < R < \operatorname{diam}(E)$, $w \in E$, at most $C(r/R)^{-\lambda}$ balls of radius r are needed to cover $E \cap B(w,R)$.

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The infimum of all $\lambda > 0$ satisfying (UA) is the *upper Assouad dimension* of E, $\overline{\dim}_A(E)$.

The supremum of all $\lambda > 0$ satisfying (LA) is called the *lower Assouad dimension* of E, $\underline{\dim}_A(E)$, in [Käenmaki-L.-Vuorinen, Indiana UMJ (to appear)].

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Thus, if $E \subset \mathbb{R}^n$ is unbounded and $\underline{\dim}_A(E) \leq n-1$, we have that $\underline{\dim}_A(E) = n - \inf\{p > 1 : E \text{ is uniformly } p\text{-fat}\}$.

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