SUFFICIENT BOUNDARY CONDITIONS FOR HARDY INEQUALITIES

JUHA LEHRBÄCK

1. Introduction

In a domain (an open and connected set) $\Omega \subset \mathbb{R}^n$, the (p, β)-Hardy inequality, for $1 < p < \infty$ and $\beta \in \mathbb{R}$, reads as

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta - p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx. \tag{1}$$

Here $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$ denotes the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$, and $u \in C_0^\infty(\Omega)$, i.e., u is a smooth test-function with a compact support in Ω. We say that $\Omega \subset \mathbb{R}^n$ *admits* the (p, β)-Hardy inequality if there exists a constant $C > 0$ so that (1) holds for all $u \in C_0^\infty(\Omega)$ with this constant. In the unweighted case $\beta = 0$ we simply speak of the p-Hardy inequality.

In this talk, I will first rewiev the history of the Hardy inequalities, starting from the one-dimensional results by G.H. Hardy et al. In higher dimensions, these inequalities were first considered by J. Necas, who proved an important result for Lipschitz domains. On the other hand, the 'smoothness' of the boundary is not essential for Hardy inequalities, as was proven by A. Ancona, J. Lewis, and A. Wannebo. The main result of this talk is a recent sufficient condition for weighted Hardy inqualities, given in terms of a Hausdorff content density condition for the boundary. This contains the above results as special cases and gives a sharp bound for the exponents that are admissible in weighted Hardy inequalities under such boundary density.

2. Hardy inequalities

2.1. **One-dimensional inequalities.** The origins of Hardy inequalities trace back to the early 20th century. In the famous 1925 paper [3], G.H. Hardy proved that the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) \, dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \, dx, \tag{2}$$

where $1 < p < \infty$, holds whenever $f \geq 0$ is measurable and, moreover, that the constant on the right-hand side is the best possible; see also [4, Section 9.8]. An excellent account on the interesting — and not that straigt-forward — progress leading to the discovery of inequality (2) can be found in [9].

This is an outline for a talk at Meijo University, Nagoya, on November 19th 2010; I thank Professor Noriaki Suzuki for the kind invitation.

The proof of (2) is rather simple, the only tools needed are integration by parts and Hölder's inequality. The same method can be applied in order to prove that if $\beta \neq p-1$, then each measurable f satisfies the one-dimensional weighted Hardy inequality (cf. [4, §330])

$$\int_0^\infty F(x)^p x^{\beta-p} dx \leq \left(\frac{p}{|p-\beta-1|}\right)^p \int_0^\infty |f(x)|^p x^\beta \, dx,$$

where

$$F(x) = \begin{cases} \int_0^x |f(t)| \, dt & \text{for } \beta < p-1 \\ \int_x^\infty |f(t)| \, dt & \text{for } \beta > p-1. \end{cases}$$

The constant on the right-hand side is again the best possible.

On the other hand, if $u\colon (0,\infty) \to \mathbb{R}$ is absolutely continuos, then using the above inequality with $f(x) = u'(x)$ gives the following theorem (this is essentially [8, Thm. 5.2]):

Theorem 2.1. *Let $1 < p < \infty$ and $\beta \neq p-1$. If $u\colon (0,\infty) \to \mathbb{R}$ is absolutely continuous with $\lim_{x\to 0} u(x) = 0 = \lim_{x\to\infty} u(x)$, then u satisfies the inequality*

$$\int_0^\infty |u(x)|^p \, x^{\beta-p} \, dx \leq \left(\frac{p}{|p-1-\beta|}\right)^p \int_0^\infty |u'(x)|^p \, x^\beta \, dx,$$

where the constant on the right-hand side is the best possible.

This is exactly the (p,β)-Hardy inequality (1) in the domain $(0,\infty) \subset \mathbb{R}$. However, if we consider inequality (1) in a bounded interval $(a,b) \subset \mathbb{R}$, it is not hard to see that in this case the inequality holds for all absolutely continuos functions with $\lim_{x\to a+} u(x) = 0 = \lim_{x\to b-} u(x)$ if and only if $\beta < p-1$.

2.2. **Higher dimensional inequalities.** Hardy inequalities were introduced to higher dimensional Euclidean spaces $\mathbb{R}^n$, $n \geq 2$, by J. Nečas, who also proved the following basic theorem in the 1960's; this theorem is the main point of reflection for our studies.

Theorem 2.2 (Nečas [15])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p,β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p-1$.*

Recall that a domain Ω is Lipschitz if the boundary can be represented locally as graphs of Lipschitz-continuous functions. The proof of Theorem 2.2 is based on this fact and the use of the one-dimensional (p,β)-Hardy inequality (1) on bounded intervals having one end point at the boundary $\partial\Omega$. Hence the bound $\beta < p-1$ is natural, but it is also the best possible, since the $(p, p-1)$-Hardy inequality fails to hold e.g. in the ball $B(0,1) \subset \mathbb{R}^n$.

However, it is nowadays well-understood that the 'smoothness' of the boundary is not that important, but it is indeed the 'thickness' or 'fatness' of the complement (or the boundary) that arises as a natural sufficient condition for Hardy inequalities. The following theorem, dating to the late 1980's, has been of fundamental importance.

Theorem 2.3 (Ancona [1] ($p = 2$), Lewis [14], Wannebo [16])**.** *Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then Ω admits the p-Hardy inequality*

Uniform p-fatness is a density condition for the variational p-capacity, but I will not give the exact definition here, as this condition can be expressed equivalently in terms of a Hausdorff content density condition, and such conditions are very closely related to the assumptions in our main results. Theorem 2.3 is thus given here only for the record; Corollary 3.4 below formulates this result using Hausdorff contents.

The result of Wannebo was actually stronger than Theorem 2.3, namely, he proved that the uniform p-fatness of Ω^c suffices in fact for (p, β)-Hardy inequalities for all $\beta \le \beta_0$, where $\beta_0 > 0$ is some small (positive) number. However, no explicit expression for β_0 was given.

3. Boundary density

Definition 3.1. The λ-*Hausdorff content* of a set $E \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\infty(E) = \inf \left\{ \sum_{i=1}^{\infty} r_i^\lambda : E \subset \bigcup_{i=1}^{\infty} B(z_i, r_i) \right\}.$$

The *Hausdorff dimension* of $E \subset \mathbb{R}^n$ is the number

$$\dim_{\mathcal{H}}(E) = \inf \left\{ \lambda > 0 : \mathcal{H}^\lambda_\infty(E) = 0 \right\}.$$

We have the following relation between uniform fatness and 'thickness' in terms of Hausdorff contents:

Proposition 3.2. *Let $1 < p < \infty$. Then a closed set $E \subset \mathbb{R}^n$ is uniformly p-fat if and only if there exists some exponent $\lambda > n - p$ and a constant $C > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(E \cap B(w, r)\big) \ge C r^\lambda \quad \textit{for every } w \in E \textit{ and all } r > 0.$$

For the idea of the proof, see e.g. the discussion in [7]. It is worth a mention that a deep result of John Lewis from [14], concerning the self-improvement of uniform p-fatness, is needed in the proof of the necessity part of Proposition 3.2.

On the other hand, if we are only interested in domains and their complements, we can use the following equivalence from [10]:

Proposition 3.3. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Then the complement Ω^c is uniformly p-fat if and only if the following 'inner' Hausdorff content density condition holds for the boundary $\partial\Omega$ with an exponent $\lambda > n - p$:*

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \ge C d_\Omega(x)^\lambda \quad \textit{for every } x \in \Omega. \tag{3}$$

Note however that the validity of condition (3) with $\lambda > n - p$ does not guarantee the uniform p-fatness of the boundary $\partial\Omega$ (!); an example of this is given by domains with outer cusps.

Combining Proposition 3.3 with Theorem 2.3, we obtain the following sufficient condition for the p-Hardy inequality.

Corollary 3.4 (Ancona–Lewis–Wannebo revisited)**.** *Let $1 < p < \infty$ and assume that condition* (3) *holds in a domain $\Omega \subset \mathbb{R}^n$ with an exponent $\lambda > n - p$. Then Ω admits the p-Hardy inequality.*

4. Main results

The following result, which extends both Theorem 2.2 and Corollary 3.4 and gives a sharp bound for β for which the (p, β)-Hardy inequality holds under the inner boundary density condition (3), was recently established in [13].

Theorem 4.1. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition* (3) *holds in Ω with an exponent $0 \le \lambda \le n-1$. Then Ω admits the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$.*

Indeed, it is easy to see that a bounded Lipschitz domain Ω satisfies (3) with $\lambda = n - 1$, and so Theorem 4.1 gives Hardy inequalities for all $\beta < p - n + (n - 1) = p - 1$; and if (3) holds with some $\lambda > n - p$, then the value $\beta = 0$ is admissible in the (p, β)-Hardy inequality.

The requirement $\lambda \le n-1$ (and thus $\beta < p-1$) is essential in Theorem 4.1, as examples from my earlier work [7] with Pekka Koskela show that the conclusion of Theorem 4.1 need not hold for $\beta \ge p - 1$ even if (3) holds with an exponent $\lambda > n - 1$. See however the discussion at Section 6 on how it is possible to exceed the 'classically sharp' bound $p - 1$ for β.

In the plane $\mathbb{R}^2$, Theorem 4.1 has the interesting consequence that besides Lipschitz domains, all simply connected domains admit the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$.

The first main ingredient in the proof of Theorem 4.1 is the use of a *pointwise* variant of the (p, β)-Hardy inequality: We say that Ω admits the pointwise (p, β)-Hardy inequality, if there exist $C > 0$ and $1 < q < p$ such that

$$|u(x)| \le C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{3d_\Omega(x)}\big(|\nabla u|^q d_\Omega{}^{\frac{\beta}{p}q}\big)(x)\Big)^{1/q} \qquad \text{for all } x \in \Omega \tag{4}$$

whenever $u \in C_0^\infty(\Omega)$. In (4) M_R is the usual restricted Hardy-Littlewood maximal operator, defined by

$$M_R f(x) = \sup_{0<r\le R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$; notation $|A|$ is used here for the Lebesgue measure of $A \subset \mathbb{R}^n$. Inequality (4) was introduced in the paper [7], following the unweighted versions of P. Hajłasz [2] and J. Kinnunen and O. Martio [6].

Using the maximal theorem and the fact that we have $1 < q < p$ in (4) it is easy to see that inequality (4) implies the usual (p, β)-Hardy inequality (1). For $\beta \le 0$, Theorem 4.1 thus follows from the following pointwise result:

Proposition 4.2. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition* (3) *holds with an exponent $0 \le \lambda \le n$. Then, if $\beta \le 0$ and $\beta < p-n+\lambda$, Ω admits the pointwise (p, β)-Hardy inequality.*

5. Proofs

One of the main tools in our proofs of Hardy inequalities is the following $(1,q)$-Poincaré inequality ($1 \le q < \infty$), which holds for every $u \in C_0^1(\mathbb{R}^n)$ with a constant $C = C(n,q) > 0$:

$$\fint_B |u - u_B|\, dx \le Cr\Big(\fint_B |\nabla u|^q\, dx\Big)^{1/q}.$$

Here we use the notation

$$u_B = \fint_B u\, dx = |B|^{-1}\int_B u\, dx.$$

For instance, it follows from the above inequality that if $B = B(w,r) \subset \mathbb{R}^n$ and $u \in C_0^1(\mathbb{R}^n)$, then

$$\begin{aligned} |u_B - u_{2B}| &\le \frac{1}{|B|}\int_B |u - u_{2B}|\, dx \le \frac{C}{|2B|}\int_{2B} |u - u_{2B}|\, dx \\ &\le Cr\Big(\fint_{2B} |\nabla u|^q\, dx\Big)^{1/q}. \end{aligned} \tag{5}$$

The proof of Proposition 4.2 is based on the following Lemma:

Lemma 5.1. *Let $1 < p < \infty$ and $\beta \le 0$, let $\Omega \subset \mathbb{R}^n$ be an open set, and take $x \in \Omega$. Denote $B(x) = \overline{B}(x, d_\Omega(x))$ and $E = \partial\Omega \cap 2B(x)$. Then, for each $\lambda > n - p + \beta$, there exist an exponent $1 < q < p$ and a constant $C > 0$ (independent of x) such that the estimate*

$$\mathcal{H}^\lambda_\infty(E)\big|u_{B(x)}\big|^q \le C d_\Omega(x)^{q - \beta\frac{q}{p} + \lambda}\fint_{3B(x)} |\nabla u(y)|^q\, d_\Omega(y)^{\beta\frac{q}{p}}\, dy \tag{6}$$

holds for every $u \in C_0^\infty(\Omega)$.

Proof. Our proof combines elements from the proofs of [7, Lemma 5.2] and [5, Thm. 5.9]. Let $\lambda > n - p + \beta$. It is easy to check that we can choose $1 < q < \infty$ so that

$$p\,\frac{n-\lambda}{p-\beta} < q < p.$$

Also denote $\beta' = \frac{q}{p}\beta$. Then $q/p > (n-\lambda)/(p-\beta)$, and we have

$$q - \beta' - n + \lambda = \tfrac{q}{p}(p-\beta) - n + \lambda > 0. \tag{7}$$

Denote $R = d_\Omega(x)$ and $B = \overline{B}(x,R)$. If $u_B = 0$ the claim (6) is trivial, so we may assume that $|u_B| > 0$, and in fact, by homogeneity, that $|u_B| = 1$.

Let us now fix $w \in E = \partial\Omega \cap 2B$, and then define $r_k = 2^{-k}R$ for $k \in \mathbb{N}$ and denote $B_k = B(w, r_k)$. Then

$$1 = |u_B| \le |u_{B_0}| + |u_{B_0} - u_B|;$$

recall that here $B = \overline{B}(x, d_\Omega(x))$ and $B_0 = \overline{B}(w, d_\Omega(x))$

If it happens that $|u_{B_0}| < 1/2$, then $|u_{B_0} - u_B| \geq 1/2$, and we calculate, using the $(1,q)$-Poincaré inequality (as in (5)), and the facts that $B, B_0 \subset 3B$ and $R^\beta \leq C d_\Omega(y)^\beta$ for every $y \in 3B \cap \Omega$ (since $\beta \leq 0$), that

$$\begin{aligned}\tfrac{1}{2} &\leq |u_{B_0} - u_B| \leq |u_{B_0} - u_{3B}| + |u_B - u_{3B}| \\ &\leq CR\Big(\fint_{3B} |\nabla u(y)|^q \, dy\Big)^{1/q} \\ &\leq CR^{1-\beta/p}\Big(\fint_{3B} |\nabla u(y)|^q d_\Omega(y)^{\beta \frac{q}{p}} \, dy\Big)^{1/q}.\end{aligned}$$

As $\mathcal{H}^\lambda_R(E) \leq CR^\lambda$ and $|u_B| = 1$, the claim (6) readily follows.

We may hence assume that $1/2 \leq |u_{B_0}| = |u(w) - u_{B_0}|$ for every $w \in E$. But then we can use (still for a fixed $w \in E$) a standard 'telescoping' argument, using again the $(1,q)$-Poincaré inequality:

$$\tfrac{1}{2} \leq |u(w) - u_{B_0}| \leq \sum_{k=0}^{\infty} |u_{B_{k+1}} - u_{B_k}| \leq C \sum_{k=0}^{\infty} r_k \left(\fint_{B_k} |\nabla u(y)|^q \, dy\right)^{1/q}. \tag{8}$$

From (8) it follows that for each $\alpha > 0$ there must exist a constant $C_1 > 0$, independent of x, u, and w, and at least one index $k_w \in \mathbb{N}$ so that

$$C_1 R^{-\alpha} {r_{k_w}}^\alpha = C_1 2^{-k_w \alpha} \leq r_{k_w} \left(\fint_{B_{k_w}} |\nabla u(y)|^q \, dy\right)^{1/q}, \tag{9}$$

as otherwise a simple calculation using geometric series would lead to a contradiction with estimate (8). Let us now choose $\alpha = \frac{1}{q}(q - \beta' - n + \lambda)$ whence $\alpha > 0$ by (7), and write $B_w = B(x_w, r_w)$ instead of $B_{k_w} = B\big(x_{k_w}, r_{k_w}\big)$ for the corresponding ball satisfying (9).

Since $\beta \leq 0$ and $w \in E$, it follows again that ${r_k}^\beta \leq C d_\Omega(y)^\beta$ for each $y \in B_k$. Thus

$$\Big(\fint_{B_w} |\nabla u(y)|^q \, dy\Big)^{1/q} \leq C r_w^{-\beta/p} |B_w|^{-1/q} \Big(\int_{B_w} |\nabla u(y)|^q d_\Omega(y)^{\beta \frac{q}{p}} \, dy\Big)^{1/q}. \tag{10}$$

Combining (9) and (10), we now obtain for each $w \in E$ a ball B_w such that

$$r_w^{\alpha - 1 + \beta/p + n/q} \leq CR^\alpha \Big(\int_{B_w} |\nabla u(y)|^q d_\Omega(y)^{\beta'} \, dy\Big)^{1/q}. \tag{11}$$

But here $\alpha - 1 + \beta/p + n/q = \lambda/q$, so by raising both sides of (11) to power q we get a useful estimate

$$r_w^\lambda \leq CR^{q - \beta' - n + \lambda} \int_{B_w} |\nabla u(y)|^q d_\Omega(y)^{\beta'} \, dy. \tag{12}$$

By a standard $5r$-covering lemma, there now exists points $w_1, w_2, \ldots \in E$ so that the balls $B_i = B(w_i, r_{w_i})$ are pairwise disjoint, but still $E \subset \bigcup_{i=1}^\infty 5B_i$.

Estimate (12) and the pairwise disjointness of the balls $B_i \subset 3B$ now yield

$$\begin{aligned}\mathcal{H}^\lambda_\infty(E) &\le \sum_{i=1}^\infty (5r_{w_i})^\lambda \le C\sum_{i=1}^\infty r_{w_i}^\lambda \\ &\le \sum_{i=1}^\infty CR^{q-\beta'-n+\lambda}\int_{B_i} |\nabla u(y)|^q d_\Omega(y)^{\beta'}\,dy \\ &\le CR^{q-\beta'+\lambda} \fint_{3B} |\nabla u(y)|^q d_\Omega(y)^{\beta'}\,dy.\end{aligned}$$

As we assumed $|u_B| = 1$ and denoted $\beta' = \beta\frac{q}{p}$, estimate (6) is finally proven. $\square$

It is now rather straight-forward to prove the pointwise (p,β)-Hardy inequalities for $\beta \le 0$:

Proof of Proposition 4.2. Let $u \in C_0^\infty(\Omega)$, $x \in \Omega$, and denote $R = d_\Omega(x)$, $B = B\big(x,R\big)$. Then

$$|u(x)| \le |u(x) - u_B| + |u_B|.$$

Choose $1 < q < p$ just as in the proof of Lemma 5.1. Using again an estimate of the type (5), with $B_k = B\big(x, r_k\big)$, $r_k = 2^{-k}R$, we obtain (since $\beta < p$)

$$\begin{aligned}|u(x) - u_B| &\le \sum_{k=0}^\infty |u_{B_{k+1}} - u_{B_k}| \le C\sum_{k=0}^\infty r_k\left(\fint_{B_k} |\nabla u(y)|^q\,dy\right)^{1/q} \\ &\le C\sum_{k=0}^\infty (2^{-k}R)^{1-\beta/p}\left(\fint_{B_k} |\nabla u(y)|^q d_\Omega(y)^{\beta\frac{q}{p}}\,dy\right)^{1/q} \\ &\le CR^{1-\beta/p}\big(M_R(|\nabla u|^q d_\Omega^{\beta q/p})(x)\big)^{1/q}.\end{aligned}$$

On the other hand, it follows from assumption (3) and Lemma 5.1, that

$$\begin{aligned}|u_B|^q &\le CR^{-\lambda}R^{q-\beta\frac{q}{p}+\lambda}\fint_{B(x,3R)} |\nabla u(y)|^q\, d_\Omega(y)^{\beta\frac{q}{p}}\,dy \\ &\le CR^{q-\beta\frac{q}{p}} M_{3R}\big(|\nabla u|^q d_\Omega^{\beta\frac{q}{p}}\big)(x).\end{aligned}$$

The pointwise (p,β)-Hardy inequality now follows by combining the above three estimates. $\square$

Let us finally give a proof for our main result concerning weighted Hardy inequalities:

Proof of Theorem 4.1. As the pointwise (p,β)-Hardy inequality always implies the usual (p,β)-Hardy inequality, Theorem 4.1 follows from Proposition 4.2 for $\beta \le 0$.

Hence, we only need to consider the case $0 < \beta < p - n + \lambda$. But now $p - \beta > n - \lambda \ge 1$, (recall that we assumed $\lambda \le n-1$) and so Proposition 4.2, applied to the unweighted case, implies that Ω admits the $(p-\beta, 0)$-Hardy inequality. The claim now follows in fact from a more general result of [11, Lemma 2.1] but in our case the calculations can be made as follows:

Let $u \in C_0^\infty(\Omega)$ and define $v = |u|^{\frac{p}{p-\beta}}$. As $0 < \beta < p$, we see that v is a Lipschitz-continuous function with a compact support in Ω, and, moreover,

$$|\nabla v(x)| = \left(\tfrac{p}{p-\beta}\right)|u(x)|^{\beta/(p-\beta)}|\nabla u(x)|$$

almost everywhere in Ω. By approximation, the $(p-\beta, 0)$-Hardy inequality holds for v as well, and so we obtain, with the help of Hölder's inequality (observe $\frac{p-\beta}{p} + \frac{\beta}{p} = 1$), that

$$\begin{aligned}
\int_\Omega |u(x)|^p d_\Omega(x)^{-(p-\beta)}\,dx \;&=\; \int_\Omega |v(x)|^{p-\beta} d_\Omega(x)^{-(p-\beta)}\,dx \\
&\le C\int_\Omega |\nabla v(x)|^{p-\beta}\,dx \;=\; C\int_\Omega |u(x)|^\beta |\nabla u(x)|^{p-\beta}\,dx \\
&= C\int_\Omega \Big(|u(x)|^\beta d_\Omega(x)^{\frac{\beta(\beta-p)}{p}}\Big)\Big(|\nabla u(x)|^{p-\beta} d_\Omega(x)^{\frac{\beta(p-\beta)}{p}}\Big)\,dx \\
&\le C\bigg(\int_\Omega |u(x)|^p d_\Omega(x)^{\beta-p}\,dx\bigg)^{\frac{\beta}{p}}\bigg(\int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta\,dx\bigg)^{\frac{p-\beta}{p}}.
\end{aligned}\tag{13}$$

The (p,β)-Hardy inequality for u follows from (13) by first dividing with the first integral term on the right-hand side (which we may assume to be non-zero), and then taking both sides to power $p/(p-\beta)$. $\square$

6. ... and beyond

Contrary to Proposition 4.2, where $\beta \le 0$, we obtain in Theorem 4.1 for $0 < \beta < p-1$ only usual Hardy inequalities, not pointwise inequalities. However, it is also possible to establish pointwise Hardy inequalities for $\beta > 0$, but with the cost of an additional *accessibility* condition. In fact, the first general sufficient condition for weighted pointwise Hardy inequalities was the following theorem from [7]:

Theorem 6.1. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that there exist $0 \le \lambda \le n$, $c \ge 1$, and $C > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(\partial^{\mathrm{vis}}_{x,c}\Omega\big) \ge C d_\Omega(x)^\lambda \qquad \text{for every } x \in \Omega. \tag{14}$$

Then Ω admits the pointwise (p,β)-Hardy inequality for all $\beta < p - n + \lambda$.

Here $\partial^{\mathrm{vis}}_{x,c}\Omega$ is called the *c-visible boundary near* $x \in \Omega$, and it consists of the points $w \in \partial\Omega$ which are accessible from x by a c-John curve γ in Ω, that is, $\gamma\colon [0,l] \to \Omega \cup \{w\}$ is parametrized by arc length, $\gamma(0) = w$, $\gamma(l) = x$, and

$$d(\gamma(t), \partial\Omega) \ge \tfrac{1}{c}t \tag{15}$$

for each $t \in [0,l]$. Besides Lipschitz domains, where (14) holds with $\lambda = n-1$, this condition is satisfied for instance in snowflake-type domains in $\mathbb{R}^n$ with $n-1 < \lambda < n$. Thus, contrary to Theorem 4.1, values $\beta > p-1$ can be reached in Theorem 6.1 — and thus also in the corresponding usual (p,β)-Hardy inequalities — provided that $\lambda > n-1$.

The proof of Theorem 6.1 is similar to the proof of Proposition 4.2, except that in the substitute of Lemma 5.1 we have to chooce the balls B_k so that instead of being centered at the boundary point w, they are centered at a John-curve joining w to x; in addition, these balls have to be well within the domain, but suitable dilatations of these balls still need to cover the point w. The details can be found in [13].

Nevertheless, there appears presently a gap concerning our knowledge on pointwise Hardy inequalities: For $0 < \beta < p - 1$ we do not know if the inner boundary density condition (3) with an exponent $\lambda > n - p + \beta$ suffices for Ω to admit the pointwise (p, β)-Hardy inequality; for $\beta \leq 0$ this is sufficient by Proposition 4.2, and for $\beta \geq p - 1$ examples from [7] show that (3) alone does not suffice, and in fact the accessibility part of Theorem 6.1 is needed even for the usual Hardy inequalities when $\beta \geq p - 1$.

Let me end by remarking that the inner boundary density condition (3) with an exponent $\lambda > n - p + \beta$ is also necessary for the pointwise (p, β)-Hardy inequality (see [12]), and thus for $\beta \leq 0$ we have the following characterization:

Corollary 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$ and $\beta \leq 0$. Then Ω admits the pointwise (p, β)-Hardy inequality* (4) *if and only if there exists an exponent $\lambda > n - p + \beta$ such that the inner density condition* (3) *holds for all $x \in \Omega$.*

References

[1] A. Ancona, 'On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$', *J. London Math. Soc.* (2) 34 (1986), no. 2, 274–290.

[2] P. Hajłasz, 'Pointwise Hardy inequalities', *Proc. Amer. Math. Soc.* 127 (1999), no. 2, 417–423.

[3] G. H. Hardy, 'Notes on some points in the integral calculus (LX)', *Messenger of Math.* 54 (1925), 150–156.

[4] G. H. Hardy, J. E. Littlewood and G. Pólya, 'Inequalities' (Second edition), Cambridge University Press, Cambridge, 1952.

[5] J. Heinonen and P. Koskela, 'Quasiconformal maps in metric spaces with controlled geometry', *Acta Math.* 181 (1998), no. 1, 1–61.

[6] J. Kinnunen and O. Martio, 'Hardy's inequalities for Sobolev functions', *Math. Res. Lett.* 4 (1997), no. 4, 489–500.

[7] P. Koskela and J. Lehrbäck, 'Weighted pointwise Hardy inequalities', *J. London Math. Soc.* 79 (2009), no. 3, 757-779.

[8] A. Kufner, 'Weighted Sobolev spaces', John Wiley & Sons, Inc., New York, 1985.

[9] A. Kufner, L. Maligranda and L. E. Persson, 'The prehistory of the Hardy inequality', *Amer. Math. Monthly* 113 (2006), no. 8, 715–732.

[10] J. Lehrbäck, 'Pointwise Hardy inequalities and uniformly fat sets', *Proc. Amer. Math. Soc.* 136 (2008), no. 6, 2193–2200.

[11] J. Lehrbäck, 'Self-improving properties of weighted Hardy inequalities', *Adv. Calc. Var.* 1 (2008), no. 2, 193–203.

[12] J. Lehrbäck, 'Necessary conditions for weighted pointwise Hardy inequalities', *Ann. Acad. Sci. Fenn. Math.* 34 (2009), no. 2, 437-446.

[13] J. Lehrbäck, 'Hardy inequalities beyond Lipschitz domains', preprint (2010), 12pp. available at: http://users.jyu.fi/~juhaleh/lip_hardy_preprint.pdf

[14] J. L. Lewis, 'Uniformly fat sets', *Trans. Amer. Math. Soc.* 308 (1988), no. 1, 177–196.

[15] J. Nečas, 'Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle', *Ann. Scuola Norm. Sup. Pisa* (3) 16 (1962), 305–326.

[16] A. Wannebo, 'Hardy inequalities', *Proc. Amer. Math. Soc.* 109 (1990), 85–95.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland

E-mail address: juha.lehrback@jyu.fi