Hardy inequalities and thickness conditions

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1. Introduction

The original p-Hardy inequality

G.H. Hardy published the following inequality in 1925:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,dx,$$

when $1 < p < \infty$ and $f \ge 0$ is measurable.

Hardy inequalities in $\mathbb{R}^n$

Taking $u(x) = \int_0^x f(t)\,dt$, the previous *p-Hardy inequality* can also be written as

$$\int_0^\infty |u(x)|^p x^{-p}\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\,dx,$$

where $1 < p < \infty$ and u is abs. continuous with $u(0) = 0$.

This can be generalized to higher dimensions in many ways; we consider the following form:

$$\int_\Omega |u(x)|^p d_\Omega(x)^{-p}\,dx \le C \int_\Omega |\nabla u(x)|^p\,dx,$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$.

The p-Hardy inequality is not always valid

If the p-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \le C \int_\Omega |\nabla u(x)|^p \, dx$$

holds for all $u \in C_0^\infty(\Omega)$ with a constant $C > 0$, we say that the domain $\Omega \subset \mathbb{R}^n$ *admits* the p-Hardy inequality.
(In this talk, we are not interested about the optimality of the constant C)

The p-Hardy inequality is not necessarily valid on all domains. For instance, using 1-dimensional tools on rays, one can calculate directly that the domain $\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ does not admit the n-Hardy inequality; but for $1 < p < n$ and $p > n$ this domain admits the p-Hardy inequality.

Our main interest is in finding (e.g. geometric) conditions which guarantee the validity of the p-Hardy inequality on a domain Ω.

Metric spaces

For simplicity, we mainly consider $\mathbb{R}^n$ in this talk, but in fact most of the considerations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \le C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the 'dimension' of X is at most $s = \log_2 C_d$)
- X supports a $(1, p)$-Poincaré inequality:

$$\fint_B |u - u_B| \, d\mu \le C_P r \Big(\fint_B g_u^p \, d\mu \Big)^{1/p}$$

  whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u: For all (or p-almost all) curves γ joining $x, y \in X$ we have $|u(x) - u(y)| \le \int_\gamma g_u \, ds$.

We use here and in the following the notation

$$u_B = \fint_B u \, d\mu = \mu(B)^{-1} \int_B u \, d\mu.$$

Sufficient conditions for Hardy inequalities

Theorem (Nečas 1962)

Let $1 < p < \infty$ and assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then Ω admits the p-Hardy inequality

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The "smoothness" of the boundary is however irrelevant here:

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The "smoothness" of the boundary is however irrelevant here:

Theorem (Ancona 1986 ($p = 2$), Lewis 1988, Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

For instance, if $\Omega \subset \mathbb{R}^n$ is bounded Lipschitz, then Ω^c is indeed uniformly p-fat for all $1 < p < \infty$.

2. Thickness conditions

When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E,\Omega) = \inf\left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E \right\}.$$

A closed set $E \subset \mathbb{R}^n$ is *uniformly p-fat* if

$$\operatorname{cap}_p\left(E \cap \overline{B}(x,r), B(x,2r)\right) \geq C \operatorname{cap}_p\left(\overline{B}(x,r), B(x,2r)\right)$$

for every $x \in E$ and all $r > 0$.
Actually, then

$$\operatorname{cap}_p\left(E \cap \overline{B}(x,r), B(x,2r)\right) \approx r^{n-p}$$

for each $x \in E$ and all $r > 0$.

Uniform fatness: self-improvement

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $q > p$, then E is also uniformly q-fat.

smaller $p \leftrightarrow$ *fatter set*

On the other hand, we have a deep result by J. Lewis:

Theorem (Lewis 1988)

If $E \subset \mathbb{R}^n$ *is uniformly* p*-fat for* $1 < p < \infty$*, then there exists some* $1 < q < p$ *such that* E *is uniformly* q*-fat.*

Mikkonen (1996) proved this in weighted $\mathbb{R}^n$ and Björn, MacManus and Shanmugalingam (2001) in metric spaces.

Hausdorff content and measure

The λ-dimensional Hausdorff δ-content of $A \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\delta(A) = \inf\left\{ \sum_{i=1}^\infty r_i^\lambda : A \subset \bigcup_{i=1}^\infty B(z_i, r_i), r_i < \delta \right\}.$$

We may in addition assume that $z_i \in A$.

The λ-dimensional Hausdorff measure is

$$\mathcal{H}^\lambda(A) = \lim_{\delta \to 0} \mathcal{H}^\lambda_\delta(A).$$

Note that for each $0 < \delta \leq \infty$ we have $\mathcal{H}^\lambda_\infty(A) \leq \mathcal{H}^\lambda_\delta(A) \leq \mathcal{H}^\lambda(A)$, but still

$$\mathcal{H}^\lambda_\infty(A) = 0 \iff \mathcal{H}^\lambda(A) = 0.$$

Hausdorff dimension

The Hausdorff dimension of A is

$$\begin{aligned}\dim_{\mathcal{H}}(A) &= \inf\{\lambda > 0 : \mathcal{H}^{\lambda}(A) = 0\} \\ &= \inf\{\lambda > 0 : \mathcal{H}^{\lambda}_{\infty}(A) = 0\}\end{aligned}$$

For us, the following difference between measure and content is important: If $\lambda < \dim_{\mathcal{H}}(A)$, then

$$\mathcal{H}^{\lambda}(A) = \infty \qquad (\ \rightarrow\ \text{useless}\ )$$

but *always*

$$\mathcal{H}^{\lambda}_{\infty}(A) \leq \operatorname{diam}(A)^{\lambda} \qquad (\ \rightarrow\ \text{useful}\ )$$

(For $\lambda = \dim_{\mathcal{H}}(A)$ the *measure* $\mathcal{H}^{\lambda}(A)$ is usually better, though)

Thickness

We say that a (closed) set $E \subset \mathbb{R}^n$ is λ-thick at $w \in E$, if there exists $C > 0$ so that

$$\mathcal{H}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \geq C r^\lambda \quad \text{for all } r > 0,$$

and that E is λ-thick, if it is λ-thick at every $w \in E$ with the same constant.

Then actually

$$\mathcal{H}^\lambda_\infty\big(E \cap \overline{B}(w,r)\big) \approx r^\lambda$$

for every $w \in E$ and all $r > 0$.

Quantitative estimates

There is an intimate connection between capacities and Hausdorff measures and contents. We state the following quantitative estimates; see for instance Reshetnyak (1969), Martio (1978/79):

Theorem

Let $E \subset \mathbb{R}^n$ be a closed set.
(a) If E is λ-thick at $w \in E$ and $\lambda > n - p$, then

$$\operatorname{cap}_p \left(E \cap \overline{B}(w,r), B(w,2r) \right) \geq C r^{n-p} \quad \text{for all } r > 0.$$

(b) If $w \in E$ and

$$\operatorname{cap}_p \left(E \cap \overline{B}(w,r), B(w,2r) \right) \geq C r^{n-p} \quad \text{for all } r > 0,$$

then E is $(n-p)$-thick at w.

Notice the subtle difference in (a) and (b).

Equivalence: Uniform fatness and thickness

Using the self-improvement and the previous quantitative estimates we obtain for $1 < p < \infty$:

$E \subset \mathbb{R}^n$ is λ-thick for some $\lambda > n - p$

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$\implies$ E is $(n - q)$-thick (and $n - q > n - p$).

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Using the self-improvement and the previous quantitative estimates we obtain for $1 < p < \infty$:

$E \subset \mathbb{R}^n$ is λ-thick for some $\lambda > n - p$

$\Longrightarrow$ E is uniformly p-fat

$\Longrightarrow$ E is uniformly q-fat for some $1 < q < p$

$\Longrightarrow$ E is $(n-q)$-thick (and $n - q > n - p$).

This can be written as

Corollary

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat if and only if E is λ-thick for some $\lambda > n - p$.

Minkowski content

Let us define a Minkowski-type content of a compact set $A \subset \mathbb{R}^n$: first set

$$\mathcal{M}_r^\lambda(A) = \inf \left\{ N r^\lambda : A \subset \bigcup_{i=1}^{N} B(z_i, r) \right\}$$

(we may again assume $z_i \in A$) and then define

$$\mathcal{M}_\infty^\lambda(A) = \inf_{r>0} \mathcal{M}_r^\lambda(A).$$

Sidenote: the (lower) Minkowski dimension of A is

$$\underline{\dim}_{\mathcal{M}}(A) = \inf\{\lambda > 0 : \mathcal{M}_\infty^\lambda(A) = 0\}.$$

Note that for each compact set $A \subset \mathbb{R}^n$

$$\mathcal{H}_\infty^\lambda(A) \le \mathcal{M}_\infty^\lambda(A).$$

From Minkowski to Hausdorff

Although Minkowski content can in general be much larger than Hausdorff content, a *uniform* estimate for $\mathcal{M}^{\lambda}_{\infty}$ yields a similar estimate for $\mathcal{H}^{\lambda'}_{\infty}$:

Lemma (L. AASFM 2009)

Let $E \subset \mathbb{R}^n$ be a closed set. Assume that there exist $0 < \lambda \leq n$ and $C_0 > 0$ such that

$$\mathcal{M}^{\lambda}_{\infty}\big(\overline{B}(w,r) \cap E\big) \geq C_0\, r^{\lambda} \qquad \text{for all } w \in E,\ \ r > 0.$$

Then, for every $0 < \lambda' < \lambda$, there exists a constant $C > 0$ such that

$$\mathcal{H}^{\lambda'}_{\infty}\big(\overline{B}(w,r) \cap E\big) \geq C\, r^{\lambda'} \qquad \text{for all } w \in E,\ \ r > 0.$$

Idea of the proof: Fix $\lambda' < \lambda$ and use the λ-Minkowski estimate repetedly to construct a Cantor type subset $C \subset E$, and then show that C is indeed λ'-thick.

Equivalence: Minkowski content

As trivially $\mathcal{H}^\lambda_\infty(E) \le \mathcal{M}^\lambda_\infty(E)$, we have a further equivalent condition for uniform fatness:

Corollary

Let $1 < p < \infty$. Then the following are equivalent for a closed set $E \subset \mathbb{R}^n$:
(a) E is uniformly p-fat
(b) E is λ-thick for some $\lambda > n - p$, i.e.

$$\mathcal{H}^\lambda_\infty(E \cap \overline{B}(w,r)) \ge r^\lambda \quad \text{for all } w \in E,\ r > 0.$$

(c) E satisfies a uniform Minkowski content estimate for some $\lambda > n - p$:

$$\mathcal{M}^\lambda_\infty(E \cap \overline{B}(w,r)) \ge r^\lambda \quad \text{for all } w \in E,\ r > 0.$$

3. Fatness and Hardy inequalities

Hardy inequalities and uniform fatness

Let us now return to the p-Hardy inequality

$$\int_{\Omega} |u(x)|^p \, d_{\Omega}(x)^{-p} \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \, dx,$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

Recall the main theorem:

Theorem (Ancona 1986 ($p = 2$), Lewis 1988, Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

However, uniform p-fatness of the complement is *necessary* for the p-Hardy inequality in $\mathbb{R}^n$ only when $p = n$ (Ancona $n = 2$, Lewis).

For instance, $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$ admits p-Hardy when $1 < p < n$ or $p > n$, but the complement is uniformly p-fat only when $p \geq n$.

Pointwise p-Hardy inequality

It is quite straight-forward to obtain the following stronger(?) pointwise inequalities from uniform p-fatness of the complement:

Theorem (Hajłasz 1999, Kinnunen-Martio 1997)

Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exists a constant $C > 0$ such that the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

Here $M_R f$ is the usual restricted Hardy-Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, defined by $M_R f(x) = \sup_{r \leq R} \fint_{B(x,r)} |f(y)|\, dy$

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A boundary Poincaré inequality

In the proof of $\left[\ \Omega^c \text{ uniformly } p\text{-fat} \ \Rightarrow \ \text{pointwise } p\text{-Hardy for } \Omega\ \right]$ the following Sobolev-type estimate due to Maz'ya plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\fint_B |u|^p\,dx \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u = 0\}, B)} \int_B |\nabla u|^p\,dx. \tag{1}$$

Now, if Ω^c is uniformly p-fat and $u \in C_0^\infty(\Omega)$, it follows from Hölder's inequality and (1) that the following 'boundary Poincaré inequality'

$$|u_B| \le \left(\fint_B |u|^p\right)^{1/p} \le C\left(r^{p-n}\int_B |\nabla u|^p\right)^{1/p} = Cr\left(\fint_B |\nabla u|^p\right)^{1/p}$$

holds for each ball $B = B(w, r)$ with $w \in \partial\Omega$.

Proof of the pointwise inequality:

The previous estimate, standard estimates (or a chaining argument) for the maximal function, and the usual Poincaré inequality now yield the pointwise p-Hardy inequality:
Let $x \in \Omega$, pick $w \in \partial\Omega$ such that $d(x, w) = d_\Omega(x)$, and write $B_x = B(x, d_\Omega(x))$, $B_w = B(w, d_\Omega(x)) \subset 2B_x$. Then

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$$
\begin{aligned}
|u(x)| &\leq |u(x) - u_{B_x}| + |u_{B_x} - u_{B_w}| + |u_{B_w}| \\
&\lesssim d_\Omega(x) M_{d_\Omega(x)} |\nabla u| + d_\Omega(x) \left(\fint_{2B_x} |\nabla u|^p \right)^{1/p} \\
&\qquad + d_\Omega(x) \left(\fint_{B_w} |\nabla u|^p \right)^{1/p} \\
&\lesssim 3 d_\Omega(x) \Big(M_{2d_\Omega(x)} |\nabla u|^p \Big)^{1/p}
\end{aligned}
$$

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Wannebo uses an inequality similar to the 'boundary Poincaré inequality' and a clever integration trick (this is not trivial, but still 'elementary').

But can we prove that the pointwise p-Hardy inequality implies the usual p-Hardy inequality?

Equivalence: Pointwise Hardy and uniform fatness

With Riikka Korte and Heli Tuominen (Math. Ann, to appear) we prove that if Ω admits the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2 d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p},$$

then the complement Ω^c *has to be* uniformly p-fat, so we obtain an equivalence between these two conditions.

(In particular, pointwise p-Hardy inequalities self-improve!)

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This equivalence means that in a proof of

Ω^c uniformly p-fat $\Rightarrow$ Ω admits p-Hardy

we have to justify why we can "integrate the above maximal function inequality with exponent 1" to obtain

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx.$$

Hence such a proof should not be 'too easy'.

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Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \le v \le 1$ and $v = 1$ in $\Omega^c \cap B$.
- If $\fint_B v \ge c$ (where $0 < c < 1$ is 'sufficiently' small), we are done by Poincaré (for $v \in C_0^\infty(2B)$):

$$c \le \fint_B v \le R\Big(\fint_{2B} |\nabla v|^p\Big)^{1/p} \;\Rightarrow\; \int_{2B} |\nabla v|^p \ge CR^{n-p}$$

- Otherwise $u = 1 - v$ must have values $\ge C_1$ in a large set $E \subset \frac{1}{4}B$; $|E| \ge C_2|B|$. Moreover, $u = 0$ on $\Omega^c \cap B$.
- $\Rightarrow$ we may use the pw p-Hardy on points $x \in E$; let r_x be the corresponding "almost" best radii ($0 < r_x < 2d_\Omega(x) < R/2$).
- "5r"-covering thm. $\Rightarrow$ we find $x_i \in E$ s.t. $B_i = B(x_i, r_i)$ are pairwise disjoint but $E \subset \bigcup 5B_i$.

From pointwise Hardy to fatness...cont'd

- Thus $R^n \le C|E| \le C \sum {r_i}^n$
- On the other hand

$$C_1^p \le |u(x_i)|^p \le C d_\Omega(x_i)^p M_{2d_\Omega(x)} |\nabla u|^p(x) \le C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

$$\Rightarrow \; {r_i}^n \le C R^p \int_{B_i} |\nabla u|^p$$

- Combining the above inequalities with the facts that $|\nabla u| = |\nabla v|$ in B and that B_i's are pairwise disjoint, we get

$$R^n \le C R^p \sum_{i=1}^{\infty} \int_{B_i} |\nabla u|^p \le C R^p \int_{2B} |\nabla v|^p$$

- Hence $\mathrm{cap}_p(\Omega^c \cap \overline{B}, 2B) \ge C R^{n-p}$, and so Ω^c is uniformly p-fat.

Conclusion

We thus have for $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ that

Ω^c uniformly p-fat $\iff$ Ω admits pointwise p-Hardy,

and the proof is based on the use of 'elementary tools'; more precisely, sophisticated machinery from potential theory is not needed.

By Wannebo's integration trick,
Ω^c uniformly p-fat $\implies$ Ω admits usual p-Hardy,

and so we finally have an 'elementary' proof for the fact that
Ω admits pointwise p-Hardy $\implies$ Ω admits usual p-Hardy.

4. Boundary conditions and weighted Hardy inequalities

Pointwise Hardy implies inner boundary density

If a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then it is easy to see that the following *inner boundary density condition* for $\partial\Omega$ holds for $\lambda = n - p$ (L. PAMS 2008):

there exists a constant $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

Idea: Let $\overline{B}(x, 2d_\Omega(x)) \cap \partial\Omega \subset \bigcup_{i=1}^{N} B(z_i, r_i)$ and use the pointwise p-Hardy for the test function

$$\varphi(y) = \min_{1 \leq i \leq N} \big\{1,\ r_i^{-1} d(y, B(z_i, r_i))\big\} \cdot \chi_\Omega(y) \cdot (\text{cut-off})$$

Pointwise Hardy from inner boundary density

Conversely, if $\Omega \subset \mathbb{R}^n$ and for some $\lambda > n - p$

$$\mathcal{H}^{\lambda}_{\infty}\big(\overline{B}(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega, \tag{2}$$

then Ω admits the pointwise p-Hardy inequality (L. PAMS 2008), (KLT 2009).

Using the self-improvement of pointwise Hardy inequalities, we have from the previous results for a domain $\Omega \subset \mathbb{R}^n$ that

Ω admits pointwise p-Hardy
$\implies$ Ω admits pointwise q-Hardy for some $1 \leq q < p$
$\implies$ $\partial\Omega$ satifies (2) with $\lambda = n - q > n - p$
$\implies$ Ω admits pointwise p-Hardy.

Inner boundary density and thickness

From the previous slide we obtain also the following interesting characterization:
Ω^c is uniformly p-fat $\Leftrightarrow$ $\partial\Omega$ satisfies inner density with some $\lambda > n - p$.

We have compared thickness conditions with uniform p-fatness
($\leftrightarrow$ λ-thickness for $\lambda > n - p$)
and pointwise p-Hardy inequalities
($\leftrightarrow$ inner λ-thickness of $\partial\Omega$ for $\lambda > n - p$)

But since $1 < p < \infty$, the relevant values of λ have been $0 \leq \lambda \leq n - 1$.

However, (Hausdorff) thickness conditions make sense for each $0 \leq \lambda \leq n$.

On the other hand, p-Hardy inequalities do not 'see' the difference between dimensions $n - 1$ and $\lambda \in (n - 1, n]$, but *weighted* Hardy inequalities do:

Weighted Hardy inequalities

Let us add a weight $d_\Omega(x)^\beta$, $\beta \in \mathbb{R}$, to the both sides of the p-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x) \quad ^{-p} \, dx \le C \int_\Omega |\nabla u(x)|^p \qquad dx$$

Weighted Hardy inequalities

Let us add a weight $d_\Omega(x)^\beta$, $\beta \in \mathbb{R}$, to the both sides of the p-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \le C \int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta \, dx$$

Weighted Hardy inequalities

Let us add a weight $d_\Omega(x)^\beta$, $\beta \in \mathbb{R}$, to the both sides of the p-Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta - p} \, dx \leq C \int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta \, dx$$

This is the *(weighted)* (p, β)*-Hardy inequality* for $u \in C_0^\infty(\Omega)$.
The following results hold for weighted Hardy inequalities:

Theorem (Nečas 1962)

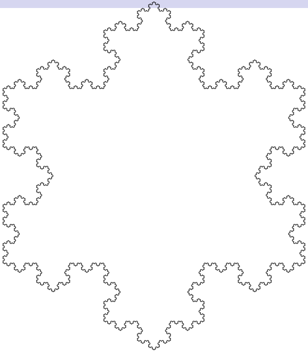
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - 1$ (sharp).

Theorem (Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then there exists some $\beta_0 > 0$ so that Ω admits the (p, β)-Hardy inequality for all $\beta < \beta_0$.

Ball and snowflake

Consider domains $B = B(0,1) \subset \mathbb{R}^2$ and a 'snowflake' domain $\Omega \subset \mathbb{R}^2$. Both B and Ω have 2-thick complements, but ∂B satisfies only inner 1-density condition whereas $\partial\Omega$ satisfies inner density condition for $\lambda = \log 4 / \log 3$.



p-Hardy inequalities do not 'see' this difference, but *weighted* Hardy inequalities do: For a fixed $1 < p < \infty$,
B admits (p, β)-Hardy iff $\beta < p - 1 \quad (= p - n + (n-1))$,
whereas Ω (should) admit (p, β)-Hardy iff $\beta < p - 2 + \lambda$.

This observation by P. Koskela was the starting point for all my research on Hardy inequalities.

Weighted pointwise Hardy inequalities

We also have the following pointwise version of the weighted (p, β)-Hardy inequality:

$$|u(x)| \leq C d_\Omega(x)^{1} \quad \Big(M_{2d_\Omega(x)}\big(|\nabla u|^q \qquad \big)(x)\Big)^{1/q}, \tag{3}$$

where we assume that $1 < q < p$ (self-improvement?).

Weighted pointwise Hardy inequalities

We also have the following pointwise version of the weighted (p, β)-Hardy inequality:

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} \Big(M_{2d_\Omega(x)} \big(|\nabla u|^q d_\Omega^{\frac{\beta}{p} q} \big)(x) \Big)^{1/q}, \qquad (3)$$

where we assume that $1 < q < p$ (self-improvement?).

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where we assume that $1 < q < p$ (self-improvement?).

We say that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality if there exist some $1 < q < p$ and a constant $C > 0$ so that (3) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$ with these q and C.

As in the unweighted case, the pointwise (p, β)-Hardy inequality implies the usual weighted (p, β)-Hardy inequality (thanks to the built-in 'self-improvement').

Accessibility

Theorem (Koskela-L. JLMS 2009)

Let $1 < p < \infty$ *and let* $\Omega \subset \mathbb{R}^n$ *be a domain. Assume that there exist* $0 \leq \lambda \leq n$, $c \geq 1$, *and* $C > 0$ *so that*

$$\mathcal{H}^{\lambda}_{\infty}(\partial^{\mathrm{vis}}_{x,c}\Omega) \geq C d_{\Omega}(x)^{\lambda} \qquad \text{for every } x \in \Omega. \tag{4}$$

Then Ω *admits the pointwise* (p,β)*-Hardy inequality whenever* $\beta < p - n + \lambda$.

Accessibility

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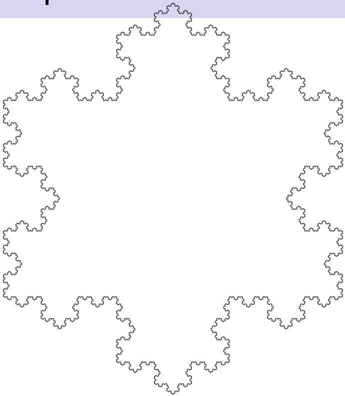
Then Ω *admits the pointwise* (p, β)*-Hardy inequality whenever* $\beta < p - n + \lambda$.

A point $w \in \partial\Omega$ is in the set $\partial^{\text{vis}}_{x,c}\Omega$, if w is *accessible* from x by a *c-John curve*, that is, there exists a curve $\gamma = \gamma_{w,x} \colon [0, l] \to \Omega$, parametrized by arc length, with $\gamma(0) = w$, $\gamma(l) = x$, and satisfying $d(\gamma(t), \partial\Omega) \ge t/c$ for every $t \in [0, l]$.
(Thus (4) is a stronger version of the inner boundary density condition introduced earlier)

Examples

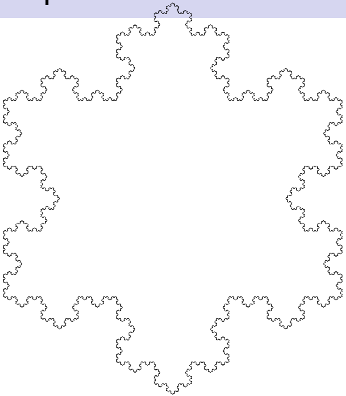


Here the boundary is λ-thick $(1 < \lambda < 2)$ and well accessible

$\Rightarrow$ (p, β)-Hardy for all

$$\beta < \underbrace{p - 2 + \lambda}_{> p-1}$$

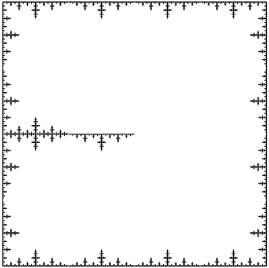
Examples



Here the boundary is λ-thick $(1 < \lambda < 2)$ and well accessible

$\Rightarrow (p, \beta)$-Hardy for all

$$\beta < \underbrace{p - 2 + \lambda}_{> p-1}$$



Here the boundary is λ-thick $(1 < \lambda < 2)$, but above the antenna in the middle the *accessible* part of the boundary is only 1-dimensional,
and indeed the (p, β)-Hardy does not hold when

$$\beta = p - 1 < p - 2 + \lambda$$

Removing accessibility

The accessibility part of the previous theorem can actually be dropped (at least) whenever $\beta \leq 0$:

Theorem (L., preprint 2010)

Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition holds with an exponent $0 \leq \lambda \leq n$. Then, if $\beta \leq 0$ and $\beta < p - n + \lambda$, Ω admits the pointwise (p, β)-Hardy inequality.

This, together with a 'shift'-property of usual Hardy inequalities (L. ACV 2008) leads to the following result:

Theorem (L. preprint 2010)

Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that the inner boundary density condition holds with an exponent $0 \leq \lambda \leq n - 1$. Then Ω admits the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$.

In other words

We can rewrite this in the spirit of Ancona–Lewis–Wannebo as

Corollary

Assume that Ω^c is uniformly q-fat for all $q > s \geq 1$. Then Ω admits the (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p - s$.

In particular, if $\Omega \subset \mathbb{R}^2$ is simply connected, then Ω admits the (p, β)-Hardy whenever $\beta < p - 1$.

The idea of the proof is (almost embarrassingly) simple: We only need to consider the case $\beta > 0$. By assumption, Ω^c is uniformly $(p - \beta)$-fat, and so Ω admits the $(p - \beta)$-Hardy inequality. Now, given $u \in C_0^\infty(\Omega)$, just use the $(p - \beta)$-Hardy inequality for the test function

$$v = |u|^{\beta/(p-\beta)},$$

and the (p, β)-inequality for u follows with simple calculations (this is the 'shift').

Conclusion and a gap

In conclusion, if $1 < p < \infty$, $\beta \in \mathbb{R}$, and $\partial\Omega \subset \mathbb{R}^n$ is inner λ-thick for $\lambda > n - p + \beta$, then Ω admits

- (p, β)-Hardy if $\beta < p - 1$;
- pointwise (p, β)-Hardy if $\beta \leq 0$;
- pointwise (p, β)-Hardy if $\partial\Omega$ is in addition accessible.

On the other hand, inner λ-thickness for $\lambda > n - p + \beta$ *does not suffice* for (p, β)-Hardy if $\beta \geq p - 1$.

Above we have a gap: Does inner λ-thickness for $\lambda > n - p + \beta$ suffice for *pointwise* (p, β)-Hardy if $0 < \beta < p - 1$?

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