NEIGHBOURHOOD CAPACITIES

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ABSTRACT. We study the behaviour of the p-capacity of a compact set E with respect to the t-neighbourhoods of E as t varies. We establish sharp upper and lower bounds for these capacities in terms of Minkowski and Hausdorff -type contents of E, respectively, and our results hold in both Euclidean and more general metric spaces. In our lower bounds the porosity of the set E plays an important role, and it is shown by examples that an assumption like this is in general necessary. In addition, we present a self-contained approach to the theory of sets of zero capacity in metric spaces.

1. INTRODUCTION

When $E \subset \mathbb{R}^n$ and $t > 0$, we denote

$$E_t = \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < t\}$$

and call E_t the *(open) t-neighbourhood* of E. The main purpose of this paper is to study the behaviour of the p-capacity of a compact set E with respect to the neighbourhood sets E_t as t varies. A very illustrative particular case of our results can be stated as follows:

Theorem 1.1. *Let $1 < p < \infty$ and $0 \leq \lambda < n$, and assume that $E \subset \mathbb{R}^n$ is an Ahlfors λ-regular compact set. If $p > n - \lambda$, then*

$$\operatorname{cap}_p(E, E_t) \approx t^{n-p-\lambda}$$

for all $0 < t < \operatorname{diam}(E)$, and if $p \leq n - \lambda$, then $\operatorname{cap}_p(E, E_t) = 0$ for all $t > 0$.

Here the (variational) p-capacity of a compact set E with respect to an open set $\Omega \supset E$ is defined as

$$\operatorname{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega), \ u \geq 1 \text{ on } E \right\}. \tag{1}$$

For the definition of Ahlfors regularity see Section 2.2. Notice also that Theorem 1.1 does not hold for n-regular sets of $\mathbb{R}^n$ (see Remark 5.3).

In $\mathbb{R}^n$, $n \geq 2$, the case $p = n$ of Theorem 1.1 and related results were essentially solved by Heikkala [5, Sect. 4] using modulus estimates. Nevertheless, our assumptions on the set E are in general slightly weaker than in the corresponding results of [5]. Earlier results concerning neighbourhood capacities can also be found in Väisälä [13] and Vuorinen [14, Sect. 6].

Theorem 1.1 follows from more general upper and lower estimates that we establish for $\operatorname{cap}_p(E, E_t)$ when E is a compact subset of a metric measure

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space (X, d, μ); see Section 2.1 for the precise setting and our requirements on the space X. In particular, we replace the smooth test functions $u \in C_0^\infty(\Omega)$ in (1) with compactly supported Lipschitz functions, as these make sense in any metric space.

The upper bounds for neighbourhood capacities are proven in terms of Minkowski-type contents of E in Section 3, and the respective lower bounds, where we need Hausdorff measures and contents, are contained in Section 5. However, if we only assume that the mesure μ is doubling, then it is convenient to use modified versions of the usual Hausdorff and Minkowski contents. The definitions and some basic properties of these are found in Section 2.2. The precise definition of the metric space version of the p-capacity is given in Section 2.3. Let us remark here that for the most part of this paper we may take $1 \le p < \infty$, but that there are also a few subtle occasions where it has to be required that $p > 1$. In Section 2.4 we recall the definition of *porous* sets, as it turns out that our lower estimates on neighbourhood capacities require the set E to be porous; the necessity of such an extra condition is illustrated in the final Section 6.

In addition to the above growth estimates, the context of these neighbourhood capacities leads one naturally to study questions related to sets of *zero capacity*. In Section 4 we present, using only 'elementary tools', a self-contained approach to some of these questions in general metric spaces; we hope that this part is also of independent interest. In particular, we obtain as a consequence a metric space proof for the known fact that if E is not of zero p-capacity, then $\operatorname{cap}_p(E, E_t) \to \infty$ as $t \to 0$.

Let us emphasize here that even though we formulate our results in the setting of a general metric space, all of our main results are, to the best of our knowledge and apart from those in Section 4, new even in the space $\mathbb{R}^n$ when $p \ne n$.

For notation we remark that throughout the paper the letter C is used to denote positive (and finite) constants whose value may change from expression to expression, and that the dependence of C on parameters $A, B, \ldots$ is expressed by writing $C = C(A, B, \ldots)$. We also denote $a \approx b$ and say that a and b are comparable if there are constants $C_1, C_2 > 0$ so that $C_1 a \le b \le C_2 a$. Finally, if a_α and b_α are such that $a_\alpha / b_\alpha \to 0$ as α tends to a limit A (usually $A = 0$ or $A = \infty$), we write $a_\alpha \ll b_\alpha$ (as $\alpha \to A$).

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2. Preliminaries

2.1. **Metric spaces.** We assume that $X = (X, d, \mu)$ is a complete metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X : d(x, y) < r\}$. For $0 < t < \infty$, we write $tB = B(x, tr)$, and $\overline{B}$ is the corresponding closed ball. When $A \subset X$, ∂A is the boundary and $\overline{A}$ the closure of A, and χ_A denotes the characteristic function of A. The distances between two points, a set and a point, or two sets, are all denoted $d(\cdot, \cdot)$.

We make the standing assumption that the measure μ is *doubling* (with respect to the metric d), i.e. that there exists a constant $C_d \geq 1$ such that

$$\mu(2B) \leq C_d\, \mu(B)$$

for all balls B of X. The doubling condition together with the completeness implies that the space X is proper, that is, closed balls of X are compact.

The doubling condition gives an upper bound for the dimension of X in the sense that there exists a constant $C = C(C_d) > 0$ such that, for $s = \log_2 C_d$, we have the estimate

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\Big(\frac{r}{R}\Big)^s \tag{2}$$

whenever $0 < r \leq R < \operatorname{diam} X$ and $y \in B(x,R)$. Note that estimate (2) can also hold for some smaller number(s) s; in the sequel we just assume that (2) holds for some s, which is is called the *doubling dimension* of X.

Another important property of a metric space that we often require is that the space X supports a *(weak)* $(1,p)$*-Poincaré inequality.* This means that we assume the existence of constants $C_p > 0$ and $\tau \geq 1$ such that for all balls $B \subset X$, all continuous functions u, and for all *upper gradients* g_u of u, we have the inequality

$$\fint_B |u - u_B|\, d\mu \leq C_p r\Big(\fint_{\tau B} g_u^p\, d\mu\Big)^{1/p}. \tag{3}$$

Here

$$u_B = \fint_B u\, d\mu = \mu(B)^{-1}\int_B u\, d\mu$$

is the integral average of u over B. Recall that a Borel function $g \geq 0$ is said to be an upper gradient of a function u (on an open set $\Omega \subset X$), if for all curves γ joining points x and y (in Ω) we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. By a curve we simply mean a non-constant, rectifiable, continuous mapping from a compact interval to X. Nevertheless, let us stress here that, unlike with doubling, we do *not* in general assume that the space X supports a Poincaré inequality, and hence we try to state clearly the occasions where it is needed.

From now on, if $C > 0$ is a constant which only depends on the doubling and Poincaré constants C_d, C_p, τ, and the doubling dimension s, we write $C = C(X)$.

Examples of metric spaces equipped with doubling measures and supporting a Poincaré inequality include (weighted) Euclidean spaces, compact Riemannian manifolds, Carnot groups, and metric graphs. See for instance Hajłasz–Koskela [4], Heinonen [6], and the references therein for more information on analysis on metric spaces based on upper gradients and Poincaré inequalities.

2.2. **Hausdorff and Minkowski.** We define λ-*Hausdorff contents* of a set $E \subset X$, for $0 < r \leq \infty$, as

$$\mathcal{H}_r^\lambda(E) = \inf\left\{ \sum_k r_k^\lambda : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \leq r \right\},$$

and the λ-*Hausdorff measure* of E is $\mathcal{H}^\lambda(E) = \lim_{r\to 0} \mathcal{H}_r^\lambda(E)$. The *Hausdorff dimension* of E is then the number

$$\dim(E) = \inf\{\lambda > 0 : \mathcal{H}^\lambda(E) = 0\}.$$

When the balls covering the set $E \subset X$ are required to be of equal radii, we obtain λ-*Minkowski contents* of E:

$$\mathcal{M}_r^\lambda(E) = \inf\left\{ N r^\lambda : E \subset \bigcup_{k=1}^N B(z_k, r),\ z_i \in E \right\}.$$

The *lower and upper Minkowski dimension* of E are then defined to be

$$\underline{\dim}_{\mathcal{M}}(E) = \inf\left\{\lambda > 0 : \liminf_{r\to 0} \mathcal{M}_r^\lambda(E) = 0\right\}$$

and

$$\overline{\dim}_{\mathcal{M}}(E) = \inf\left\{\lambda > 0 : \limsup_{r\to 0} \mathcal{M}_r^\lambda(E) = 0\right\},$$

respectively.

Notice that for each compact set E we have

$$\dim_{\mathcal{H}}(E) \leq \underline{\dim}_{\mathcal{M}}(E) \leq \overline{\dim}_{\mathcal{M}}(E),$$

where all inequalities can be strict; see for instance Mattila [12, Ch. 5]. But if $\underline{\dim}_{\mathcal{M}}(E) = \overline{\dim}_{\mathcal{M}}(E)$, we simply write $\dim_{\mathcal{M}}(E) = \overline{\dim}_{\mathcal{M}}(E)$. For many sufficiently regular sets all of these dimensions agree. For instance, a set E is said to be *(Ahlfors) λ-reqular* if

$$\mathcal{H}^\lambda(E \cap B(x, r)) \approx r^\lambda$$

whenever $x \in E$ and $0 < r < \operatorname{diam}(E)$. For compact Ahlfors regular sets we have the following result (see e.g. [12, Thm. 5.7.]):

Lemma 2.1. *Assume that E is compact and Ahlfors λ-regular. Then*

$$\overline{\dim}_{\mathcal{M}}(E) = \dim_{\mathcal{H}}(E) = \lambda.$$

In general metric spaces it is often more convenient to use modified versions of $\mathcal{H}_r^\lambda$ and $\mathcal{M}_r^\lambda$, namely the *Hausdorff and Minkowski contents of codimension q*. The former is defined for a set $E \subset X$ by

$$\widetilde{\mathcal{H}}_r^q(E) = \inf\left\{ \sum_k \mu(B_k)\, r_k^{-q} : E \subset \bigcup_k B_k,\ x_k \in E,\ 0 < r_k \leq r \right\},$$

where we write $B_k = B(x_k, r_k)$, and the latter by

$$\widetilde{\mathcal{M}}_r^q(E) = \inf\left\{ r^{-q} \sum_k \mu(B(x_k, r)) : E \subset \bigcup_k B(x_k, r),\ x_k \in E \right\}.$$

Then, naturally, the *Hausdorff measure of codimension q* is defined as

$$\widetilde{\mathcal{H}}^q(E) = \lim_{r\to 0} \widetilde{\mathcal{H}}_r^q(E).$$

Note that in a s-regular space $\widetilde{\mathcal{H}}^q_r \approx \mathcal{H}^{s-q}_r$ and $\widetilde{\mathcal{M}}^q_r \approx \mathcal{M}^{s-q}_r$. On the other hand, in a general metric space X with doubling dimension s we have the following one-sided estimate for compact sets:

Lemma 2.2. *Let $E \subset X$ be a compact set. Then*

$$\mathcal{H}^{s-q}_r(E) \le C(X,E)\widetilde{\mathcal{H}}^q_r(E)$$

for each $0 < r < \operatorname{diam}(E)$.

Proof. We may clearly assume that $E \neq \emptyset$. Fix $w_0 \in E$ and write $B_E = B(w_0, 2\operatorname{diam}(E))$. Let $E \subset \bigcup B_k$, $B_k = B(x_k, r_k)$, $r_k \le r < \operatorname{diam}(E)$. Then $B_k \subset B_E$ for all k, and so, by the doubling estimate (2), we have $r_k^s \le C \operatorname{diam}(E)^s \mu(B_k)/\mu(B_E)$. Thus

$$\begin{aligned}\mathcal{H}^{s-q}_r(E) &\le \sum_k r_k^{s-q} \le C \sum_k \operatorname{diam}(E)^s \mu(B_k)\mu(B_E)^{-1} r_k^{-q} \\ &= C(E,X) \sum_k \mu(B_k) r_k^{-q},\end{aligned}$$

and taking the infimum over all such covers yields the claim. □

Remark 2.3. By a similar argument we see that if $E \subset X$ is compact, then

$$\mathcal{M}^{s-q}_r(E) \le C(X,E)\widetilde{\mathcal{M}}^q_r(E)$$

for all $0 < r < \operatorname{diam}(E)$.

We will also use the following simple upper bound for the contents $\widetilde{\mathcal{M}}$:

Lemma 2.4. *Let $E \subset X$ be a compact set. Then*

$$\widetilde{\mathcal{M}}^q_r(E) \le C r^{-q} \mu(E_{\operatorname{diam}(E)})$$

for all $0 < r < \operatorname{diam}(E)$, where the constant $C > 0$ only depends on the doubling constant of X.

Proof. Cover E with balls $B_i = B(w_i, r)$, $i = 1, \ldots, N$, $w_i \in E$, in such a way that the balls $(1/5)B_i$ are pairwise disjoint (see e.g. [6] for this 'basic' or '$5r$'-covering theorem). Then, using the doubling property of μ, we see that

$$\widetilde{\mathcal{M}}^q_r(E) \le r^{-q} \sum_{i=1}^N \mu\big(B_i\big) \le C r^{-q} \sum_{i=1}^N \mu\big((1/5)B_i\big) \le C r^{-q} \mu(E_{\operatorname{diam}(E)}).$$

This proves the claim. □

For $\widetilde{\mathcal{H}}^q$ we have the following result, which is well-known for the usual Hausdorff measures and contents (see e.g. [7]); the converse of this claim is of course trivial since $\widetilde{\mathcal{H}}^q_r$ is decreasing in r.

Lemma 2.5. *Assume that $E \subset X$ is a compact set with $\widetilde{\mathcal{H}}^q_R(E) = 0$ for some $0 < R < \infty$. Then also $\widetilde{\mathcal{H}}^q(E) = 0$.*

Proof. We may again assume that $E \neq \emptyset$. Fix $w_0 \in E$, denote $B_0 = B(w_0, R + \operatorname{diam}(E))$, and let $\varepsilon > 0$. It suffices to show that $\widetilde{\mathcal{H}}^q_\varepsilon(E) < \varepsilon$. By the doubling estimate (2) we have for each ball $B = B(w, r)$ with $w \in E$ and $0 < r < R$, and for every $q \geq s$, that

$$\mu(B)r^{-q} \geq C_1\mu(B_0)(R + \operatorname{diam}(E))^{-q} > 0.$$

Hence, if $\widetilde{\mathcal{H}}^q_r(E) = 0$, we must have $q < s$.

By assumption, there now exists balls $B_i = B(w_i, r_i)$, $w_i \in E$ and $r_i \leq R$, so that $E \subset \bigcup_i B_i$ and

$$\sum_i \mu(B_i)r_i^{-q} < \min\Big\{C_1\mu(B_0)(R + \operatorname{diam}(E))^{-s}\,\varepsilon^{s-q}, \varepsilon\Big\}.$$

From this we obtain for each i the trivial estimate

$$\mu(B_i)r_i^{-q} < C_1\mu(B_0)(R + \operatorname{diam}(E))^{-s}\,\varepsilon^{s-q}. \tag{4}$$

But now it follows from (2) with the help of (4) that

$$r_i^{s-q} \leq r_i^{-q}C_1^{-1}(R + \operatorname{diam}(E))^s\mu(B_i)/\mu(B_0) < \varepsilon^{s-q}.$$

Since $q < s$, we have $r_i < \varepsilon$ for each i. Hence $\widetilde{\mathcal{H}}^q_\varepsilon(E) < \varepsilon$. □

2.3. **Capacity.** For simplicity, we define the p-capacity in metric spaces using Lipschitz functions, and only for compact sets. Recall that when $\Omega \subset X$, a function $u\colon \Omega \to \mathbb{R}$ is said to be *(L-)Lipschitz*, if

$$|u(x) - u(y)| \leq Ld(x, y) \qquad \text{for all } x, y \in \Omega.$$

The set of all Lipschitz functions $u\colon \Omega \to \mathbb{R}$ is denoted $\operatorname{Lip}(\Omega)$, and $\operatorname{Lip}_0(\Omega)$ is the set of Lipschitz functions $u \in \operatorname{Lip}(\Omega)$ with a compact support in Ω; the support of a function $u\colon \Omega \to \mathbb{R}$, denoted $\operatorname{spt}(u)$, is the closure of the set where u is non-zero.

The pointwise Lipschitz constant of a function $u\colon \Omega \to \mathbb{R}$ at $x \in \Omega$ is

$$\operatorname{Lip}(u; x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}.$$

It is not hard to see that this defines an upper gradient g for a Lipschitz function $u\colon \Omega \to \mathbb{R}$ by $g(x) = \operatorname{Lip}(u; x)$.

When $\Omega \subset X$ is an open set and E is a compact subset of Ω, the *p-capacity* of E with respect to Ω, for $1 \leq p < \infty$, is defined to be

$$\operatorname{cap}_p(E, \Omega) = \inf\Big\{\int_\Omega g_u^p\, d\mu : 0 \leq u \in \operatorname{Lip}_0(\Omega),\ u = 1 \text{ in } E\Big\}.$$

Note that the infimum is taken over *all* upper gradients of Lipschitz functions u satisfying the above conditions. If there are no such functions, we set $\operatorname{cap}_p(E, \Omega) = \infty$.

If the space X supports a $(1, p)$-Poincaré inequality (and μ is doubling, as usual), the above definition of the capacity coincides with the more abstract and more general definition of the relative (*Newtonian*) Sobolev capacity; see Costea [1] for the definitions and the details, and in particular his Remark 3.4 for the above-mentioned equivalence of the definitions. From this equivalence we also conclude that in $\mathbb{R}^n$ the above 'Lipschitz' capacity agrees with the 'smooth' capacity defined in (1).

For more information on capacities we refer to [1] and the book [7, Ch. 2] by Heinonen, Kilpeläinen, and Martio. The latter only deals with weighted

Euclidean spaces, but many of the underlying ideas are similar in more general metric spaces as well.

2.4. **Porous sets.** We say that a set $E \subset X$ is *α-porous* (for $0 < \alpha < 1$), if for every $w \in E$ and all $0 < r < \operatorname{diam}(E)$ there exists a point $y \in X$ such that $B(y, \alpha r) \subset B(w, r) \cap (X \setminus E)$.

It is well-known that in a s-regular metric space X a set $E \subset X$ is porous if and only if the *Assouad dimension* $\dim_{\mathcal{A}}(E)$ is bounded from above away from s; see Luukkainen [11, Thm 5.2] ($X = \mathbb{R}^n$) and David–Semmes [2, Lemma 5.8] (general X); consult also [9] and the references therein for recent more general results related to upper bounds of dimensions of porous sets in metric spaces. Recall that $\dim_{\mathcal{A}}(E)$ is the infimum of all numbers $\beta > 0$ for which there exists $C(\beta) \geq 1$ such that each subset $F \subset E$ can be covered by at most $C(\beta)\varepsilon^{-\beta}$ balls of radius $r = \varepsilon \operatorname{diam}(F)$ whenever $0 < \varepsilon < 1/2$. It is easy to verify that $\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{A}}(E)$ for each set E, and $\overline{\dim}_{\mathcal{M}}(E) \leq \dim_{\mathcal{A}}(E)$ for each bounded set E. Moreover, if E is s-regular, then $\dim_{\mathcal{A}}(E) = s$. See the paper [11] for a thorough discussion on the Assouad dimension.

3. Upper bounds

In this section we obtain upper estimates for neighbourhood capacities with the help of Minkowski contents. The proofs of these estimates are rather straight-forward, and, for the most part, we only need to assume that the space X satisfies the doubling condition. In addition, we also formulate the growth estimates in the case when the measures of balls (centered at the compact set E) have uniform upper bounds in terms of the radii; this holds, in particular, in Ahlfors regular spaces. Nevertheless, no Poincaré inequalities are needed in this section.

Let us begin with a simple observation which gives a 'unversal' upper bound for the growth of neighbourhood capacities; this can be viewed as a generalization of a result of Vuorinen [14, Lemma 6.27], in which only the the case $p = n$ was concerned.

Proposition 3.1. *Let $E \subset X$ be a compact set. Then there exists a constant $C = C(X, E) > 0$ such that*

$$\operatorname{cap}_p(E, E_t) \leq C t^{-p}$$

for every $0 < t < \operatorname{diam}(E)$.

Proof. Fix $0 < t < \operatorname{diam}(E)$ and let $0 < \varepsilon < 1$. Then the function

$$u(x) = \max\left\{0, 1 - (1+\varepsilon)t^{-1} \operatorname{dist}(x, E)\right\}$$

is clearly an admissible test function for the capacity $\operatorname{cap}_p(E, E_t)$, and

$$g_u = (1+\varepsilon)t^{-1} \chi_{E_{(t/(1+\varepsilon))} \setminus E}$$

is an upper gradient of u. Thus

$$\operatorname{cap}_p(E, E_t) \leq \int_{E_t} g_u^p \, d\mu \leq (1+\varepsilon)^p t^{-p} \mu(E_{\operatorname{diam}(E)}),$$

and letting $\varepsilon \to 0$ proves the claim with the constant $C = C(X, E) = \mu(E_{\operatorname{diam}(E)}) < \infty$. □

In the next lemma we obtain more precise upper bounds for the growth of neighbourhood capacities in terms of Minkowski contents.

Lemma 3.2. (a) *Let $1 \le p < \infty$. Then there exists a constant $C = C(X,p) > 0$ such that*

$$\operatorname{cap}_p(E, E_t) \le C\widetilde{\mathcal{M}}^q_{t/3}(E)t^{q-p}$$

whenever $E \subset X$ is a compact set and $0 < r < \operatorname{diam}(E)$.
(b) *If in addition $\mu(B(w,r)) \le cr^d$ whenever $w \in E$ and $0 < r < \operatorname{diam}(E)$, then*

$$\operatorname{cap}_p(E, E_t) \le C\mathcal{M}^\lambda_{t/3}(E)t^{d-p-\lambda}.$$

for all $0 < r < \operatorname{diam}(E)$, where $C = C(X,p,c) > 0$.

Proof. Let $0 < t < \operatorname{diam}(E)$, cover the compact set E with balls $B_i = B(w_i, t/3)$, $w_i \in E$, $i = 1, \ldots, N$, and define

$$u(x) = \max_{1 \le i \le N} \big\{0, 1 - 2t^{-1}\operatorname{dist}(x, B_i)\big\}.$$

Then u is a Lipschitz function, $u = 1$ in E, and u is supported in E_t. Moreover, u has an upper gradient g_u such that

$$g_u(x)^p \le \sum_{i=1}^N (t/2)^{-p}\chi_{2B_i}(x)$$

for a.e. $x \in E_t$. Using the doubling condition we thus have

$$\operatorname{cap}_p(E, E_t) \le \int_{E_t} g_u^p \, d\mu \le \sum_{i=1}^N \mu(2B_i)(t/2)^{-p} \le C\sum_{i=1}^N \mu(B_i)(t/3)^{-p}.$$

Taking the infimum over all such covers yields

$$\operatorname{cap}_p(E, E_t) \le C\widetilde{\mathcal{M}}^p_{t/3}(E) = C\widetilde{\mathcal{M}}^q_{t/3}(E)t^{q-p},$$

since the second equality holds for all q. Part (a) follows.

(b) Now, if $\mu(B(w,r)) \le cr^d$ for all balls with a center point $w \in E$ and $r < \operatorname{diam}(E)$ we have that $\widetilde{\mathcal{M}}^q_r(E) \le c\mathcal{M}^{d-q}_r(E)$ for $0 < r < \operatorname{diam}(E)$. Thus it follows from part (a), by taking $q = d - \lambda$, that

$$\operatorname{cap}_p(E, E_t) \le \mathcal{M}^\lambda_{t/3}(E)t^{d-p-\lambda},$$

which is the claim. □

If we now know that the Minkowski contents of E remain bounded as $r \to 0$, we obtain the following extension of Proposition 3.1:

Proposition 3.3. (a) *Let $1 \le p < \infty$ and assume that $E \subset X$ is a compact set satisfying $\limsup_{r\to 0} \widetilde{\mathcal{M}}^q_r(E) < \infty$. Then there exists a constant $C = C(X, E, p, q) > 0$ such that*

$$\operatorname{cap}_p(E, E_t) \le Ct^{q-p}$$

for all $0 < t < \operatorname{diam}(E)$.
(b) *If $E \subset X$ is a compact set satisfying $\limsup_{r\to 0} \mathcal{M}^\lambda_r(E) < \infty$, and if*

in addition $\mu(B(w,r)) \leq cr^d$ *whenever* $w \in E$ *and* $0 < r < \operatorname{diam}(E)$*, then there exists a constant* $C = C(X, E, p, d, \lambda, c) > 0$ *such that*

$$\operatorname{cap}_p(E, E_t) \leq C t^{d-p-\lambda}$$

for all $0 < t < \operatorname{diam}(E)$*. This is true, in particular, if* $\lambda > \overline{\dim}_{\mathcal{M}}(E)$*.*

Proof. (a) By assumption, there exists $t_0 > 0$ such that

$$\widetilde{\mathcal{M}}^q_t(E) \leq 2 \limsup_{r\to 0} \mathcal{M}^\lambda_r(E) =: A \quad \text{for all } 0 < t \leq t_0.$$

On the other hand, by Lemma 2.4

$$\widetilde{\mathcal{M}}^q_r(E) \leq C_1 r^{-q} \mu(E_{\operatorname{diam}(E)}) \quad \text{for all } 0 < r < \operatorname{diam}(E).$$

Hence, for $t_0 < t < \operatorname{diam}(E)$, we have

$$\widetilde{\mathcal{M}}^q_t(E) \leq C_2 t_0^{-q} \qquad \text{with } C_2 = C_1 \mu(E_{\operatorname{diam}(E)}).$$

The claim, with the constant $C = C_3 \max\{A, C_2 t_0^{-q}\}$, where C_3 is the constant from Lemma 3.2, follows from the above estimates and Lemma 3.2(a).
(b) This part is an immediate consequence of part (a), since now we have that $\limsup_{r\to 0} \widetilde{\mathcal{M}}^q_r(E) < \infty$ for $q = d - \lambda$. □

4. Sets of zero capacity

In this section we give slight generalizations for some of the well-known results concerning (compact) sets of zero capacity (see e.g. [1] and [7]). The novelty for our approach comes mainly from the replacement of the usual Hausdorff measures $\mathcal{H}^\lambda$ with the measures $\widetilde{\mathcal{H}}^q$ and the fact that we get a (more or less) self-contained exposition with minimal assumptions on the space X.

We say that a compact set $E \subset X$ is of zero capacity, $\operatorname{cap}_p(E) = 0$, if $\operatorname{cap}_p(E, \Omega) = 0$ for all open sets $\Omega \supset E$; otherwise we write $\operatorname{cap}_p(E) > 0$. It is immediate that $\operatorname{cap}_p(E) = 0$ if and only if $\operatorname{cap}_p(E, E_t) = 0$ for all $0 < t < \operatorname{diam}(E)$.

The following results illustrate the intimate connection between capacities and Hausdorff measures.

Proposition 4.1. *Let* $1 < p < \infty$ *and assume that a compact set* $E \subset X$ *satisfies* $\widetilde{\mathcal{H}}^p(E) < \infty$*. Then* $\operatorname{cap}_p(E) = 0$*.*

Remark 4.2. Proposition 4.1 fails to hold when $p = 1$. Indeed, it was shown by Kinnunen et al. in [10] that if μ is doubling and X supports a $(1,1)$-Poincaré inequality, then $\operatorname{cap}_1(E, X) \approx \widetilde{\mathcal{H}}^1_\infty(E)$ for all compact sets $E \subset X$. See also [3, Sect. 5.6.3].

We state below an 'almost converse' to Proposition 4.1, but we postpone the proof until the next section, as the proof is closely related to the lower bounds for neighbourhood capacities. Note also that here we need to assume a Poincaré inequality.

Proposition 4.3. *Let* $1 \leq p < \infty$ *and assume that* X *supports a* $(1,p)$*-Poincaré inequality. Let* $E \subsetneq X$ *be a compact set with* $\operatorname{cap}_p(E) = 0$*. Then* $\widetilde{\mathcal{H}}^q(E) = \mathcal{H}^{s-q}(E) = 0$ *for all* $q < p$*, and so in particular* $\dim_{\mathcal{H}}(E) \leq s - p$*.*

We now start to prove that $\widetilde{\mathcal{H}}^p(E) < \infty$ implies $\operatorname{cap}_p(E) = 0$. The following simple (but useful) fact is the first step into this direction.

Lemma 4.4. *Let $1 \le p < \infty$ and let $E \subset X$ be a compact set. Assume further that $\Omega \supset E$ is open. Then*

$$\operatorname{cap}_p(E, \Omega) \le C \widetilde{\mathcal{H}}^p_r(E)$$

for all $0 < r < d(E, X \setminus \Omega)/2$, where actually $C = C_d$ is the doubling constant; here we interpret $d(E, \emptyset) = \infty$.

Proof. Fix $0 < r < \operatorname{dist}(E, X \setminus \Omega)/2$, and cover E with balls $B_i = B(w_i, r_i)$, where $w_i \in E$ and $0 < r_i \le r$. Define

$$u(x) = \max_{1 \le i \le N} \left\{0, 1 - {r_i}^{-1} \operatorname{dist}(x, B_i)\right\},$$

so that u is a Lipschitz function, $u = 1$ in E, and u is supported in Ω. Moreover, u has an upper gradient g_u such that

$$g_u(x)^p \le \sum_{i=1}^N r_i^{-p} \chi_{2B_i}(x)$$

for a.e. $x \in \Omega$. Using the doubling condition we obtain

$$\operatorname{cap}_p(E, \Omega) \le \int_\Omega g_u^p \, d\mu \le \sum_{i=1}^N \mu(2B_i) r_i^{-p} \le C_d \sum_{i=1}^N \mu(B_i) r_i^{-p},$$

and the the claim follows by taking the infimum over all such covers of E. □

Notice that the proof of Lemma 4.4 is almost identical to the proof of Lemma 3.2. In fact, a similar proof (proving a similar statement) already appears in the paper [15, pp. 335–336] by H. Wallin, and the idea is there credited to L. Carleson. It is also worth a mention that Lemma 4.4 could be proven by using general properties of capacities, in particular the subadditivity; see for instance [1] and [7] for this approach. Nevertheless, our proof above does not rely on such general theory.

We now proceed to prove Proposition 4.1:

Proof of Proposition 4.1. Let Ω be an open set containing E. By Lemma 4.4 we have, for small enough $r > 0$, that

$$\operatorname{cap}_p(E, \Omega) \le C \widetilde{\mathcal{H}}^p_r(E) \le C \widetilde{\mathcal{H}}^p(E) < \infty. \tag{5}$$

The claim is now obvious if $\widetilde{\mathcal{H}}^p(E) = 0$, and hence we only have to deal with the case $0 < \widetilde{\mathcal{H}}^p(E) < \infty$. As the constant C above is independent of Ω, we formulate this case as a separate lemma:

Lemma 4.5. *Let $1 < p < \infty$ and let $E \subset X$ be a compact set. Assume that there exists a constant $0 < M < \infty$ such that*

$$\operatorname{cap}_p(E, \Omega) < M \quad \text{for all open } \Omega \supset E.$$

Then $\operatorname{cap}_p(E) = 0$.

Proof. Let us give a direct constructive proof for this lemma in the spirit of Evans and Gariepy [3, Sect. 4.7.2]; alternative proofs (using compactness properties of Sobolev spaces) can be found e.g. in [1] or [7].

As noted at the beginning of this section, it is sufficient to prove that $\mathrm{cap}_p(E, E_t) = 0$ for all $0 < t < \mathrm{diam}(E)$. Thus, fix $0 < t_0 < \mathrm{diam}(E)$, and choose $u_1 \in \mathrm{Lip}_0(E_{t_0})$ such that $u_1|_E = 1$ and u_1 has an upper gradient g_1 with $\int g_1^p < M$. Write $V_1 = \{u_1 > 1/2\} \supset E$ and let $t_1 = \mathrm{dist}(E, X \setminus V_1)/2 > 0$. We now define $v_1 = \min\{2u_1, 1\}$, and it is then clear that $v_1|_{V_1} = 1$ and $h_1 = 2g_1\chi_{X\setminus V_1}$ is an upper gradient of v_1. In particular, $\int h_1^p < 2^pM$.

We then take $u_2 \in \mathrm{Lip}_0(E_{t_1})$ with $u_2|_E = 1$ and $\int g_2^p < M$ for an upper gradient g_2 of u_2; note that $\mathrm{spt}(u_2) \subset V_1$. Write $V_2 = \{u_2 > 1/2\} \supset E$ and let $t_2 = \mathrm{dist}(E, X \setminus V_2)/2 > 0$. Define as above $v_2 = \min\{2u_2, 1\}$, whence $v_2|_{V_2} = 1$ and $h_2 = 2g_2\chi_{X\setminus V_2}$ is an upper gradient of v_2 with $\int h_2^p < 2^pM$.

Continuing this way we find numbers

$$t_0 > t_1 > t_2 > \cdots > t_k > t_{k+1} > \cdots > 0$$

and functions $v_k \in \mathrm{Lip}_0(E_{t_k})$ with their respective upper gradients h_k satisfying the following properties: (i) $v_k|_E = 1$, (ii) the supports $\mathrm{spt}(h_k)$ are pairwise disjoint, and (iii) $\int h_k^p < 2^pM$ for each $k = 1, 2, 3\ldots$

Now define $\varphi_j = j^{-1}\sum_{k=1}^{j} v_k$. Then clearly $\varphi_j \in \mathrm{Lip}_0(E_{t_0})$ and, by (i), $\varphi_j|_E = 1$ for each $j = 1, 2, \ldots$ Moreover, $\psi_j = j^{-1}\sum_{k=1}^{j} h_k$ is an upper gradient of φ_j. Using the properties (ii) and (iii) of the functions h_k we easily calculate that

$$\int_{E_1} \psi_j^p \, d\mu = j^{-p} \sum_{k=1}^{j} \int_{E_1} h_k^p \, d\mu < j^{1-p} 2^p M \xrightarrow{j\to\infty} 0$$

which leads to the desired conclusion $\mathrm{cap}_p(E, E_{t_0}) = 0$; note that here $p > 1$ is essential. □

Returning to the main question we conclude from (5) and Lemma 4.5 that if $\widetilde{\mathcal{H}}^p(E) < \infty$, then indeed $\mathrm{cap}_p(E) = 0$. This finishes the proof of Proposition 4.1. □

Since $\mathrm{cap}_p(E, E_t)$ is non-increasing in t, the following result is an immediate consequence of Lemma 4.5; see also the note [13] by Väisälä for a different proof in the Euclidean case.

Corollary 4.6. *Let $1 < p < \infty$. If $E \subset X$ is a compact set with $\mathrm{cap}_p(E) > 0$, then $\mathrm{cap}_p(E, E_t) \to \infty$ as $t \to 0$.*

Heikkala [5, Thm 4.6] showed that, in general, the growth towards ∞ in Corollary 4.6 can be arbitrarily slow. However, in the next section we obtain quantitative lower bounds for the growth of $\mathrm{cap}_p(E, E_t)$ under some additional conditions on the set E (and the space X, as well). Before that, let us end this section with the following lemma, which will not be needed in the sequel, but which illuminates the character of sets of zero capacity in a striking way; see also [7, Lemma 2.9]. Note however that—contrary to the beginning of this section, with the exception of Proposition 4.3—we need here the help of a Poincaré inequality, and that it is also convenient to assume that there is enough of the space X outside E.

Lemma 4.7. *Let $1 \le p < \infty$, and assume that X supports a $(1,p)$-Poincaré inequality. Let $E \subset X$ be a compact set, and assume that* $\operatorname{diam}(E) < \operatorname{diam}(X)/3$. *Then* $\operatorname{cap}_p(E) = 0$ *if and only if there exists some* $0 < t_0 < \operatorname{diam}(E)$ *such that* $\operatorname{cap}_p(E, E_{t_0}) = 0$.

Proof. The necessity is of course trivial. Hence, assume that $\operatorname{cap}_p(E, E_{t_0}) = 0$ for some $0 < t_0 < \operatorname{diam}(E)$ (whence $\operatorname{cap}_p(E, E_t) = 0$ for all $t_0 < t < \operatorname{diam}(E)$ as well), and let $0 < t < t_0$. Pick a sequence $u_k \in \operatorname{Lip}_0(E_{t_0})$ with the respective upper gradients g_k such that $\int g_k^p \to 0$ as $k \to \infty$. It then follows with the help of the $(1,p)$-Poincaré inequality (and the fact $\operatorname{diam}(X) > 3 \operatorname{diam}(E)$) that the functions u_k satisfy a Sobolev type inequality (see for instance [4, Thm 13.1]). Then, in particular, $\int |u_k| \to 0$ as $k \to \infty$, and hence there exits a subsequence, also denoted u_k, such that $u_k \to 0$ for a.e. $x \in E_{t_0}$.

Now choose $\psi \in \operatorname{Lip}_0(E_t)$ with an upper gradient g_ψ such that $0 \le \psi \le 1$, $\psi = 1$ in E, and $g_\psi \le L < \infty$ a.e. in E_t, and define $\varphi_k = \min\{\psi, u_k\}$; then also $\varphi_k \in \operatorname{Lip}_0(E_t)$. If we fix $\varepsilon > 0$, it is easy to find $\rho > 0$ such that $\mu(\{0 < \psi < \rho\}) < \varepsilon$. Furthermore, if we denote $A_k = \{|u_k| > \rho\}$, there exits (by Egorov's theorem) $k_0 \in \mathbb{N}$ so that $\mu(A_k) < \varepsilon$ for all $k \ge k_0$. For such k, the function φ_k has an upper gradient g_{φ_k} which is zero outside E_t and coincides with g_k at least a.e. in the set $E_t \setminus (A_k \cup \{0 < \psi < \rho\})$ (where $u_k \le \rho \le \psi$). It follows that, for k large enough,

$$\int_{E_t} g_{\varphi_k}^p \, d\mu \le \int_{E_t} g_k^p \, d\mu + \int_{\{0<\psi<\rho\} \cup A_k} g_\psi^p \, d\mu \le \int_{E_{t_0}} g_k^p \, d\mu + 2\varepsilon L^p,$$

and so, letting first $k \to \infty$ and then $\varepsilon \to 0$, we obtain $\operatorname{cap}_p(E, E_t) = 0$. Thus $\operatorname{cap}_p(E, E_t) = 0$ for all $0 < t < \operatorname{diam}(E)$, and so $\operatorname{cap}_p(E) = 0$. □

5. Lower bounds

The lower bounds for neighbourhood capacities are a bit more involved than the upper bounds, and here we actually need to have more information on the geometry of both E and the space X near E. More precisely, we need to assume that E is porous, and that a $(1,p)$-Poincaré inequality holds in X, at least for all balls of radii $0 < r < \operatorname{diam}(E)$ and centered at E. Our main estimate is formulated in the following lemma:

Lemma 5.1. *Let $1 \le p < \infty$ and let $E \subset X$ be a compact α-porous set. Assume further that the $(1,p)$-Poincaré inequality* (3) *is valid for all balls $B(w,r)$ with $w \in E$ and* $0 < r < \operatorname{diam}(E)$, *and let $q < p$. Then there exists a constant $C = C(X, E, p, q) > 0$ such that the estimate*

$$\operatorname{cap}_p(E, E_t) \ge C \widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}t}(E) t^{q-p}$$

holds for all $0 < t < \operatorname{diam}(E)$.

Proof. Take $q < p$ and let $0 < t < \operatorname{diam}(E)$. Fix $w \in E$ for a moment, define $r_k = 2^{1-k}\alpha^{-1}t$ for $k \in \mathbb{N}$, and denote $B_k = B(w, r_k)$. Also, let u be a test function for the capacity $\operatorname{cap}_p(E, E_t)$. Since E was assumed to be α-porous, there exists $B(y, 2t) \subset B_0 \cap (X \setminus E)$, and so $u = 0$ in $\tilde{B} = B(y, t)$. Hence we obtain, using the doubling estimate (2), that

$$|u_{B_0}| = \mu(B_0)^{-1} \int_{B_0\setminus\tilde{B}} u \le 1 - \frac{\mu(\tilde{B})}{\mu(B_0)}$$
$$\le 1 - \left(\frac{t}{2\alpha^{-1}t}\right)^s = 1 - (\alpha/2)^s,$$

and as $u \ge 0$ and $u(w) = 1$, we have

$$|u(w) - u_{B_0}| = (\alpha/2)^s > 0.$$

From this lower bound we infer, with a standard 'telescoping' argument for the balls B_k using the $(1,p)$-Poincaré inequality (cf. e.g. [6]), that

$$1 \le C \sum_{k=0}^{\infty} r_k \left(\fint_{\tau B_k} g_u^p \, d\mu \right)^{1/p}. \tag{6}$$

Here we use a trick inspired by the proof of [8, Thm. 5.9]. Namely, it follows from (6) that if $\delta > 0$, then there must exist a constant $C_1 > 0$, independent of u and w, and at least one index $k_w \in \mathbb{N}$ such that

$$r_{k_w} \left(\fint_{\tau B_{k_w}} g_u^p \, d\mu \right)^{1/p} \ge C_1 2^{-k_w \delta} = C_1 t^{-\delta} {r_{k_w}}^{\delta}. \tag{7}$$

We now choose $\delta = (p-q)/p > 0$ and write $B_w = B(x_w, r_w)$ instead of $B_{k_w} = B\big(x_{k_w}, r_{k_w}\big)$ for the corresponding ball in (7).

By raising both sides of (7) to power p we thus obtain for each $w \in E$ a ball B_w such that

$$r_w^{-q} \mu(\tau B_w) \le C t^{p-q} \int_{\tau B_w} g_u(y)^p \, d\mu. \tag{8}$$

Using (again) the basic '$5r$'-covering theorem from [6], we obtain points $w_i \in E$, $i = 1, 2, \dots$, such that the balls $\tau B_i = B(w_i, \tau r_{w_i})$ are pairwise disjoint, but still $E \subset \bigcup_{i=1}^{\infty} 5\tau B_i$. Note that the radii of these covering balls are no more than $10\tau\alpha^{-1}t$. From estimate (8), the doubling property, the pairwise disjointness of the balls τB_i, and the fact that $g_u = 0$ outside E_t we infer

$$\widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}t}(E) \le \sum_{i=1}^{\infty} \mu(5\tau B_i)(5\tau r_{w_i})^{-q} \le C \sum_{i=1}^{\infty} \mu(\tau B_i) r_{w_i}^{-q}$$
$$\le C \sum_{i=1}^{\infty} t^{p-q} \int_{\tau B_i} g_u(y)^p \, d\mu$$
$$\le C t^{p-q} \int_{E_t} g_u(y)^p \, d\mu.$$

This proves the desired estimate. □

Remark 5.2. Using Lemma 2.2 we see immediately that if E is as in Lemma 5.1 and $\lambda > s - p$, then

$$\operatorname{cap}_p(E, E_t) \ge C \mathcal{H}^{\lambda}_{10\tau\alpha^{-1}t}(E) t^{s-p-\lambda}$$

for all $0 < t < \operatorname{diam}(E)$.

In particular, if $E \subset X$ is porous and $\mathrm{cap}_p(E) = 0$, then $\widetilde{\mathcal{H}}^q(E) = \mathcal{H}^{s-q}(E) = 0$ for all $q < p$, and thus $\dim_{\mathcal{H}}(E) \le s - p$. Actually, with a slight modification to the proof of Lemma 5.1, we obtain a proof for this same fact without the assumption that E is porous; this was stated above in Proposition 4.3. Let us outline here the main ideas:

Proof of Proposition 4.3. Let $q < p$. We assume here that $E \subsetneq X$. Take $y \in X \setminus E$ and denote $T = \mathrm{dist}(y, E)/2 > 0$. As $\mathrm{cap}_p(E) = 0$, we find $u_k \in \mathrm{Lip}_0(E_T)$ with upper gradients g_k such that $u_k|_E = 1$ and $\int g_k^p \to 0$ as $k \to \infty$. Now fix $w_0 \in E$ and write $B_E = B(w_0, \mathrm{diam}(E) + 3T)$. With the $(1, p)$-Poincaré inequality it is then not hard to see that, for each $w \in E$,

$$\begin{aligned}|(u_k)_{B(w,T)}| &\le |(u_k)_{B(w,T)} - (u_k)_{B_E}| + |(u_k)_{B_E} - (u_k)_{B(y,T)}| \\ &\le C(X, E, T, p) \int_{B_E} g_k^p \, d\mu,\end{aligned}$$

whence (by passing to a subsequence, if necessary) we may assume that $|(u_k)_{B(w,T)}| < 1/2$, and so $|u_k(w) - (u_k)_{B(w,T)}| \ge 1/2$ for each $w \in E$. It is now possible to continue just as in the proof of Lemma 5.1, starting from estimate (6), with the balls B_k now being $2^{1-k}B(w, T)$. At the end we obtain the conclusion that

$$\widetilde{\mathcal{H}}^q_{5\tau T}(E) \le CT^{p-q} \int_{E_T} g_k^p \, d\mu \overset{k\to\infty}{\longrightarrow} 0,$$

and thus also $\widetilde{\mathcal{H}}^q(E) = 0$ by Lemma 2.5. □

Remark 5.3. Lemma 5.1 does not (in general) hold without the assumption that E is porous. Indeed, for the trivial example of a ball $B \subset \mathbb{R}^n$ we have

$$\mathrm{cap}_p(B, B_t) \approx t^{n-p-(n-1)} = t^{1-p} \ll t^{-p} \approx \mathcal{H}^n_{5t}(B) t^{n-p-n}$$

as $t \to 0$, and, more generally, for a snowflake-type domain $S_\lambda \subset \mathbb{R}^n$ (whose boundary is a λ-regular set) with $\dim_{\mathcal{H}}(\partial S_\lambda) = \lambda \in (n-1, n)$, that

$$\mathrm{cap}_p(S, (S_\lambda)_t) \approx t^{n-p-\lambda} \ll t^{-p} \approx \mathcal{H}^n_{5t}(S_\lambda) t^{n-p-n}$$

as $t \to 0$. In these cases the compact sets B and S_λ are n-regular, and so they have, in particular, positive Lebesgue measure. Nevertheless, we show in Section 6 with a more sophisticated example that this fact plays no essential role in the failure of Lemma 5.1.

The following result is an immediate consequence of Lemma 5.1 and our results for sets of zero capacity:

Corollary 5.4. *Let $1 < p < \infty$ and assume that X is a doubling metric space supporting the $(1, p)$-Poincaré inequality. Let $E \subset X$ be a compact α-porous set with $0 < \widetilde{\mathcal{H}}^q(E) < \infty$. Then, for all $0 < t < \mathrm{diam}(E)$, we have*

$$\mathrm{cap}_p(E, E_t) \ge Ct^{q-p} \quad \text{if } p > q,$$

and

$$\mathrm{cap}_p(E) = 0 \quad \text{if } p \le q.$$

Proof. The latter claim follows from Proposition 4.1 (and Hölder's inequality for $p < q$). For $p > q$ the estimate follows from Lemma 5.1, since $\widetilde{\mathcal{H}}^q(E) > 0$ implies by Lemma 2.5 that

$$0 < \widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}\operatorname{diam}(E)}(E) \leq \widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}t}(E)$$

for all $0 < t < \operatorname{diam}(E)$. □

Combining Corollary 5.4 and Proposition 3.3(b), we obtain, for sufficiently nice subsets of an s-regular space, the following two-sided estimate for neighbourhood capacities:

Theorem 5.5. *Let $1 < p < \infty$ and assume that X is an s-regular metric space supporting the $(1,p)$-Poincaré inequality. Let $E \subset X$ be a porous set with*

$$0 < \mathcal{H}^\lambda(E) \leq \limsup_{t\to 0} \mathcal{M}^\lambda_t(E) < \infty.$$

Then, for all $0 < t < \operatorname{diam}(E)$, we have

$$\operatorname{cap}_p(E, E_t) \approx t^{s-p-\lambda} \quad \text{if } p > s - \lambda,$$

and

$$\operatorname{cap}_p(E) = 0 \quad \text{if } p \leq s - \lambda.$$

Notice that Theorem 1.1 is now a special case of Theorem 5.5, since for $0 \leq \lambda < n$ a λ-regular subset of $\mathbb{R}^n$ is necessarily porous (see Section 2.4 and also [11] for more details). As can be seen from Lemmas 3.2 and 5.1, it is possible to weaken the assumptions of Theorem 5.5 by only assuming the (upper) regularity condition and the $(1,p)$-Poincaré inequality for all balls centered at E and with radii no more than $\operatorname{diam}(E)$.

6. A more intriguing example

In this section we construct, for a given $\lambda \in (1,2)$, a Cantor-type set $E \subset \mathbb{R}^2$ so that $\mathcal{H}^\lambda_\infty(E) > 0$ and $\dim_{\mathcal{H}}(E) < 2$ (we can even take $\dim_{\mathcal{H}}(E) = \lambda$), but for which there exists a sequence $t_k \to 0$ such that, for all $1 \leq p < \infty$,

$$\operatorname{cap}_p\left(E, E_{t_k}\right) \ll t_k^{2-p-\lambda} \quad \text{as } k \to \infty.$$

We then know, by Lemma 5.1, that E can not be porous, and in particular this shows that our results from Section 5 do not (necessarily) hold for non-porous sets, and also that such non-porous counterexamples need not have positive measure (compare to Remark 5.3). We also remark that higher dimensional examples can be easily constructed along the same lines.

6.1. **Construction.** The idea is to use a typical 'alternating' Cantor-type construction, where we have (a) 'thick' generations of squares to guarantee the loss of porosity (or equivalently giving Assouad dimension 2 for the resulting set E), and (b) 'thin' generations which keep the Hausdorff dimension of E in control, in particular bounded away from 2.

Let us first fix $\lambda \in (1,2)$ and $0 < \delta < \lambda - 1$. Also pick

$$0 < d < \min\left\{1 - 2 \cdot 4^{-1/\lambda}, 1/3\right\}$$

and take $\lambda < \tilde{\lambda} < 2$. Our construction consists of the first step (I), and the general step (II), both of which are divided in two parts (a) and (b) in accordance with the above description of their purposes.

(Ia) We begin by removing from the unit square $I_0 = [0,1]^2$ the strips $((1-d)/2, (1+d)/2) \times [0,1]$ and $[0,1] \times ((1-d)/2, (1+d)/2)$ of width d, and so we obtain four new squares $I_1, \ldots, I_4$ of side-length $l_1 = (1-d)/2 > 4^{-1/\lambda}$. Set $n_1 = 0$ and $m_1 = 1$.

(Ib) After this we run the usual λ-dimensional self-similar Cantor construction on each I_j, $j \in \{1,2,3,4\}$ as follows: we first remove from each square I_j strips of equal width, symmetrically with respect to the square I_j, in such a way that we are left with 16 squares $I_{j_1 j_2}$, $j_1, j_2 \in \{1,2,3,4\}$ of side length $l_2 = 4^{-1/\lambda} l_1$. We then continue the same process with the squares $I_{j_1 j_2}$ and remove from these symmetric strips so that we are left with 4^3 squares of side length $l_3 = 4^{-2/\lambda} l_1$. We repeat this process until we have $4^{1+\tilde{m}_1}$ squares of side length $l_{1+\tilde{m}_1} = 4^{-m_1/\lambda} l_1$, where $\tilde{m}_1$ is so large that $4^{1+\tilde{m}_1} l_{1+\tilde{m}_1}^{\tilde{\lambda}} < 1$; it is obvious that such $\tilde{m}_1$ exists, since $\lambda < \tilde{\lambda}$ (see also the end of step (IIb)). Finally, define $n_2 = n_1 + m_1 + \tilde{m}_1$ $(= 1 + \tilde{m}_1)$.

Steps (Ia) and (Ib) thus provide the basic step for our recursive construction.

(IIa) We begin the general kth step of the construction by assuming that we have 4^{n_k} squares $I_{j_1 j_2 \ldots j_{n_k}}$, $j_i \in \{1,2,3,4\}$, of side length l_{n_k}. Now choose $0 < t_k < l_{n_k}$ to be so small that $t_k^{1-\lambda+\delta} \geq l_{n_k} 4^{n_k}$ (recall that $\lambda - \delta > 1$). Remove from the squares $I_{j_1 j_2 \ldots j_{n_k}}$ symmetric strips of width dt_k, so that we obtain 4^{n_k+1} squares of side length l_{n_k+1}. From these new squares we remove strips of width $d^2 t_k$, and then from the next generation of squares again strips of width $d^3 t_k$, and so forth, until we have $4^{n_k+m_k}$ squares of side length $l_{n_k+m_k}$, where m_k is chosen to be so large that $2^{m_k} \geq 2 l_{n_k}/t_k$. Notice here that as $d < 1/3$, it is possible to keep on removing the strips as described above (in particular $\sum_i 2^{i-1} d^i < 1$).

(IIb) After these m_k steps we switch again to the λ-dimensional self-similar Cantor construction starting from the $(n_k + m_k)$th generation of squares. That is, we first remove strips so that we obtain new squares of side-length $l_{n_k+m_k+1} = 4^{-1/\lambda} l_{n_k+m_k}$, then remove strips from these smaller squares to obtain squares of side-length $l_{n_k+m_k+2} = 4^{-2/\lambda} l_{n_k+m_k}$, and continue until we have $4^{n_k+m_k+\tilde{m}_k}$ squares of side-length

$$l_{n_k+m_k+\tilde{m}_k} = 4^{-m_k/\lambda} l_{n_k+m_k},$$

where $\tilde{m}_k$ is so large that $4^{n_k+m_k+\tilde{m}_k} (l_{n_k+m_k+\tilde{m}_k})^{\tilde{\lambda}} < 1$. This choice is possible, because from $\lambda < \tilde{\lambda}$ we conclude that

$$4^{n_k+m_k+m} (l_{n_k+m_k+m})^{\tilde{\lambda}} = 4^{n_k+m_k} (l_{n_k+m_k})^{\tilde{\lambda}} 4^{m(1-\tilde{\lambda}/\lambda)} \stackrel{m\to\infty}{\longrightarrow} 0.$$

This finishes the kth step of the construction, and we write

$$n_{k+1} = n_k + m_k + \tilde{m}_k.$$

6.2. **Justification.** Let us now show that if we define E to be the intersection of all the generations of the squares in the above construction, i.e,

$$E = \bigcap_{l=1}^{\infty} \bigcup_{j_i \in \{1,2,3,4\}} I_{j_1 j_2 \ldots j_l},$$

then the compact set E has the desired properties.

First of all, we claim that

$$\operatorname{cap}_p\left(E, E_{t_k}\right) \ll t_k^{2-p-\lambda} \quad \text{as } k \to \infty. \tag{9}$$

For this purpose, we write

$$E^{n_k} = \bigcup_{j_i \in \{1,2,3,4\}} I_{j_1 j_2 \ldots j_{n_k}},$$

so that E^{n_k} is the union of all the squares from the n_kt level of the construction, and define $u_k \in \operatorname{Lip}(\mathbb{R}^2)$ by $u_k(x) = \max\{0, 1 - 2t_k^{-1} \operatorname{dist}(x, E^{n_k})\}$. It is immediate that $u_k = 1$ in E. Moreover, since $|\nabla u_k|$ is at most $2/t_k$ in the union of the '$t_k/2$-annuli' around the squares of the n_kth generation, and vanishes elsewhere, we calculate that

$$\int |\nabla u_k|^p \, dx \le C 4^{n_k} l_{n_k} (t_k/2)^{1-p} \le C t_k^{1-\lambda+\delta} t_k^{1-p} \ll t_k^{2-p-\lambda},$$

where the second inequality follows from the choice of t_k in the construction, and the last estimate is valid since $\delta > 0$; here ∇u_k is of course the distributional gradient of u_k. Hence it suffices to show that u_k is indeed an admissible test-function for $\operatorname{cap}_p(E, E_{t_k})$, that is, $u_k \in \operatorname{Lip}_0(E_{t_k})$.

To this end, fix a square $I' = I_{j_1 j_2 \ldots j_{n_k}}$ from the n_kth generation. It is clearly sufficient to show that $I' \subset E_{t_k}$. Now, in the part (IIa) of the construction we divide I' into 4^{m_k} smaller squares by removing strips of width $dt_k, d^2 t_k, \ldots, d^{m_k} t_k$. As $d < 1/3$, it is clear that if $x \in I'$ lies in one of these removed strips, we can actually find $w \in E$ so that $d(x, w) < t_k$; in particular $x \in E_{t_k}$. On the other hand, if $x \in I'$ is not in one of the strips, there exists a square I'' from the $(n_k + m_k)$th generation such that $x \in I''$. Since by the choice of m_k we have $2^{m_k} \ge 2 l_{n_k}/t_k$, and obviously $l_{n_k+m_k} \le l_{n_k}/2^{m_k}$, we infer $l_{n_k+m_k} \le t_k/2$, and thus $I'' \subset E_{t_k}$. We have thereby shown that indeed $I' \subset E_{t_k}$, and so we conclude that estimate (9) is valid.

It is also evident that $\mathcal{H}^\lambda_\infty(E) > 0$. Indeed, if instead of the above 'biphasic' construction we only run the usual λ-dimensional self-similar Cantor construction, starting from the unit square, we obtain a set E' with $0 < \mathcal{H}^\lambda(E') < \infty$. Now, by our convenient choice $d < 1 - 2 \cdot 4^{-1/\lambda}$, all the removed strips in our construction are surely narrower than the corresponding strips in that 'unperturbed' λ-dimensional Cantor construction. Hence it is obvious that $\mathcal{H}^\lambda(E) \ge \mathcal{H}^\lambda(E') > 0$, and thus also $\mathcal{H}^\lambda_\infty(E) > 0$.

We are left to justify that $\dim_{\mathcal{H}}(E) \le \tilde{\lambda} < 2$. Let $r > 0$ and pick $k \in \mathbb{N}$ so that $l_{n_k} < r$. Then the set E is covered by the squares of the n_kth generation, which by construction satisfy $4^{n_k} (l_{n_k})^{\tilde{\lambda}} < 1$. It follows that $\mathcal{H}^{\tilde{\lambda}}_r(E) \le C$, where C is independent of r. We conclude that $\mathcal{H}^{\tilde{\lambda}}(E) \le C$, whence $\dim_{\mathcal{H}}(E) \le \tilde{\lambda}$.

Remark 6.1. Actually, it is possible to obtain even $\dim_{\mathcal{H}}(E) = \lambda$ with a slight modification to the above construction. Namely, pick a decreasing sequence $2 > \lambda_k \to \lambda$, and use in the part (IIb) of the kth step the number λ_k instead of $\tilde{\lambda}$. It is then straight-forward to verify (just as above) that $\dim_{\mathcal{H}}(E) \leq \lambda_k$ for each k.

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