Neighborhood capacities

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1. Introduction

Setting

Let (X, d, μ) be a metric measure space. When $E \subset X$ and $t > 0$, we denote

$$E_t = \{x \in X : \mathrm{dist}(x, E) < t\}$$

and call E_t the *(open) t-neighborhood* of E. Our main purpose is to study the p-capacities $\mathrm{cap}_p(E, E_t)$ of a compact set E as t varies (especially when $t \to 0$).

Here

$$\mathrm{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega |Du|^p \, d\mu : 0 \le u \in \mathrm{Lip}_0(\Omega), \ u = 1 \text{ in } E \right\}$$

for $\Omega \subset X$ open and $E \subset \Omega$ compact, $\mathrm{Lip}_0(\Omega)$ is the set of compactly supported Lipschitz-functions in Ω, and the function Du is a 'suitable gradient' of $u \in \mathrm{Lip}_0(\Omega)$ (in $\mathbb{R}^n$ we have $Du = \nabla u$).

Sets of non-zero capacity

A compact set $E \subset X$ is of zero capacity, $\mathrm{cap}_p(E) = 0$, if $\mathrm{cap}_p(E, \Omega) = 0$ for all open sets $\Omega \supset E$; otherwise we write $\mathrm{cap}_p(E) > 0$.

It is immediate that $\mathrm{cap}_p(E) = 0$ if and only if $\mathrm{cap}_p(E, E_t) = 0$ for all $0 < t < \mathrm{diam}(E)$.

For sets of non-zero capacity (in $\mathbb{R}^n$) we have the following result by Väisälä:

Theorem (Väisälä (MMJ, 1975))

Let $1 < p < \infty$. If $E \subset \mathbb{R}^n$ is a compact set with $\mathrm{cap}_p(E) > 0$, then $\mathrm{cap}_p(E, E_t) \to \infty$ as $t \to 0$.

(The same actually holds in a 'reasonable' metric space setting as well)

More history

We also have the following results for the n-capacity in $\mathbb{R}^n$:

Theorem (Vuorinen (CGAQRM, 1985))

If $E \subset \mathbb{R}^n$ is a compact set, then $\operatorname{cap}_n(E, E_t) \leq Ct^{-n}$ for every $0 < t < \operatorname{diam}(E)$. Moreover, t^{-n} is the best asymptotics that one can have for a general set.

Theorem (Heikkala (AASCFD, 2002))

If $E \subset \mathbb{R}^n$ is a compact λ-Ahlfors regular set for $0 \leq \lambda < n$, then $\operatorname{cap}_n(E, E_t) \approx t^{-\lambda}$ for every $0 < t < \operatorname{diam}(E)$.

These results were obtained using modulus estimates. Porosity of E was assumed in the lower bound, but this follows from the λ-regularity for $\lambda < n$.

A result for p-capacity

It is now natural to ask how $\operatorname{cap}_p(E, E_t)$ behaves, when (e.g.) E is λ-regular.

Theorem (L. 2010)

Let $1 < p < \infty$ and $0 \leq \lambda < n$, and assume that $E \subset \mathbb{R}^n$ is an Ahlfors λ-regular compact set. If $p > n - \lambda$, then

$$\operatorname{cap}_p(E, E_t) \approx t^{n-\lambda-p}$$

for all $0 < t < \operatorname{diam}(E)$, and if $p \leq n - \lambda$, then $\operatorname{cap}_p(E, E_t) = 0$ for all $t > 0$.

This result follows from more general upper and lower bounds that we establish separately with weaker assumptions; these results hold in general metric spaces as well.

2. Preliminaries

Metric spaces

We assume that $X = (X, d, \mu)$ is a metric mesure space satisfying the following (standard) assumptions:

- measure μ is *doubling*: $\mu(2B) \le C_d \mu(B)$ for each ball $B \subset X$
- X supports a *(weak) p-Poincaré inequality*:

$$\fint_B |u - u_B| \, d\mu \le Cr \Big(\fint_{\tau B} g_u^p \, d\mu \Big)^{1/p}$$

  whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u: For all (or p-almost all) curves γ joining $x, y \in X$

$$|u(x) - u(y)| \le \int_\gamma g_u \, ds. \tag{1}$$

For instance, if $u \in \mathrm{Lip}(\Omega)$, then (1) holds with

$$g_u(x) = \mathrm{Lip}(u; x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}.$$

Metric spaces II

If μ is doubling, then there exists a number $0 \leq s < \infty$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\left(\frac{r}{R}\right)^s \tag{2}$$

whenever $0 < r \leq R < \operatorname{diam} X$ and $y \in B(x,R)$; this holds certainly for $s = \log_2 C_d$.

In the following we fix some s for which (2) holds, and call this the *doubling dimension* of X.

Measure μ is called *(Ahlfors) s-regular*, if $\mu(B(x,r)) \approx r^s$ for every $x \in X$ and $0 < r < \operatorname{diam}(X)$.

More generally, a set $E \subset X$ is said to be *(Ahlfors) λ-reqular* if

$$\mathcal{H}^\lambda(E \cap B(x,r)) \approx r^\lambda$$

whenever $x \in E$ and $0 < r < \operatorname{diam}(E)$.
(Here $\mathcal{H}^\lambda$ is the usual Hausdorff measure $\rightarrow$)

Hausdorff and Minkowski

We define the *Hausdorff and Minkowski contents* of dimension λ as

$$\mathcal{H}^\lambda_r(E) = \inf\left\{ \sum_k r_k^\lambda : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \le r \right\},$$

and

$$\mathcal{M}^\lambda_r(E) = \inf\left\{ N r^\lambda : E \subset \bigcup_{k=1}^N B(x_k, r),\ x_i \in E \right\},$$

respectively.

It is immediate that $\mathcal{H}^\lambda_r(E) \le \mathcal{M}^\lambda_r(E)$ for each compact $E \subset X$.

The λ-*Hausdorff measure* of E is $\mathcal{H}^\lambda(E) = \lim_{r\to 0} \mathcal{H}^\lambda_r(E)$.

Dimensions

The *Hausdorff dimension* of $E \subset X$ is

$$\dim_{\mathcal{H}}(A) = \inf\{\lambda > 0 : \mathcal{H}^{\lambda}(A) = 0\}$$

The *lower and upper Minkowski dimension* of $E \subset X$ are defined to be

$$\underline{\dim}_{\mathcal{M}}(E) = \inf\left\{\lambda > 0 : \liminf_{r \to 0} \mathcal{M}_r^{\lambda}(E) = 0\right\}$$

and

$$\overline{\dim}_{\mathcal{M}}(E) = \inf\left\{\lambda > 0 : \limsup_{r \to 0} \mathcal{M}_r^{\lambda}(E) = 0\right\},$$

respectively.

Notice that for each compact set $E \subset X$ we have

$$\dim_{\mathcal{H}}(E) \leq \underline{\dim}_{\mathcal{M}}(E) \leq \overline{\dim}_{\mathcal{M}}(E),$$

where all inequalities can be strict. If $\underline{\dim}_{\mathcal{M}}(E) = \overline{\dim}_{\mathcal{M}}(E)$, we simply write $\dim_{\mathcal{M}}(E) = \overline{\dim}_{\mathcal{M}}(E)$.

Hausdorff and Minkowski revisited

In general metric spaces it is often convenient to use modified versions of $\mathcal{H}_r^\lambda$ and $\mathcal{M}_r^\lambda$, namely the following *Hausdorff and Minkowski contents of codimension q*:

$$\widetilde{\mathcal{H}}_r^q(E) = \inf \left\{ \sum_k \mu(B_k)\, r_k^{-q} : E \subset \bigcup_k B_k,\ x_k \in E,\ 0 < r_k \le r \right\},$$

where we write $B_k = B(x_k, r_k)$, and

$$\widetilde{\mathcal{M}}_r^q(E) = \inf \left\{ r^{-q} \sum_k \mu(B(x_k, r)) : E \subset \bigcup_k B(x_k, r),\ x_k \in E \right\}.$$

Note that in a s-regular space $\widetilde{\mathcal{H}}_r^q \approx \mathcal{H}_r^{s-q}$ and $\widetilde{\mathcal{M}}_r^q \approx \mathcal{M}_r^{s-q}$.

Porous sets

A set $E \subset X$ is α-*porous* (for $0 < \alpha < 1$), if for every $w \in E$ and all $0 < r < \text{diam}(E)$ there exists a point $y \in X$ such that

$$B(y, \alpha r) \subset B(w, r) \cap (X \setminus E).$$

It is well-known that in an s-regular metric space X a set $E \subset X$ is porous if and only if the *Assouad dimension* $\dim_{\mathcal{A}}(E) < s$.

Recall that for a λ-regular set $E \subset X$ we have $\dim_{\mathcal{H}}(E) = \dim_{\mathcal{A}}(E) = \lambda$.

3. Upper bounds

A general upper bound

Lemma

Let $1 \le p < \infty$*. Then there exists a constant* $C = C(X, p) > 0$ *such that*

$$\operatorname{cap}_p(E, E_t) \le C\widetilde{\mathcal{M}}^q_{t/3}(E)t^{q-p}$$

whenever $E \subset X$ *is a compact set and* $0 < r < \operatorname{diam}(E)$*.*

Idea: Cover E with balls $B_i = B(w_i, t/3)$, $w_i \in E$, $i = 1, \ldots, N$, and define

$$u(x) = \max_{1 \le i \le N} \{0, 1 - 2t^{-1} \operatorname{dist}(x, B_i)\}.$$

Then $u \in \operatorname{Lip}_0(E_t)$, $u = 1$ in E, and u has an upper gradient g_u such that

$$g_u(x)^p \le \sum_{i=1}^{N} (t/2)^{-p} \chi_{2B_i}(x) \quad \text{for a.e. } x \in E_t \quad \ldots$$

A general upper bound, cont'd..

$$\ldots\ g_u(x)^p \le \sum_{i=1}^{N} (t/2)^{-p} \chi_{2B_i}(x) \text{ for a.e. } x \in E_t.$$

Thus, by the doubling condition,

$$\mathrm{cap}_p(E, E_t) \le \int_{E_t} g_u^p \, d\mu \le \sum_{i=1}^{N} (t/2)^{-p} \mu(2B_i) \le C(t/3)^{-p} \sum_{i=1}^{N} \mu(B_i).$$

Taking the infimum over all such covers yields the claim

$$\mathrm{cap}_p(E, E_t) \le C\widetilde{\mathcal{M}}^p_{t/3}(E) = C\widetilde{\mathcal{M}}^q_{t/3}(E) t^{q-p}.$$

(the second equality holds for all q)

Notice that no Poincaré inequalities are needed here.

Upper bound under bounded volume growth

As a simple corollary of the previous result we obtain:

Lemma

Let $1 \le p < \infty$ and assume that $\mu(B(w,r)) \le cr^d$ whenever $w \in E$ and $0 < r < \operatorname{diam}(E)$. Then there exists a constant $C = C(X,p,c) > 0$ such that

$$\operatorname{cap}_p(E, E_t) \le C\mathcal{M}^{\lambda}_{t/3}(E)t^{d-\lambda-p}$$

for all $0 < r < \operatorname{diam}(E)$.

Idea: If $\mu(B(w,r)) \le cr^d$ for all $w \in E$ and $r < \operatorname{diam}(E)$, we have $\widetilde{\mathcal{M}}^q_r(E) \le c\mathcal{M}^{d-q}_r(E)$ for all $0 < r < \operatorname{diam}(E)$. Thus the previous lemma with $q = d - \lambda$ gives

$$\operatorname{cap}_p(E, E_t) \le C\widetilde{\mathcal{M}}^q_{t/3}(E)t^{q-p} \le C\mathcal{M}^{\lambda}_{t/3}(E)t^{d-\lambda-p}.$$

In particular...

Proposition

(a) Let $1 \le p < \infty$ *and assume that* $E \subset X$ *is a compact set satisfying* $\limsup_{r\to 0} \widetilde{\mathcal{M}}_r^q(E) < \infty$. *Then there exists a constant* $C = C(X, E, p, q) > 0$ *such that*

$$\operatorname{cap}_p(E, E_t) \le C t^{q-p}$$

for all $0 < t < \operatorname{diam}(E)$.
(b) If $E \subset X$ *is a compact set satisfying* $\limsup_{r\to 0} \mathcal{M}_r^\lambda(E) < \infty$, *and if in addition* $\mu(B(w, r)) \le c r^d$ *whenever* $w \in E$ *and* $0 < r < \operatorname{diam}(E)$, *then there exists a constant* $C = C(X, E, p, d, \lambda, c) > 0$ *such that*

$$\operatorname{cap}_p(E, E_t) \le C t^{d-\lambda-p}$$

for all $0 < t < \operatorname{diam}(E)$. *This is true, in particular, if* $\lambda > \overline{\dim}_{\mathcal{M}}(E)$.

4. Lower bounds

A general lemma

Our lower bounds for neighborhood capacities follow from the next general lemma:

Lemma

Let $1 \leq p < \infty$ *and let* $E \subset X$ *be a compact* α*-porous set. Assume that the* $(1, p)$*-Poincaré inequality is valid for all balls* $B(w, r)$ *with* $w \in E$ *and* $0 < r < \operatorname{diam}(E)$*, and let* $q < p$*. Then there exists a constant* $C = C(X, E, p, q) > 0$ *such that*

$$\operatorname{cap}_p(E, E_t) \geq C \widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}t}(E) t^{q-p}$$

for all $0 < t < \operatorname{diam}(E)$*.*

(Here $\tau \geq 1$ is the dilatation constant from the weak Poincaré inequality.)

Idea of the proof I

Take $q < p$ and let $0 < t < \operatorname{diam}(E)$. Fix $w \in E$ and define $r_k = 2^{1-k}\alpha^{-1}t$, $B_k = B(w, r_k)$. Let $u \geq 0$ be a test function for the capacity $\operatorname{cap}_p(E, E_t)$. From the α-porosity it follows that $u = 0$ in $\tilde{B} = B(y, t)$ for some $y \in B_0$. Hence, using doubling,

$$|u_{B_0}| = \mu(B_0)^{-1} \int_{B_0 \setminus \tilde{B}} u \leq 1 - \frac{\mu(\tilde{B})}{\mu(B_0)}$$
$$\leq 1 - C\left(\frac{t}{2\alpha^{-1}t}\right)^s = 1 - C(\alpha/2)^s.$$

As $u \geq 0$ and $u(w) = 1$, we have $|u(w) - u_{B_0}| = C(\alpha/2)^s > 0$. From this it follows (using 'telescoping') that

$$1 \leq C \sum_{k=0}^{\infty} r_k \left(\fint_{\tau B_k} g_u^p \, d\mu \right)^{1/p}.$$

Idea of the proof II

But if

$$1 \leq C \sum_{k=0}^{\infty} r_k \left(\fint_{\tau B_k} g_u^p \, d\mu \right)^{1/p},$$

then, for $\delta := (p-q)/p > 0$ there exists $C_1 > 0$ and an index $k_w \in \mathbb{N}$ such that

$$r_{k_w} \left(\fint_{\tau B_{k_w}} g_u^p \, d\mu \right)^{1/p} \geq C_1 2^{-k_w \delta} = C t^{-\delta} {r_{k_w}}^{\delta}.$$

Write from now on $B_w = B(x_w, r_w)$ instead of $B_{k_w} = B\big(x_{k_w}, r_{k_w}\big)$.

We thus obtain for each $w \in E$ a ball B_w such that

$$r_w^{-q} \mu(\tau B_w) \leq C t^{p-q} \int_{\tau B_w} g_u(y)^p \, d\mu. \tag{3}$$

Idea of the proof III

With the basic '5r'-covering theorem we now obtain points $w_i \in E$, $i = 1, 2, \dots$, such that the balls $\tau B_i = B(w_i, \tau r_{w_i})$ are pairwise disjoint, but still $E \subset \bigcup_{i=1}^{\infty} 5\tau B_i$. Note that the radii of these covering balls are no more than $10\tau\alpha^{-1}t$. Using the previous estimate with doubling, the pairwise disjointness of the balls τB_i, and the fact that $g_u = 0$ outside E_t, we infer

$$\begin{aligned}
\widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}t}(E) &\leq \sum_{i=1}^{\infty} \mu(5\tau B_i)(5\tau r_{w_i})^{-q} \leq C \sum_{i=1}^{\infty} \mu(\tau B_i) r_{w_i}^{-q} \\
&\leq C \sum_{i=1}^{\infty} t^{p-q} \int_{\tau B_i} g_u(y)^p \, d\mu \\
&\leq C t^{p-q} \int_{E_t} g_u(y)^p \, d\mu,
\end{aligned}$$

proving the Lemma.

Corollary

The following result is an immediate consequence of our main Lemma and general results for sets of zero capacity:

Corollary

Let $1 < p < \infty$ *and assume that* X *is a doubling metric space supporting the* $(1, p)$*-Poincaré inequality. Let* $E \subset X$ *be a compact* α*-porous set with* $0 < \widetilde{\mathcal{H}}^q(E) < \infty$*. Then, for all* $0 < t < \operatorname{diam}(E)$*, we have*

$$\operatorname{cap}_p(E, E_t) \geq C t^{q-p} \quad \text{if } p > q,$$

and

$$\operatorname{cap}_p(E) = 0 \quad \text{if } p \leq q.$$

Here we only need to know that $\widetilde{\mathcal{H}}^q(E) > 0$ implies

$$0 < \widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}\operatorname{diam}(E)}(E) \leq \widetilde{\mathcal{H}}^q_{10\tau\alpha^{-1}t}(E) \text{ for all } 0 < t < \operatorname{diam}(E).$$

Another corollary

Combining the previous result and the upper bounds for neighborhood capacities in a regular space, we obtain the following version of our main result:

Corollary

Let $1 < p < \infty$ and assume that X is a s-regular metric space supporting the $(1, p)$-Poincaré inequality. Let $E \subset X$ be a compact α-porous set with

$$0 < \mathcal{H}^{\lambda}(E) \leq \limsup_{t \to 0} \mathcal{M}_t^{\lambda}(E) < \infty.$$

Then, for all $0 < t < \operatorname{diam}(E)$, we have

$$\operatorname{cap}_p(E, E_t) \approx t^{s-\lambda-p} \quad \text{if } p > s - \lambda,$$

and

$$\operatorname{cap}_p(E) = 0 \quad \text{if } p \leq s - \lambda.$$

'Trivial' counterexamples

Our lower bounds do not hold without the assumption that E is porous. Indeed, for a ball $B \subset \mathbb{R}^n$ we have

$$\operatorname{cap}_p(B, B_t) \approx t^{n-(n-1)-p} = t^{1-p} \ll t^{-p} \approx \mathcal{H}^n_{5t}(B) t^{n-n-p}$$

as $t \to 0$, and, more generally, for a snowflake-type domain $S_\lambda \subset \mathbb{R}^n$ with $\dim_{\mathcal{H}}(\partial S_\lambda) = \lambda \in (n-1, n)$, that

$$\operatorname{cap}_p(S, (S_\lambda)_t) \approx t^{n-\lambda-p} \ll t^{-p} \approx \mathcal{H}^n_{5t}(S_\lambda) t^{n-n-p}$$

as $t \to 0$.

These compact sets B and S_λ are n-regular (and of positive Lebesgue measure).

Nevertheless, this fact plays no essential role in the failure of the lower bounds, as we will see.

5. An example

What can we do?

Given $\lambda \in (1, 2)$, it is possible to construct a Cantor-type set $E \subset \mathbb{R}^2$ so that $\mathcal{H}^\lambda_\infty(E) > 0$ and $\dim_{\mathcal{H}}(E) < 2$ (we can even take $\dim_{\mathcal{H}}(E) = \lambda$), but for all $1 \leq p < \infty$

$$\operatorname{cap}_p\left(E, E_{t_k}\right) \ll t_k^{2-\lambda-p} \quad \text{as } k \to \infty,$$

for a suitable sequence $t_k \to 0$.

Higher dimensional examples can easily be constructed along the same lines.

Notice that such a set E can not be porous.

Idea of the construction

The idea is to use a typical ‘alternating’ Cantor-type construction, where we have

(a) ‘thick’ generations of squares to guarantee the loss of porosity (or equivalently giving Assouad dimension 2 for the resulting set E),

and

(b) ‘thin’ generations which keep the Hausdorff dimension of E in control, in particular bounded away from 2.

Some details I

Fix $0 < \delta < \lambda - 1$ and $\lambda < \widetilde{\lambda} < 2$. After running the alternating construction k times we have 4^{n_k} squares of side length l_{n_k}. Let E^{n_k} denote the union of all these squares.

(a) 'thick': Choose t_k so that $t_k^{1-\lambda+\delta} \geq 4^{n_k} l_{n_k}$, and define $u_k(x) = \max\{0, 1 - 2t_k^{-1} \operatorname{dist}(x, E^{n_k})\}$. Then

$$\int |\nabla u_k|^p \, dx \leq C 4^{n_k} l_{n_k} (t_k/2)^{1-p} \leq C t_k^{1-\lambda+\delta} t_k^{1-p} \ll t_k^{2-\lambda-p}.$$

Now, we only need to guarantee, that u_k is an admissible test function for $\operatorname{cap}_p(E, E_{t_k})$ (where E is the resulting Cantor set).

To this end, we remove from the squares in E^{n_k} 'very narrow' ($< t_k$) strips, and continue in this way untill we have $4^{n_k+m_k}$ squares of side length $l_{n_k+m_k} < t_k$, so that indeed $E^{n_k} \subset E_{t_k}$, and thus $u_k \in \operatorname{Lip}_0(E_{t_k})$.

Some details II

(b) 'thin': After this we take the $4^{n_k+m_k}$ squares and run the usual λ-dimensional Cantor construction on these for $\tilde{m}_k$ steps, where $\tilde{m}_k$ is so large that

$$4^{n_k+m_k+\tilde{m}_k}(l_{n_k+m_k+\tilde{m}_k})^{\tilde{\lambda}} < 1.$$

(This is possible because $\lambda < \tilde{\lambda}$).

From this it ultimately follows, that $\dim_{\mathcal{H}}(E) \leq \tilde{\lambda} < 2$.

We then set $n_{k+1} = n_k + m_k + \tilde{m}_k$, and continue from step (a).

It is now evident, that, for all k, $\mathcal{H}^{\lambda}_{ct_k}(E) > C_1$ for some $C_1 > 0$, since E is 'larger' than the standard λ-dimensional Cantor set. Hence it is **not** true that

$$\operatorname{cap}_p(E, E_t) \geq C\mathcal{H}^{\lambda}_{ct}(E)t^{2-\lambda-p}$$

for all $0 < t < \operatorname{diam}(E)$, showing the failure of our lower bounds.

6. On sets of zero capcity

Hausdorff and capacity

The following result holds with very weak assumptions on the space X; we basically only need doubling.

Proposition

Let $1 < p < \infty$ and assume that a compact set $E \subset X$ satisfies $\widetilde{\mathcal{H}}^p(E) < \infty$. Then $\mathrm{cap}_p(E) = 0$.

It is actually very easy to show that if $\Omega \supset E$ is open, then

$$\mathrm{cap}_p(E, \Omega) \leq C\widetilde{\mathcal{H}}^p_r(E)$$

for all $0 < r < d(E, X \setminus \Omega)/2$, where $C = C_d$ is the doubling constant (this holds for $p = 1$ as well).

The proof is just like our previous proof for the upper bounds of neighborhood capacities.

The 'hard' part

It requires a bit more work to prove the following Lemma:

Lemma

Let $1 < p < \infty$ *and let* $E \subset X$ *be a compact set. Assume that there exists a constant* $0 < M < \infty$ *such that*

$$\operatorname{cap}_p(E, \Omega) < M \qquad \text{for all open } \Omega \supset E.$$

Then $\operatorname{cap}_p(E) = 0$.

Idea: Fix $0 < t_0 < \operatorname{diam}(E)$, and choose $u_1 \in \operatorname{Lip}_0(E_{t_0})$ such that $u_1|_E = 1$ and u_1 has an upper gradient g_1 with $\int g_1^p < M$. Write $V_1 = \{u_1 > 1/2\} \supset E$, let $t_1 = \operatorname{dist}(E, X \setminus V_1)/2 > 0$, and define $v_1 = \min\{2u_1, 1\}$. Then $v_1|_{V_1} = 1$ and $h_1 = 2g_1\chi_{X \setminus V_1}$ is an upper gradient of v_1. Moreover, $\int h_1^p < 2^p M$.

The 'hard' part 2

Then take $u_2 \in \text{Lip}_0(E_{t_1})$ with $u_2|_E = 1$ and $\int g_2^p < M$ for an upper gradient g_2 of u_2; note that $\text{spt}(u_2) \subset V_1$. Write $V_2 = \{u_2 > 1/2\} \supset E$ and let $t_2 = \text{dist}(E, X \setminus V_2)/2 > 0$. Define as above $v_2 = \min\{2u_2, 1\}$, whence $v_2|_{V_2} = 1$ and $h_2 = 2g_2\chi_{X\setminus V_2}$ is an upper gradient of v_2 with $\int h_2^p < 2^p M$.

This way we find numbers $t_0 > t_1 > \cdots > t_k > \cdots > 0$ and functions $v_k \in \text{Lip}_0(E_{t_k})$ with upper gradients h_k satisfying: (i) $v_k|_E = 1$, (ii) the supports $\text{spt}(h_k)$ are pairwise disjoint, and (iii) $\int h_k^p < 2^p M$ for each k.

Define $\varphi_j = j^{-1}\sum_{k=1}^j v_k$. Then clearly $\varphi_j \in \text{Lip}_0(E_{t_0})$ and $\varphi_j|_E = 1$ for each j. Moreover, $\psi_j = j^{-1}\sum_{k=1}^j h_k$ is an upper gradient of φ_j. Using the properties (ii) and (iii) of the functions h_k we easily calculate

$$\text{cap}_p(E, E_{t_0}) \le \int_{E_1} \psi_j^p \, d\mu = j^{-p}\sum_{k=1}^{j}\int_{E_1} h_k^p \, d\mu < j^{1-p}2^p M \xrightarrow{j\to\infty} 0.$$

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