Quasiadditivity of variational capacity

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1. Introduction

What is ... Quasiadditivity?

Let X be a space and let $\mathrm{Cap}(\cdot)$ be a notion of *capacity*, defined for (compact) sets $E \subset X$.

Capacities are countably *subadditive*, i.e. if $E = \bigcup_{i\in\mathbb{N}} E_i$, then

$$\mathrm{Cap}(E) \leq \sum_{i\in\mathbb{N}} \mathrm{Cap}(E_i).$$

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In this talk we are interested in the converse inequality (up to a constant). More precisely, if $U \subset X$ and $\mathcal{W} = \mathcal{W}(U) = \{Q_i\}_{i\in\mathbb{N}}$ is a decomposition or a covering of U, we ask if there exists a constant $1 \le A < \infty$ such that

$$\sum_{i\in\mathbb{N}} \mathrm{Cap}(E \cap Q_i) \le A\,\mathrm{Cap}(E)$$

for every (compact) $E \subset U$. If this is the case, we say that the capacity Cap is *quasiadditive* with respect to $\mathcal{W}$.

Our setting

Let $X = (X, d, \mu)$ be a (sufficienty nice) metric measure space, and let $\Omega \subset X$ be an open set.

We consider the *variational p-capacity*

$$\operatorname{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega |Du|^p \, d\mu : u \in N_0^{1,p}(\Omega), \; u = 1 \text{ in } E \right\}$$

for compact $E \subset \Omega$. Here $N_0^{1,p}(\Omega)$ is the space of (Newtonian) Sobolev functions u with $u = 0$ in $X \setminus \Omega$, and the function $|Du|$ is the 'length of the gradient' of u.
(in $\mathbb{R}^n$ think of $W_0^{1,p}(\Omega)$ or $C_0^\infty(\Omega)$ with $Du = \nabla u$ being the (weak) gradient).

In fact, one may replace $N_0^{1,p}(\Omega)$ by $\operatorname{Lip}_0(\Omega)$, the set of compactly supported Lipschitz-functions in Ω, whence $|Du|$ is the pointwise Lipschitz constant.

Quasiadditivity of variational capacity

The variational p-capacity $\operatorname{cap}_p(\cdot, \Omega)$ is *quasiadditive* with respect to a decomposition or a covering $\mathcal{W}$ of Ω, if there exists a constant $A > 0$ such that for each compact $E \subset \Omega$ we have

$$\sum_{Q \in \mathcal{W}} \operatorname{cap}_p(E \cap Q, \Omega) \leq A \operatorname{cap}_p(E, \Omega).$$

Quasiadditivity of variational capacity

The variational p-capacity $\operatorname{cap}_p(\cdot,\Omega)$ is *quasiadditive* with respect to a decomposition or a covering $\mathcal{W}$ of Ω, if there exists a constant $A>0$ such that for each compact $E\subset\Omega$ we have

$$\sum_{Q\in\mathcal{W}} \operatorname{cap}_p(E\cap Q,\Omega)\le A\operatorname{cap}_p(E,\Omega).$$

First remarks:

- If $\mathcal{W}$ is not infinite, the question is not very interesting :)
- There are certainly 'bad' decompositions (I don't even mention bad coverings)
- Whitney-type decompositions/coverings make sense
- The capacity $\operatorname{cap}_p(\cdot,\Omega)$ is not always quasiadditive w.r.t. a Whitney decomposition of Ω (even when $\Omega\subset\mathbb{R}^n$).

Aikawa & Riesz

Theorem (Aikawa (1991, Math. Scand.))

Let $1 \le p < \infty$ *and suppose* $F \subset \mathbb{R}^n$ *is a closed set satisfying*

$\dim_{\mathcal{A}}(F) < n - \alpha p$.

Then the Riesz capacity $R_{\alpha,p}$ *is quasiadditive with respect to the Whitney decomposition* $\mathcal{W} = \mathcal{W}(\mathbb{R}^n \setminus F)$*, i.e. for every* $E \subset \mathbb{R}^n$

$$\sum_{Q_i \in \mathcal{W}} R_{\alpha,p}(E \cap Q_i) \le A R_{\alpha,p}(E).$$

Note: The dimension $\dim_{\mathcal{A}}$ was defined especially for this result; always $\dim_{\mathcal{H}}(E) \le \dim_{\mathcal{A}}(E)$.

Earlier considerations on quasiadditivity, in the case $F = \{0\}$ and ‘annular decompositions’, were made by Landkof (1972) and Adams (1978).

Riesz capacity vs. variational capacity

Recall that the Riesz capacity $R_{\alpha,p}(E)$ is defined, for $1 < p < \infty$ and $0 < \alpha < n$, as follows:

$$R_{\alpha,p}(E) = \inf \left\{ ||f||_p^p : \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \geq 1 \text{ for all } x \in E,\ f \geq 0 \right\}.$$

For $\alpha = 1$,

$$R_{1,p}(E)\text{ `} \sim \text{' } \mathrm{cap}_p(E, \mathbb{R}^n \setminus F),$$

(always $R_{1,p}(E) \leq \mathrm{cap}_p(E, \Omega)$) so quasiadditivity for the variational p-capacity under the condition $\dim_{\mathcal{A}}(F) < n - p$ is at least plausible.

Questions:

- Quasiadditivity for the variational capacity, $1 < p < \infty$?
- Quasiadditivity in metric measure spaces?
- The relation between quasiadditivity and the Hardy inequality:

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx,$$

  where $\Omega \subset \mathbb{R}^n$ is open, $d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$, and the constant $C > 0$ is independent of $u \in C_0^\infty(\Omega)$.
- Is Aikawa's condition $\dim_{\mathcal{A}}(\Omega^c) < n - p$ sufficient for the quasiadditivity of variational capacity in Ω? Is it necessary?
- What is Aikawa's dimension $\dim_{\mathcal{A}}$?

2. Preliminaries

Metric spaces: doubling

We assume that $X = (X, d, \mu)$ is a metric mesure space satisfying the following (standard) assumptions:

(1) Measure μ is *doubling*: There exists $c_d > 0$ such that $\mu(2B) \leq C_d \mu(B)$ for each ball $B \subset X$.

Measure μ is called *(Ahlfors) Q-regular*, if there is $C > 0$ such that

$$\frac{1}{C} r^Q \leq \mu(B(x, r)) \leq C r^Q$$

for every $x \in X$ and all $0 < r < \operatorname{diam}(X)$. A regular measure is certainly doubling.

Metric spaces: Poincaré

(2) X supports a *(weak) p-Poincaré inequality*:

$$\fint_B |u - u_B| \, d\mu \le Cr \Big(\fint_{\lambda B} g_u^p \, d\mu \Big)^{1/p}$$

whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u:

For all (or p-almost all) curves γ joining $x, y \in X$

$$|u(x) - u(y)| \le \int_\gamma g_u \, ds. \tag{1}$$

We use above the notation

$$u_B := \frac{1}{\mu(B)} \int_B u \, d\mu =: \fint_B u \, d\mu.$$

For instance, if $u \in \mathrm{Lip}(\Omega)$, then (1) holds with

$$g_u(x) = \mathrm{Lip}(u; x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}.$$

Aikawa dimension

Aikawa's notion of dimension is given in terms of integrals of the distance function. In a Q-regular metric space $X = (X, \mu, d)$ the definition is as follows:

Definition (Aikawa)

Let $E \subset X$. The *Aikawa dimension* $\dim_{\mathcal{A}}(E)$ is the infimum of those $t > 0$ for which there exists a constant c_t such that

$$\int_{B(x,r)} d(y,E)^{t-Q}\, d\mu \leq c_t r^t$$

for every $x \in E$ and all $0 < r < \infty$.

We use the convention that if the set $E \subset X$ has positive measure, then $\dim_{\mathcal{A}}(E) = Q$, and thus for each $E \subset X$ we have $0 \leq \dim_{\mathcal{A}}(E) \leq Q$.

Assouad dimension

Definition (Assouad)

Let $E \subset X$. The *Assouad dimension* $\dim_{\mathcal{AS}}(E)$ is the infimum of all $\beta > 0$ for which the following covering property holds: There exists $c_\beta \geq 1$ such that, for all $0 < \varepsilon < 1/2$, each subset $F \subset E$ can be covered by at most $c_\beta \varepsilon^{-\beta}$ balls of radius $r = \varepsilon \operatorname{diam}(F)$.

Recall, for instance, that in a Q-regular metric space X a set $E \subset X$ is porous if and only if $\dim_{\mathcal{A}}(E) < Q$.

There are many equivalent definitions for the Assouad dimension since this same concept has appeared on many occasions under different names. (See the paper by Luukkainen for a historical account).

Recently, one more equivalence was added to the list:

Aikawa and Assouad

Theorem (L.-Tuominen, 2011, JJMS (to appear))

Assume that X is Q-regular. Then $\dim_{\mathcal{A}}(E) = \dim_{\mathcal{AS}}(E)$ for each $E \subset X$.

In particular, we have that $\dim_{\mathcal{H}}(E)\big(\leq \overline{\dim}_{\mathcal{M}}(E)\big) \leq \dim_{\mathcal{A}}(E)$ for all (bounded) sets, where the inequalities can be strict. On the other hand, for regular sets $\dim_{\mathcal{H}} = \dim_{\mathcal{A}}$.

If the measure is only doubling, it is useful to define a related *codimension*: The *Aikawa co-dimension* $\operatorname{codim}_{\mathcal{A}}(E)$ is the supremum of all $q > 0$ for which there exists a constant c_q such that

$$\int_{B(x,r)} d(y,E)^{-q}\, d\mu \leq c_q r^{-q} \mu(B(x,r))$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(E)$.
If μ is Q-regular, then $\operatorname{codim}_{\mathcal{A}}(E) = Q - \dim_{\mathcal{A}}(E)$ for every $E \subset X$.

(Newtonian) Sobolev spaces

The *(Newtonian) Sobolev space* is defined as

$$N^{1,p}(X) = \{u : X \to [-\infty, \infty] \, : \, \|u\|_{N^{1,p}(X)} < \infty\},$$

where

$$\|u\|_{N^{1,p}(X)} := \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p} ;$$

here the infimum is taken over all upper gradients g of u, and we identify functions $u, v \in N^{1,p}$ if $\|u - v\|_{N^{1,p}(X)} = 0$.
If $\Omega \subset \mathbb{R}^n$ is an open set, then $N^{1,p}(\Omega) = W^{1,p}(\Omega)$.

Now the variational p-capacity of (a measurable set) E with respect to Ω is

$$\operatorname{cap}_p(E, \Omega) := \inf_u \inf_{g_u} \int_X g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ with $u = 1$ on E, and over all upper gradients g_u of u. (If there are no such functions u, we say that $\operatorname{cap}_p(E, \Omega) = \infty$.)

Whitney balls

When $\Omega \subset X$ is open and $0 < c < 1$, we fix a Whitney type covering $\mathcal{W}_c(\Omega) = \{B_i\}_{i \in I}$ consisting of closed balls $B_i = B(x_i, c\,\mathrm{dist}(x_i, X \setminus \Omega))$, $x_i \in \Omega$, such that the balls $(1/5)B_i$ are pairwise disjoint (use the $5r$-covering theorem). We often write $r_i = c d_\Omega(x_i) = c\,\mathrm{dist}(x_i, X \setminus \Omega))$.

We need to be able to dilate the Whitney balls without having too much overlap; this is always possible for sufficiently small c:

Lemma

Let $\Omega \subset X$ be an open set. Fix $L \geq 1$ and let $\mathcal{W}_c(\Omega) = \{B_i\}_{i \in I}$ be a Whitney-type covering of Ω with $c \leq (3L)^{-1}$. Then the overlap of the balls LB_i is bounded by a uniform constant.

Capacity of Whitney balls

We need for Whitney balls $B_i \in \mathcal{W}_c(\Omega)$ the following capacity estimate (WBCE)

$$c_1 r_i^{-p} \mu(B_i) \leq \mathrm{cap}_p(B_i, \Omega) \leq c_2 r_i^{-p} \mu(B_i),$$

where c_1, c_2 depend on $\mathcal{W}_c(\Omega)$ but not on B_i.

In the upper bound only doubling is needed, but the lower bound is not always true, and so some information on the 'geometry' of X is required, e.g. that

- $1 < p < Q$, μ is Q-regular, $\mu(X) = \infty$ and X supports a p-Poincaré inequality;

  or
- The p-Hardy inequality is valid for all $u \in \mathrm{Lip}_0(\Omega)$ (or $u \in N_0^{1,p}(\Omega)$).

3. Results: Quasiadditivity, Maz'ya, and Hardy

The Maz'ya connection

From now on, we assume doubling and p-Poincaré.

Lemma (The Main Lemma)

Assume that $\Omega \subset X$ is an open set with a Whitney-type covering $\mathcal{W}_c(\Omega) = \{B_i\}_{i \in I}$ (for a sufficiently small c). Then $\operatorname{cap}_p(\cdot, \Omega)$ is quasiadditive with respect to $\mathcal{W}_c$ if and only if there exists $C > 0$ such that

$$\int_E d_\Omega(x)^{-p} \, d\mu \le C \operatorname{cap}_p(E, \Omega) \tag{2}$$

for each compact set $E \subset \Omega$. (In the 'only if'-part we need WBCE).

The latter part of the Main Lemma (eq. (2)) is a so-called *Maz'ya-type characterization* for the p-Hardy inequality, and thus we obtain the following corollaries:

The Hardy connection

Corollary

Let $\mathcal{W}_c(\Omega)$ be a Whitney covering of an open $\Omega \subset X$ with a suitably small parameter $0 < c < 1$. Then $\operatorname{cap}_p(\cdot, \Omega)$ is quasiadditive with respect to $\mathcal{W}_c(\Omega)$ if and only if Ω admits the p-Hardy inequality. (WBCE in the 'only if'-part)

Here we say that an open set $\Omega \subset X$ admits the p-Hardy inequality if

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, d\mu \le C \int_\Omega g_u(x)^p \, d\mu$$

for all $u \in N_0^{1,p}(\Omega)$. Since this holds if and only if

$$\int_E d_\Omega(x)^{-p} \, d\mu \le C \operatorname{cap}_p(E, \Omega)$$

for each compact set $E \subset \Omega$ (Maz'ya in $\mathbb{R}^n$, Korte–Shanmugalingam in X), we obtain the above corollary from the Main Lemma.

The Hardy connection and uniform fatness

Corollary

Let $\mathcal{W}_c(\Omega)$ be a Whitney covering of an open $\Omega \subset X$ with a suitably small parameter $0 < c < 1$. If $X \setminus \Omega$ is uniformly p-fat, then $\operatorname{cap}_p(\cdot, \Omega)$ is quasiadditive with respect to $\mathcal{W}_c(\Omega)$.

Recall that a closed set $E \subset X$ is uniformly p-fat if for every $x \in E$ and all $r > 0$

$$\operatorname{cap}_p\left(E \cap \overline{B}(x,r), B(x,2r)\right) \geq C \operatorname{cap}_p\left(\overline{B}(x,r), B(x,2r)\right) \quad \left(\approx r^{Q-p} \right),$$

or equivalently, in a Q-regular space,

$$\mathcal{H}^\lambda_\infty(E \cap B(x,r)) \geq C r^\lambda \text{ for some } \lambda > Q - p.$$

Since uniform p-fatness of the complement $X \setminus \Omega$ implies the p-Hardy inequality in Ω (Ancona, Lewis, Wannebo in $\mathbb{R}^n$, Björn–MacManus–Shanmugalingam in X (cf. also Korte–L.–Tuominen)), the above result follows from the previous corollary.

Small boundary parts

The following implies a counterpart for the Aikawa theorem:

Lemma

Assume that μ is Q-regular, $\mu(X) = \infty$, $1 < p < Q$, and that $\dim_{\mathcal{A}}(X \setminus \Omega) < Q - p$. Furthermore, assume that Ω satisfies a 'uniform John-type' condition. Then there is $A > 0$ such that

$$\int_E d_\Omega(x)^{-p}\, d\mu \leq A \operatorname{cap}_p(E, \Omega)$$

for all compact $E \subset \Omega$.

By a 'uniform John-type' condition we mean that each Whitney ball can be joined to an arbitrarily large Whitney ball using a c-John curve γ; that is, $\gamma\colon [0, l(\gamma)] \to X$ satisfies $d(\gamma(t), X \setminus \Omega) \geq ct$ for all $t \in [0, l(\gamma)]$.

Using the Main Lemma, we thus obtain quasiadditivity (and the p-Hardy inequality) from the assumptions of the above Lemma.

4. Tools from nonlinear potential theory

Minimizers and superminimizers

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-minimizer*, if

$$\int_{\operatorname{spt}\varphi} g_u^p \, d\mu \le \int_{\operatorname{spt}\varphi} g_{u+\varphi}^p \, d\mu \tag{3}$$

for all $\varphi \in \mathrm{Lip}_0(\Omega)$; here we use the *minimal weak upper gradients* for u and $u + \varphi$.

If (3) holds for all $0 \le \varphi \in \mathrm{Lip}_0(\Omega)$, then u is a *p-superminimizer*.

p-potentials

Let $E \subset \Omega$ be compact. The *p-potential* of E is the function $u_E \in N_0^{1,p}(\Omega)$ with $0 \le u_E \le 1$ and $u_E = 1$ (q.e.) in E, which satisfies

$$\int_\Omega g_{u_E}^p \, d\mu = \operatorname{cap}_p(E, \Omega).$$

The existence of such functions (in metric space setting) was proven by Shanmugalingam.

All p-potentials are *p-superminimizers*, and so they satisfy the following weak Harnack inequality (due to Kinnunen–Shanmugalingam in metric spaces), which is crucial for us.

Lemma (Weak Harnack)

There exists constants $H > 0$, $C_H \ge 1$, and $q > 0$ such that if $u \ge 0$ is a p-superminimizer in $C_H B \subset \Omega$, then $\left(\fint_{2B} u^q \, d\mu \right)^{1/q} \le H \inf_B u.$

5. Proofs

For balls

The following easy consequence of (WBCE) is our first link between quasiadditivity and Maz'ya-type conditions:

Let $\Omega \subset X$ be an open set with Whitney covering $\mathcal{W}_c(\Omega) = \{B_i\}_{i\in I}$ and let $U \subset \Omega$ be a union of Whitney balls, i.e., $U = \bigcup_{i\in I_0} B_i$, $B_i \in \mathcal{W}_c(\Omega)$. Then

$$\int_U d_\Omega(x)^{-p}\, d\mu \approx \sum_{i\in I_0} r_i^{-p}\mu(B_i) \approx \sum_{i\in I_0} \operatorname{cap}_p(B_i,\Omega).$$

Thus if either side is less than $\operatorname{cap}_p(U,\Omega)$ (up to a constant), then so is the other side as well.

From balls to general sets in quasiadditivity

Let us now show that the quasiadditivity for unions of balls implies the quasiadditivity for general sets:

Lemma

Let $\Omega \subset X$ be an open set with a Whitney-type covering $\mathcal{W}_c$ where $c \leq \min\{(C_H)^{-1}, (30\lambda)^{-1}\}$. Assume that there is $C_1 > 0$ such that if $U = \bigcup_{i=1}^{N} B_i$, $B_i \in \mathcal{W}_c$, then

$$\sum_{i=1}^{N} \operatorname{cap}_p(B_i, \Omega) \leq C_1 \operatorname{cap}_p(U, \Omega).$$

Then the capacity $\operatorname{cap}_p(\cdot, \Omega)$ is quasiadditive with respect to $\mathcal{W}_c$, i.e., there exists a constant $A > 0$ such that

$$\sum_{i \in I} \operatorname{cap}_p(E \cap B_i, \Omega) \leq A \operatorname{cap}_p(E, \Omega)$$

for all compact $E \subset \Omega$.

Our main tools in the proof of the previous lemma are the weak Harnack inequality and the following Sobolev type inequality, proved in $\mathbb{R}^n$ by Maz'ya and in X by J. Björn.

Lemma

There is a constant $C > 0$ such that for each $u \in N^{1,p}(X)$ and for all balls $B \subset X$ we have

$$\fint_{2B} |u|^p \, d\mu \le \frac{C}{\operatorname{cap}_p(B \cap \{u = 0\}, 2B)} \int_{10\lambda B} g_u^p \, d\mu,$$

where λ is from the $(1, p)$-Poincaré inequality.

From balls to general sets in Maz'ya

The following result is a 'Maz'ya' analog of the previous quasiadditivity lemma:

Lemma

Let $\Omega \subset X$ be an open set with a Whitney-type covering $\mathcal{W}_c$ where $c \le \min\{(C_H)^{-1}, (30\lambda)^{-1}\}$. Assume that

$$\int_U d_\Omega(x)^{-p}\, d\mu \le C_0 \operatorname{cap}_p(U, \Omega)$$

whenever $U \subset \Omega$ is a (finite) union of Whitney balls. Then there exists a constant $C > 0$ such that

$$\int_E d_\Omega(x)^{-p}\, d\mu \le C \operatorname{cap}_p(E, \Omega)$$

whenever $E \subset \Omega$ is compact.

Conclusion

Theorem

Let $\Omega \subset X$ be an open set and let $\mathcal{W}_c(\Omega) = \{B_i\}_{i \in I}$ be a Whitney-type covering of Ω with $c \leq \min\{(C_H)^{-1}, (30\lambda)^{-1}\}$. Then the following conditions are equivalent for $1 < p < \infty$:

(a) Quasiadditivity for compact $E \subset \Omega$:

$$\sum_{i \in I} \operatorname{cap}_p(E \cap B_i, \Omega) \leq A \operatorname{cap}_p(E, \Omega)$$

(b) Quasiadditivity for unions of Whitney balls.

(c) Maz'ya for unions of Whitney balls.

(d) Maz'ya for compact $E \subset \Omega$:

$$\int_E d_\Omega(x)^{-p} \, d\mu \leq C \operatorname{cap}_p(E, \Omega)$$

(e) p-Hardy inequality for all $u \in N_0^{1,p}(\Omega)$.

Note:

$$(a) \overset{\text{(Harnack)}}{\Longleftrightarrow} (b) \overset{\text{(WBCE)}}{\Longleftrightarrow} (c) \overset{\text{(Harnack)}}{\Longleftrightarrow} (d) \overset{\text{(Maz'ya)}}{\Longleftrightarrow} (e)$$

A lemma towards Aikawa

To prove our counterpart for the Aikawa result, we need to show the following:

Lemma

Assume that μ is Q-regular, $\mu(X) = \infty$, $1 < p < Q$, and that $\dim_{\mathcal{A}}(X \setminus \Omega) < Q - p$. Furthermore, assume that Ω satisfies a uniform John condition. Then there is $A > 0$ such that

$$\int_U d_\Omega(x)^{-p}\, d\mu \leq A\, \mathrm{cap}_p(U, \Omega)$$

whenever U is a finite union of Whitney balls.

The necessity of the John condition is not known.

6. Hardy, again...

A necessary condition

The connection between Hardy inequalities and the Aikawa dimension was brought up in (L. MM 2008), but a similar concept of dimension appeared in this context already in the unpublished works of Wannebo in the 80's.

In metric spaces we have the following dichotomy concerning the dimension of the complement of a domain admitting the p-Hardy inequality:

Theorem (Koskela–Zhong (2003), L.–Tuominen (2011))

Let $1 < p < \infty$ and assume that a domain $\Omega \subset X$ admits the p-Hardy inequality. Then there exists an $\varepsilon > 0$, depending only on the given data, such that for each ball $B \subset X$ either

(i) $\dim_{\mathcal{H}}(2B \cap \Omega^c) \geq Q - p + \varepsilon \qquad \big(\operatorname{codim}_{\mathcal{H}}(2B \cap \Omega^c) \leq p - \varepsilon\big)$

or

(ii) $\dim_{\mathcal{A}}(B \cap \Omega^c) \leq Q - p - \varepsilon \qquad \big(\operatorname{codim}_{\mathcal{A}}(B \cap \Omega^c) \geq p + \varepsilon\big)$.

An 'almost-characterization' for Hardy

Assume that μ is Q-regular, $\mu(X) = \infty$, and $1 < p < Q$. Assume that $\Omega \subset X$ is a domain such that either

(i) $\dim_{\mathcal{H}}(\Omega^c) > Q - p$

or

(ii) $\dim_{\mathcal{A}}(\Omega^c) < Q - p$

Then Ω admits the p-Hardy inequality. (This is also necessary!)

An 'almost-characterization' for Hardy

Assume that μ is Q-regular, $\mu(X) = \infty$, and $1 < p < Q$. Assume that $\Omega \subset X$ is a domain such that either

(i) $\dim_{\mathcal{H}}(\Omega^c) > Q - p$ ('uniformly': (*))

or

(ii) $\dim_{\mathcal{A}}(\Omega^c) < Q - p$ (+John(??))

Then Ω admits the p-Hardy inequality.

(*): $\mathcal{H}^{\lambda}_{\infty}(\Omega^c \cap B(x,r)) \geq Cr^{\lambda}$ for some $\lambda > Q - p$ (all $x \in \Omega^c$ and $r > 0$). A condition like this is certainly needed, even a local bound for the dimension is not sufficient alone.

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Assume that μ is Q-regular, $\mu(X) = \infty$, and $1 < p < Q$. Assume that $\Omega \subset X$ is a domain such that either

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(*): $\mathcal{H}^{\lambda}_{\infty}(\Omega^c \cap B(x,r)) \geq Cr^{\lambda}$ for some $\lambda > Q - p$ (all $x \in \Omega^c$ and $r > 0$). A condition like this is certainly needed, even a local bound for the dimension is not sufficient alone.

A 'suitable combination of (i) and (ii)' works as well, but the necessity of the John condition is what keeps bothering us..

Some references

D. ADAMS, 'Quasi-additivity and sets of finite L^p-capacity', *Pacific J. Math.* 79 (1978), no. 2, 283–291.

H. AIKAWA, 'Quasiadditivity of Riesz capacity', *Math. Scand.* 69 (1991), no. 1, 15–30.

H. AIKAWA AND M. ESSÉN, 'Potential theory—selected topics', Lecture Notes in Mathematics, 1633, Springer-Verlag, Berlin, 1996.

A. BJÖRN AND J. BJÖRN 'Nonlinear Potential Theory on Metric Spaces', EMS Tracts in Mathematics 17, European Mathematical Society, Zurich, 2011.

J. KINNUNEN AND N. SHANMUGALINGAM, 'Regularity of quasi-minimizers on metric spaces', *Manuscripta Math.* 105 (2001), no. 3, 401–423.

R. KORTE, J. LEHRBÄCK AND H. TUOMINEN, 'The equivalence between pointwise Hardy inequalities and uniform fatness', *Math. Ann.* 351 (2011), no. 3, 711–731.

P. KOSKELA AND X. ZHONG, 'Hardy's inequality and the boundary size', *Proc. Amer. Math. Soc.* 131 (2003), no. 4, 1151–1158.

J. LEHRBÄCK AND H. TUOMINEN, 'A note on the dimensions of Assouad and Aikawa', *J. Japan Math. Soc.* (to appear)

J. LUUKKAINEN, 'Assouad dimension: antifractal metrization, porous sets, and homogeneous measures', *J. Korean Math. Soc.* 35 (1998), no. 1, 23–76.

More references

H. Aikawa, 'Quasiadditivity of capacity and minimal thinnes', *Ann. Acad. Sci. Fenn. Ser. A I Math.* 18 (1993), no. 1, 65–75.

J. Björn, P. MacManus and N. Shanmugalingam, 'Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces', *J. Anal. Math.* 85 (2001), 339–369.

J. Heinonen, 'Lectures on analysis on metric spaces', Universitext, Springer, 2001.

R. Korte and N. Shanmugalingam, 'Equivalence and self-improvement of p-fatness and Hardy's inequality, and...', *Math. Z.* 264 (2010), no. 1, 99–110.

N.S. Landkof, 'Foundations of modern potential theory', Springer, 1972.

J. Lehrbäck, 'Weighted Hardy inequalities and the size of the boundary', *Manuscripta Math.* 127 (2008), no. 2, 249–273.

V. G. Maz'ya, 'Sobolev Spaces', Springer, 1985.

N. Shanmugalingam, 'Newtonian spaces: an extension of Sobolev spaces to metric measure spaces', *Rev. Mat. Iberoamericana* 16 (2000), no. 2, 243–279.

N. Shanmugalingam, 'Harmonic functions on metric spaces', *Illinois J. Math.* 45 (2001), no. 3, 1021–1050.

A. Wannebo, 'Hardy inequalities', *Proc. Amer. Math. Soc.* 109 (1990), 85–95.