Pointwise Hardy inequalities and uniform fatness

Juha Lehrbäck
partially based on a joined work with
Riikka Korte and Heli Tuominen

University of Jyväskylä

ROMFIN 2009, 17.8.2009, Turku

Original inequalities

G.H. Hardy 1925:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\,dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p\,dx,$$

when $1 < p < \infty$ and $f \ge 0$ is measurable.

Original inequalities

G.H. Hardy 1925:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \, dx,$$

when $1 < p < \infty$ and $f \ge 0$ is measurable.
Another form:

$$\int_0^\infty |u(x)|^p \, x^{-p} \, dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |u'(x)|^p \, dx,$$

where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.

Hardy inequalities in $\mathbb{R}^n$

Again: $\int_0^\infty |u(x)|^p \, x^{-p} \, dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p \, dx$,
where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.

Hardy inequalities in $\mathbb{R}^n$

Again: $\int_0^\infty |u(x)|^p \, x^{-p} \, dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p \, dx$,
where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.
This can be generalized to higher dimensions in many ways; we consider the following form:

Hardy inequalities in $\mathbb{R}^n$

Again: $\int_0^\infty |u(x)|^p x^{-p}\, dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\, dx$,
where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.
This can be generalized to higher dimensions in many ways; we consider the following form:

$$\int_\Omega |u(x)|^p\, d_\Omega(x)^{-p}\, dx \le C \int_\Omega |\nabla u(x)|^p\, dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

Hardy inequalities in $\mathbb{R}^n$

Again: $\int_0^\infty |u(x)|^p \, x^{-p} \, dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p \, dx$,
where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.
This can be generalized to higher dimensions in many ways; we consider the following form:

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.
If (1) holds for all $u \in C_0^\infty(\Omega)$ with the same constant $C = C(\Omega, p, \beta) > 0$, we say that $\Omega \subset \mathbb{R}^n$ *admits* the p-Hardy inequality.

Hardy inequalities in $\mathbb{R}^n$

Again: $\int_0^\infty |u(x)|^p \, x^{-p} \, dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p \, dx$,
where $1 < p < \infty$ and u is abs. continuous, $u(0) = 0$.
This can be generalized to higher dimensions in many ways; we consider the following form:

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{-p} \, dx \le C \int_\Omega |\nabla u(x)|^p \, dx, \qquad (1)$$

where $\Omega \subset \mathbb{R}^n$ is open, $u \in C_0^\infty(\Omega)$, and $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.
If (1) holds for all $u \in C_0^\infty(\Omega)$ with the same constant $C = C(\Omega, p, \beta) > 0$, we say that $\Omega \subset \mathbb{R}^n$ *admits* the p-Hardy inequality.
(We do not care here about the optimality of the constant C)

Sufficient conditions

Theorem (Nečas 1962)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the p-Hardy inequality for all $1 < p < \infty$.

The "smoothness" of the boundary is however irrelevant here:

Sufficient conditions

Theorem (Nečas 1962)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the p-Hardy inequality for all $1 < p < \infty$.

The "smoothness" of the boundary is however irrelevant here:

Theorem (Ancona 1986 ($p = 2$), Lewis 1988, Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

Sufficient conditions

Theorem (Nečas 1962)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then Ω admits the p-Hardy inequality for all $1 < p < \infty$.

The "smoothness" of the boundary is however irrelevant here:

Theorem (Ancona 1986 ($p = 2$), Lewis 1988, Wannebo 1990)

Let $\Omega \subset \mathbb{R}^n$ be a domain such that the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$ is uniformly p-fat. Then Ω admits the p-Hardy inequality.

(If $\Omega \subset \mathbb{R}^n$ is bounded Lipschitz, then Ω^c is indeed uniformly p-fat for all $1 < p < \infty$)

Uniform fatness

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, if it satisfies a uniform capacity density condition. Precisely:

Uniform fatness

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, if it satisfies a uniform capacity density condition. Precisely:
When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E,\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E \right\}.$$

Uniform fatness

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, if it satisfies a uniform capacity density condition. Precisely:
When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E,\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } E \right\}.$$

A closed set $E \subset \mathbb{R}^n$ is *uniformly p-fat* if

$$\operatorname{cap}_p \left(E \cap \overline{B}(x,r), B(x,2r)\right) \geq C \operatorname{cap}_p \left(\overline{B}(x,r), B(x,2r)\right)$$

for every $x \in E$ and all $r > 0$.

Uniform fatness

A closed set $E \subset \mathbb{R}^n$ is uniformly p-fat, if it satisfies a uniform capacity density condition. Precisely:
When $\Omega \subset \mathbb{R}^n$ is a domain and $E \subset \Omega$ is a compact subset, the *(variational) p-capacity* of E (relative to Ω) is

$$\operatorname{cap}_p(E, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in C_0^\infty(\Omega), \; u \geq 1 \text{ on } E \right\}.$$

A closed set $E \subset \mathbb{R}^n$ is *uniformly p-fat* if

$$\operatorname{cap}_p \left(E \cap \overline{B}(x, r), B(x, 2r) \right) \geq C \operatorname{cap}_p \left(\overline{B}(x, r), B(x, 2r) \right)$$

for every $x \in E$ and all $r > 0$.
Actually,

$$\operatorname{cap}_p(\overline{B}(x, r), B(x, 2r)) = C(n, p) r^{n-p}$$

for each ball $B(x, r) \subset \mathbb{R}^n$.

Uniform fatness: self-improvement

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $q > p$, then E is also uniformly q-fat.

Uniform fatness: self-improvement

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $q > p$, then E is also uniformly q-fat.

smaller p $\leftrightarrow$ fatter set

Uniform fatness: self-improvement

It is easy to see that if a set $E \subset \mathbb{R}^n$ is uniformly p-fat and $q > p$, then E is also uniformly q-fat.

smaller $p \leftrightarrow$ *fatter set*

On the other hand, we have a deep result by J. Lewis:

Theorem (Lewis 1988)

If $E \subset \mathbb{R}^n$ is uniformly p-fat for $1 < p < \infty$, then there exists some $1 < q < p$ such that E is uniformly q-fat.

Uniform fatness: "geometric" characterization

Uniform fatness is equivalent to the following Hausdorff content density condition:

Uniform fatness: "geometric" characterization

Uniform fatness is equivalent to the following Hausdorff content density condition:

There exists some $\lambda > n - p$ and a constant $C > 0$ such that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap B(w,r)\big) \geq C r^{\lambda} \quad \text{for all } w \in E \text{ and all } r > 0.$$

Uniform fatness: "geometric" characterization

Uniform fatness is equivalent to the following Hausdorff content density condition:
There exists some $\lambda > n - p$ and a constant $C > 0$ such that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap B(w,r)\big) \geq Cr^{\lambda} \quad \text{for all } w \in E \text{ and all } r > 0.$$

Recall that the λ-Hausdorff content of $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^{\lambda}_{\infty}(A) = \inf\left\{ \sum_{i=1}^{\infty} r_i^{\lambda} : A \subset \bigcup_{i=1}^{\infty} B(z_i, r_i) \right\}.$$

Uniform fatness: “geometric” characterization

Uniform fatness is equivalent to the following Hausdorff content density condition:
There exists some $\lambda > n - p$ and a constant $C > 0$ such that

$$\mathcal{H}^\lambda_\infty\big(E \cap B(w,r)\big) \geq Cr^\lambda \quad \text{for all } w \in E \text{ and all } r > 0.$$

Recall that the λ-Hausdorff content of $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^\lambda_\infty(A) = \inf\left\{ \sum_{i=1}^\infty r_i^\lambda : A \subset \bigcup_{i=1}^\infty B(z_i, r_i) \right\}.$$

It is now immediate that every non-empty $E \subset \mathbb{R}^n$ is unif. p-fat for all $p > n$, and an m-dimensional subspace $L \subset \mathbb{R}^n$ is is unif. p-fat for all $p > n - m$.

Pointwise p-Hardy inequality

Uniform p-fatness of the complement yields actually stronger(?) inequalities:

Pointwise p-Hardy inequality

Uniform p-fatness of the complement yields actually stronger(?) inequalities:

Theorem (Hajłasz 1999, Kinnunen-Martio 1997)

Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exists a constant $C > 0$ such that the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2 d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

Pointwise p-Hardy inequality

Uniform p-fatness of the complement yields actually stronger(?) inequalities:

Theorem (Hajłasz 1999, Kinnunen-Martio 1997)

Let $1 < p < \infty$ and assume that the complement of a domain $\Omega \subset \mathbb{R}^n$ is uniformly p-fat. Then there exists a constant $C > 0$ such that the pointwise p-Hardy inequality

$$|u(x)| \leq C d_\Omega(x) \big(M_{2 d_\Omega(x)} \big(|\nabla u|^p \big)(x) \big)^{1/p}$$

holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$.

Here $M_R f$ is the usual restricted Hardy-Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, defined by $M_R f(x) = \sup_{r \leq R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy$

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

$$|u(x)| \leq C d_\Omega(x) \big(M_{2 d_\Omega(x)} \big(|\nabla u|^q \big)(x) \big)^{1/q}$$

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

$$|u(x)|^{p} \leq C d_{\Omega}(x)^{p} \big(M_{2d_{\Omega}(x)} \big(|\nabla u|^{q} \big)(x) \big)^{p/q}$$

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

$$|u(x)|^p d_\Omega(x)^{-p} \quad \leq C \quad \left(M_{2d_\Omega(x)}\left(|\nabla u|^q\right)(x)\right)^{p/q}$$

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

$$\int_\Omega |u(x)|^p d_\Omega(x)^{-p}\,dx \le C \int_\Omega \big(M_{2d_\Omega(x)}(|\nabla u|^q)(x)\big)^{p/q}\,dx$$

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

$$\int_\Omega |u(x)|^p d_\Omega(x)^{-p}\,dx \le C \int_\Omega \big(M_{2d_\Omega(x)}\big(|\nabla u|^q\big)(x)\big)^{p/q}\,dx$$
$$\le C \int_\Omega \big(|\nabla u|^q\big)^{p/q}\,dx$$

Pointwise implies integral

By the maximal theorem it is easy to see that a pointwise q-Hardy inequality (4) implies the usual p-Hardy inequality for all $p > q$:

$$\begin{aligned} \int_\Omega |u(x)|^p d_\Omega(x)^{-p}\, dx &\le C \int_\Omega \big(M_{2d_\Omega(x)}\big(|\nabla u|^q\big)(x)\big)^{p/q}\, dx \\ &\le C \int_\Omega |\nabla u|^p\, dx \end{aligned}$$

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat
  $\Rightarrow$ Ω admits the pointwise q-Hardy.
  $\Rightarrow$ Ω admits the p-Hardy.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat
  $\Rightarrow$ Ω admits the pointwise p-Hardy.
  $\Rightarrow$ Ω admits the p'-Hardy for all $p' > p$.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat
  $\Rightarrow$ Ω admits the pointwise p-Hardy.
  $\Rightarrow$ Ω admits the p'-Hardy for all $p' > p$.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad \Rightarrow \quad \Omega$ admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat
  $\Rightarrow \Omega$ admits the pointwise p-Hardy.
  $\Rightarrow \Omega$ admits the p'-Hardy for all $p' > p$.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\Rightarrow$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\Rightarrow$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow$ Ω admits the pointwise p-Hardy.
  $\Rightarrow$ Ω admits the p'-Hardy for all $p' > p$.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow \Omega$ admits the pointwise q-Hardy.
  $\Rightarrow \Omega$ admits the p'-Hardy for all $p' > p$.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow$ Ω admits the pointwise q-Hardy.
  $\Rightarrow$ Ω admits the p-Hardy.

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow$ Ω admits the pointwise q-Hardy.
  $\Rightarrow$ Ω admits the p-Hardy.
- Is it true that: Pointwise p-Hardy $\quad\Rightarrow\quad$ p-Hardy ??

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow$ Ω admits the pointwise q-Hardy.
  $\Rightarrow$ Ω admits the p-Hardy.
- Is it true that: Pointwise p-Hardy $\quad\Rightarrow\quad$ p-Hardy ??
- We have: Ω^c unif. n-fat $\quad\Leftrightarrow\quad$ Ω admits the n-Hardy (in $\mathbb{R}^n$).
  (Ancona $n = 2$, Lewis)

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\Rightarrow$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\Rightarrow$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow$ Ω admits the pointwise q-Hardy.
  $\Rightarrow$ Ω admits the p-Hardy.
- Is it true that: Pointwise p-Hardy $\Rightarrow$ p-Hardy ??
- We have: Ω^c unif. n-fat $\Leftrightarrow$ Ω admits the n-Hardy (in $\mathbb{R}^n$).
  (Ancona $n = 2$, Lewis)
- But: Ω^c unif. p-fat $\not\Leftarrow$ Ω admits the p-Hardy if $1 < p < n$.
  (a punctured ball $B(0, r) \setminus \{0\} \subset \mathbb{R}^n$ admits the p-Hardy for all $p \neq n$, but is not unif. p-fat for $p \leq n$.)

Revision and questions:

- By Ancona–Lewis–Wannebo:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω admits the p-Hardy.
- By Hajłasz–Kinnunen–Martio:
  Ω^c unif. p-fat $\quad\Rightarrow\quad$ Ω^c unif. q-fat, $q < p$ (Lewis)
  $\Rightarrow$ Ω admits the pointwise q-Hardy.
  $\Rightarrow$ Ω admits the p-Hardy.
- Is it true that: Pointwise p-Hardy $\quad\Rightarrow\quad$ p-Hardy ??
- We have: Ω^c unif. n-fat $\quad\Leftrightarrow\quad$ Ω admits the n-Hardy (in $\mathbb{R}^n$).
  (Ancona $n = 2$, Lewis)
- But: Ω^c unif. p-fat $\quad\not\Leftarrow\quad$ Ω admits the p-Hardy if $1 < p < n$.
  (a punctured ball $B(0, r) \setminus \{0\} \subset \mathbb{R}^n$ admits the p-Hardy for all $p \neq n$, but is not unif. p-fat for $p \leq n$.)
- Does the converse hold for pointwise inequalities ??

Metric spaces

For simplicity, we only consider $\mathbb{R}^n$ in this talk, but in fact all of the considrations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

Metric spaces

For simplicity, we only consider $\mathbb{R}^n$ in this talk, but in fact all of the considrations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \leq C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the "dimension" of X is at most $s = \log_2 C_d$)

Metric spaces

For simplicity, we only consider $\mathbb{R}^n$ in this talk, but in fact all of the considrations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \le C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the "dimension" of X is at most $s = \log_2 C_d$)
- X supports a (weak) p-Poincaré inequality:

$$\fint_B |u - u_B| \, d\mu \le C_P r \Big(\fint_{\tau B} g_u^p \, d\mu \Big)^{1/p}$$

whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u:

Metric spaces

For simplicity, we only consider $\mathbb{R}^n$ in this talk, but in fact all of the considrations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \leq C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the "dimension" of X is at most $s = \log_2 C_d$)
- X supports a (weak) p-Poincaré inequality:

$$\fint_B |u - u_B| \, d\mu \leq C_P r \left(\fint_{\tau B} g_u^p \, d\mu \right)^{1/p}$$

whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u: For all (or p-almost all) curves γ joining $x, y \in X$

$$|u(x) - u(y)| \leq \int_\gamma g_u \, ds.$$

Metric spaces

For simplicity, we only consider $\mathbb{R}^n$ in this talk, but in fact all of the considrations and results hold (with minor modifications) in a complete metric measure space $X = (X, d, \mu)$, provided that

- μ is *doubling*: $\mu(2B) \le C_d \mu(B)$ for each ball $B \subset X$ (it follows from this that the "dimension" of X is at most $s = \log_2 C_d$)
- X supports a (weak) p-Poincaré inequality:

$$\fint_B |u - u_B| \, d\mu \le C_P r \Big(\fint_{\tau B} g_u^p \, d\mu \Big)^{1/p}$$

whenever $u \in L^1_{\mathrm{loc}}(X)$ and g_u is an (or a weak) *upper gradient* of u: For all (or p-almost all) curves γ joining $x, y \in X$

$$|u(x) - u(y)| \le \int_\gamma g_u \, ds.$$

In particular, the self-improvement of uniform p-fatness was proved in this setting by Björn, MacManus and Shanmugalingam (2001).

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have ?$= p$, then (2) is true by ALW!

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have ?$= p$, then (2) is true by ALW!
Regarding (1):

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have $? = p$, then (2) is true by ALW!
Regarding (1):

- For usual p-Hardy, this only holds for $p = n = ?$

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have ?$= p$, then (2) is true by ALW!
Regarding (1):

- For usual p-Hardy, this only holds for $p = n =$?
- However, if Ω admits the p-Hardy, then Ω^c can not contain (isolated) parts of dimension $n - p$ (Koskela–Zhong, 2003)

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have ?$= p$, then (2) is true by ALW!
Regarding (1):

- For usual p-Hardy, this only holds for $p = n =$?
- However, if Ω admits the p-Hardy, then Ω^c can not contain (isolated) parts of dimension $n - p$ (Koskela–Zhong, 2003)
- Moreover, it is easy to see that
  pointwise p-Hardy $\Rightarrow$ pointwise p'-Hardy for all $p' > p$ (Hölder).

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have ?$= p$, then (2) is true by ALW!
Regarding (1):

- For usual p-Hardy, this only holds for $p = n =$?
- However, if Ω admits the p-Hardy, then Ω^c can not contain (isolated) parts of dimension $n - p$ (Koskela–Zhong, 2003)
- Moreover, it is easy to see that
  pointwise p-Hardy $\Rightarrow$ pointwise p'-Hardy for all $p' > p$ (Hölder).
- thus, if Ω admits the pointwise p-Hardy, then Ω^c can not contain (isolated) parts of dimension $\leq n - p$, so that Ω^c must be "quite fat"

Main questions:

1. Pointwise p-Hardy $\Rightarrow$ uniform ?-fatness ??
2. Pointwise p-Hardy $\Rightarrow$ p-Hardy ??

Notice that if in (1) we have ?$= p$, then (2) is true by ALW!
Regarding (1):

- For usual p-Hardy, this only holds for $p = n =$?
- However, if Ω admits the p-Hardy, then Ω^c can not contain (isolated) parts of dimension $n - p$ (Koskela–Zhong, 2003)
- Moreover, it is easy to see that
  pointwise p-Hardy $\Rightarrow$ pointwise p'-Hardy for all $p' > p$ (Hölder).
- thus, if Ω admits the pointwise p-Hardy, then Ω^c can not contain (isolated) parts of dimension $\leq n - p$, so that Ω^c must be "quite fat"
- This makes (1) plausible, at least for some ?.

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

(1) Ω^c unif. p-fat

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

(1) Ω^c unif. p-fat
$\Rightarrow$ (2) Ω admits the pointwise q-Hardy for some $q < p$

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

(1) Ω^c unif. p-fat
$\Rightarrow$ (2) Ω admits the pointwise q-Hardy for some $q < p$
$\Rightarrow$ (3) there exists $C > 0$ so that for $\lambda = n - q$ we have the following *inner boundary density condition*:

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

(1) Ω^c unif. p-fat
$\Rightarrow$ (2) Ω admits the pointwise q-Hardy for some $q < p$
$\Rightarrow$ (3) there exists $C > 0$ so that for $\lambda = n - q$ we have the following *inner boundary density condition*:

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

$\Rightarrow$ (4) there exists $C > 0$ so that for $\lambda = n - q$

$$\mathcal{H}^{\lambda}_{\infty}\big(\Omega^c \cap B(w, r)\big) \geq C r^{\lambda} \qquad \text{for every } w \in \Omega^c \text{ and all } r > 0.$$

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

(1) Ω^c unif. p-fat
$\Rightarrow$ (2) Ω admits the pointwise q-Hardy for some $q < p$
$\Rightarrow$ (3) there exists $C > 0$ so that for $\lambda = n - q$ we have the following *inner boundary density condition*:

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \quad \text{for every } x \in \Omega.$$

$\Rightarrow$ (4) there exists $C > 0$ so that for $\lambda = n - q > n - p$

$$\mathcal{H}^\lambda_\infty\big(\Omega^c \cap B(w, r)\big) \geq C r^\lambda \qquad \text{for every } w \in \Omega^c \text{ and all } r > 0.$$

$\Rightarrow$ (1) Ω^c unif. p-fat.

From pointwise Hardy to fatness, pt. 1

A partial answer to question 1. (in $\mathbb{R}^n$) was given in (L, PAMS 2008):

(1) Ω^c unif. p-fat
$\Rightarrow$ (2) Ω admits the pointwise q-Hardy for some $q < p$
$\Rightarrow$ (3) there exists $C > 0$ so that for $\lambda = n - q$ we have the following *inner boundary density condition*:

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \quad \text{for every } x \in \Omega.$$

$\Rightarrow$ (4) there exists $C > 0$ so that for $\lambda = n - q > n - p$

$$\mathcal{H}^\lambda_\infty\big(\Omega^c \cap B(w, r)\big) \geq C r^\lambda \qquad \text{for every } w \in \Omega^c \text{ and all } r > 0.$$

$\Rightarrow$ (1) Ω^c unif. p-fat.
(recall that (1)$\Rightarrow$(2) and (4)$\Leftrightarrow$(1) were previously known)

A closer look

So, in particular we obtain:

Ω admits the pointwise p-Hardy for some $1 < p < \infty$
$\Rightarrow \Omega^c$ unif. p'-fat for all $p' > p$

A closer look

So, in particular we obtain:

Ω admits the pointwise p-Hardy for some $1 < p < \infty$
$\Rightarrow \Omega^c$ unif. p'-fat for all $p' > p$
(thus almost the converse; would want Ω^c unif. p-fat)

A closer look

So, in particular we obtain:

Ω admits the pointwise p-Hardy for some $1 < p < \infty$
$\Rightarrow \Omega^c$ unif. p'-fat for all $p' > p$
(thus almost the converse; would want Ω^c unif. p-fat)

On the other hand:
Ω admits the pointwise p-Hardy
$\Rightarrow$ for $\lambda = n - p$

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

A closer look

So, in particular we obtain:

Ω admits the pointwise p-Hardy for some $1 < p < \infty$
$\Rightarrow \Omega^c$ unif. p'-fat for all $p' > p$
(thus almost the converse; would want Ω^c unif. p-fat)

On the other hand:
Ω admits the pointwise p-Hardy
$\not\Leftarrow$ for $\lambda = n - p$

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

A closer look

So, in particular we obtain:

Ω admits the pointwise p-Hardy for some $1 < p < \infty$
$\Rightarrow \Omega^c$ unif. p'-fat for all $p' > p$
(thus almost the converse; would want Ω^c unif. p-fat)

On the other hand:
Ω admits the pointwise p-Hardy
$\Rightarrow$ for $\lambda = n - p$

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega.$$

Idea of $\Rightarrow$: Let $B(x, 2d_{\Omega}(x)) \cap \partial\Omega \subset \bigcup_{i=1}^{N} B(z_i, r_i)$ and use the pointwise p-Hardy for test function

$$\varphi(y) = \min_{1 \leq i \leq N} \big\{1,\ r_i^{-1} d(y, B(z_i, 2r_i))\big\} \cdot (\text{cut-off})$$

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{2}$$

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{2}$$

?? there exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(w, r) \cap \partial\Omega\big) \geq C r^{\lambda} \quad \text{for every } r > 0,\ w \in \partial\Omega \tag{3}$$

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{2}$$

$\Leftarrow$ there exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(w, r) \cap \partial\Omega\big) \geq C r^{\lambda} \quad \text{for every } r > 0,\ w \in \partial\Omega \tag{3}$$

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{2}$$

$\not\Rightarrow$ there exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(w, r) \cap \partial\Omega\big) \geq C r^{\lambda} \quad \text{for every } r > 0,\ w \in \partial\Omega \tag{3}$$

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{2}$$

$\not\Rightarrow$ there exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(w, r) \cap \partial\Omega\big) \geq C r^{\lambda} \quad \text{for every } r > 0,\ w \in \partial\Omega \tag{3}$$

Reason: think of a "cusp"-domain in $\mathbb{R}^3$:
(2) holds for all $\lambda \leq 2$, but (3) only holds for $\lambda \leq 1$.

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(x, 2d_{\Omega}(x)) \cap \partial\Omega\big) \geq C d_{\Omega}(x)^{\lambda} \quad \text{for every } x \in \Omega. \tag{2}$$

$\Rightarrow$ there exists a constat $C > 0$ so that

$$\mathcal{H}^{\lambda}_{\infty}\big(B(w, r) \cap \Omega^c\big) \geq C r^{\lambda} \quad \text{for every } r > 0,\ w \in \Omega^c \quad (\partial\Omega) \tag{3}$$

Inner boundary density and complement density

Let us take another look at the following density conditions:

There exists a constat $C > 0$ so that

$$\mathcal{H}^\lambda_\infty\big(B(x, 2d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda \quad \text{for every } x \in \Omega. \tag{2}$$

$\Rightarrow$ there exists a constat $C > 0$ so that

$$\mathcal{H}^\lambda_\infty\big(B(w, r) \cap \Omega^c\big) \geq C r^\lambda \quad \text{for every } r > 0,\ w \in \Omega^c \quad (\partial\Omega) \tag{3}$$

Idea of $\Rightarrow$: If $|B(w, r) \cap \Omega^c| \geq \frac{1}{2}|B(w, r)|$, then (3) holds.
Otherwise use (2) with a covering argument to show that actually in this case

$$\mathcal{H}^\lambda_\infty\big(B(w, r) \cap \partial\Omega\big) \geq C r^\lambda \quad \text{for every } r > 0,\ w \in \partial\Omega.$$

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$
$\Rightarrow$ (4) complement density for $\lambda = n - p$

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$
$\Rightarrow$ (4) complement density for $\lambda = n - p$
$\Rightarrow$ (5) unif. p'-fatness for all $p' > p$.

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$
$\Rightarrow$ (4) complement density for $\lambda = n - p$
$\Rightarrow$ (5) unif. p'-fatness for all $p' > p$.
Something is lost along the way. But where?

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$
$\Rightarrow$ (4) complement density for $\lambda = n - p$
$\Rightarrow$ (5) unif. p'-fatness for all $p' > p$.
Something is lost along the way. But where?

- (4)$\Rightarrow$(5) does not hold for $p' = p$. Is this where we lose the game?

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get

(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy

$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$

$\Rightarrow$ (4) complement density for $\lambda = n - p$

$\Rightarrow$ (5) unif. p'-fatness for all $p' > p$.

Something is lost along the way. But where?

- (4)$\Rightarrow$(5) does not hold for $p' = p$. Is this where we lose the game? Not really.

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$
$\Rightarrow$ (4) complement density for $\lambda = n - p$
$\Rightarrow$ (5) unif. p'-fatness for all $p' > p$.
Something is lost along the way. But where?

- (4)$\Rightarrow$(5) does not hold for $p' = p$. Is this where we lose the game? Not really.
- (2)$\Rightarrow$(3) does not invert.

A moment of insight...

If we want to stay at the "p"-level, and not use the self-improvement, we get
(1) unif. p-fatness $\Rightarrow$ (2) pointwise p-Hardy
$\Rightarrow$ (3) inner boundary density for $\lambda = n - p$
$\Rightarrow$ (4) complement density for $\lambda = n - p$
$\Rightarrow$ (5) unif. p'-fatness for all $p' > p$.
Something is lost along the way. But where?

- (4)$\Rightarrow$(5) does not hold for $p' = p$. Is this where we lose the game? Not really.
- (2)$\Rightarrow$(3) does not invert. This is crucial.

Once we pass from capacity to Hausdorff content, something is inevitably lost.

...turns into a plan..

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;

...turns into a plan..

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;
that is, invert $(1) \Rightarrow (2)$.

...turns into a plan....and a positive result

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;
that is, invert (1) $\Rightarrow$ (2).
This we were able to do with Riikka Korte and Heli Tuominen:

...turns into a plan....and a positive result

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;
that is, invert (1) $\Rightarrow$ (2).
This we were able to do with Riikka Korte and Heli Tuominen:

Theorem (KLT, 2009)

Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality if and only if Ω^c is uniformly p-fat.

...turns into a plan....and a positive result

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;
that is, invert (1) $\Rightarrow$ (2).
This we were able to do with Riikka Korte and Heli Tuominen:

Theorem (KLT, 2009)

Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality if and only if Ω^c is uniformly p-fat.

Notice here the inclusion of the case $p = 1$; on the contrary, the 1-Hardy inequality does not hold even in the smoothests of domains.

...turns into a plan....and a positive result

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;
that is, invert (1) $\Rightarrow$ (2).
This we were able to do with Riikka Korte and Heli Tuominen:

Theorem (KLT, 2009)

Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality if and only if Ω^c is uniformly p-fat.

Notice here the inclusion of the case $p = 1$; on the contrary, the 1-Hardy inequality does not hold even in the smoothests of domains.
This result was proven also in the metric space setting. In addition, the previous content results from $\mathbb{R}^n$ were generalized to metric spaces

...turns into a plan....and a positive result

Hence, if we are trying to find a *sharp* relation between uniform p-fatness and the pointwise p-Hardy inequality, we have to forget Hausdorff contents, and only use p-capacity;
that is, invert (1) $\Rightarrow$ (2).
This we were able to do with Riikka Korte and Heli Tuominen:

Theorem (KLT, 2009)

Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality if and only if Ω^c is uniformly p-fat.

Notice here the inclusion of the case $p = 1$; on the contrary, the 1-Hardy inequality does not hold even in the smoothests of domains.
This result was proven also in the metric space setting. In addition, the previous content results from $\mathbb{R}^n$ were generalized to metric spaces (but for Hausdorff content of *co-dimension* p, corresponding to gauge function $h(B(x,r)) = \mu(B)r^{-p}$)

Consequences

The above theorem has some interesting consequences:

Consequences

The above theorem has some interesting consequences:

Corollary (LKT, 2009)

If $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then there is $1 < q < p$ so that Ω admits the pointwise q-Hardy inequality, too.

Consequences

The above theorem has some interesting consequences:

Corollary (LKT, 2009)

If $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then there is $1 < q < p$ so that Ω admits the pointwise q-Hardy inequality, too.

Corollary (LKT, 2009)

If $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then Ω admits the usual p-Hardy inequality.

Consequences

The above theorem has some interesting consequences:

Corollary (LKT, 2009)

If $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then there is $1 < q < p$ so that Ω admits the pointwise q-Hardy inequality, too.

Corollary (LKT, 2009)

If $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise p-Hardy inequality, then Ω admits the usual p-Hardy inequality.

(This finally justifies our notion of "pointwise p-Hardy inequality"!!)

A small side-step: Uniformly perfect sets

A set $E \subset \mathbb{R}^n$ is *uniformly perfect*, if $\#E \geq 2$ and there exists $c \geq 1$ such that for all $x \in E,\ r > 0$

$$E \cap B(x, cr) \setminus B(x, r) \neq \emptyset$$

(if $E \setminus B(x, cr) \neq \emptyset$.)

A small side-step: Uniformly perfect sets

A set $E \subset \mathbb{R}^n$ is *uniformly perfect*, if $\#E \geq 2$ and there exists $c \geq 1$ such that for all $x \in E$, $r > 0$

$$E \cap B(x, cr) \setminus B(x, r) \neq \emptyset$$

(if $E \setminus B(x, cr) \neq \emptyset$.)
For unbounded sets, uniform perfecness is equivalent to uniform n-fatness (Sugawa ($n = 2$) 2003, Korte–Shanmugalingam, 2009; see also Järvi–Vuorinen 1996 for related results).

A small side-step: Uniformly perfect sets

A set $E \subset \mathbb{R}^n$ is *uniformly perfect*, if $\#E \geq 2$ and there exists $c \geq 1$ such that for all $x \in E, \ r > 0$

$$E \cap B(x, cr) \setminus B(x, r) \neq \emptyset$$

(if $E \setminus B(x, cr) \neq \emptyset$.)
For unbounded sets, uniform perfecness is equivalent to uniform n-fatness (Sugawa ($n = 2$) 2003, Korte–Shanmugalingam, 2009; see also Järvi–Vuorinen 1996 for related results).
Now, by the previous theorem we also have the equivalence:

A small side-step: Uniformly perfect sets

A set $E \subset \mathbb{R}^n$ is *uniformly perfect*, if $\#E \geq 2$ and there exists $c \geq 1$ such that for all $x \in E$, $r > 0$

$$E \cap B(x, cr) \setminus B(x, r) \neq \emptyset$$

(if $E \setminus B(x, cr) \neq \emptyset$.)
For unbounded sets, uniform perfecness is equivalent to uniform n-fatness (Sugawa ($n = 2$) 2003, Korte–Shanmugalingam, 2009; see also Järvi–Vuorinen 1996 for related results).
Now, by the previous theorem we also have the equivalence:
Ω admits the pointwise n-Hardy
$\Leftrightarrow$ Ω^c is uniformly perfect and unbounded

A small side-step: Uniformly perfect sets

A set $E \subset \mathbb{R}^n$ is *uniformly perfect*, if $\#E \geq 2$ and there exists $c \geq 1$ such that for all $x \in E,\ \ r > 0$

$$E \cap B(x, cr) \setminus B(x, r) \neq \emptyset$$

(if $E \setminus B(x, cr) \neq \emptyset$.)
For unbounded sets, uniform perfecness is equivalent to uniform n-fatness (Sugawa ($n = 2$) 2003, Korte–Shanmugalingam, 2009; see also Järvi–Vuorinen 1996 for related results).
Now, by the previous theorem we also have the equivalence:
Ω admits the pointwise n-Hardy
$\Leftrightarrow$ Ω^c is uniformly perfect and unbounded
($\Leftrightarrow$ Ω admits n-Hardy)

A boundary Poincaré inequality

In the proof of [unif. p-fat $\Rightarrow$ pointwise p-Hardy], the following Sobolev-type estimate due to Maz'ja plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\frac{1}{|B|}\int_B |u|^p\,dx \le \frac{C}{\mathrm{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)}\int_B |\nabla u|^p\,dx. \tag{4}$$

A boundary Poincaré inequality

In the proof of [unif. p-fat $\Rightarrow$ pointwise p-Hardy], the following Sobolev-type estimate due to Maz'ja plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\frac{1}{|B|}\int_B |u|^p\,dx \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)}\int_B |\nabla u|^p\,dx. \tag{4}$$

Now, if Ω^c is unif. p-fat and $u \in C_0^\infty(\Omega)$, it follows from (4) that

$$\int_B |u|^p\,dx \le Cr^p \int_B |\nabla u|^p\,dx. \tag{5}$$

(a "boundary Poincaré inequality")

A boundary Poincaré inequality

In the proof of [unif. p-fat $\Rightarrow$ pointwise p-Hardy], the following Sobolev-type estimate due to Maz'ja plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\frac{1}{|B|}\int_B |u|^p\,dx \leq \frac{C}{\mathrm{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)}\int_B |\nabla u|^p\,dx. \tag{4}$$

Now, if Ω^c is unif. p-fat and $u \in C_0^\infty(\Omega)$, it follows from (4) that

$$\int_B |u|^p\,dx \leq Cr^p \int_B |\nabla u|^p\,dx. \tag{5}$$

(a "boundary Poincaré inequality")
This, combined with standard estimates (or a chaining argument), yields the pointwise p-Hardy inequality.

A boundary Poincaré inequality

In the proof of [unif. p-fat $\Rightarrow$ pointwise p-Hardy], the following Sobolev-type estimate due to Maz'ja plays a key role: for $u \in C^\infty(\mathbb{R}^n)$

$$\frac{1}{|B|}\int_B |u|^p\,dx \le \frac{C}{\operatorname{cap}_p(\frac{1}{2}B \cap \{u=0\}, B)}\int_B |\nabla u|^p\,dx. \tag{4}$$

Now, if Ω^c is unif. p-fat and $u \in C_0^\infty(\Omega)$, it follows from (4) that

$$\int_B |u|^p\,dx \le Cr^p \int_B |\nabla u|^p\,dx. \tag{5}$$

(a "boundary Poincaré inequality")
This, combined with standard estimates (or a chaining argument), yields the pointwise p-Hardy inequality.
Remark: Once we obtain [pointwise p-Hardy $\Leftrightarrow$ unif. p-fat], we may conclude that the validity of the p-Poincaré inequality (5) for all $u \in C_0^\infty(\Omega)$ is equivalent with the two other "p"-properties

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \leq v \leq 1$ and $v \geq 1$ in $B \cap E$.

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \le v \le 1$ and $v \ge 1$ in $B \cap E$.
- If $⨍_B v \ge C$, we are done by Poincaré:

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \le v \le 1$ and $v \ge 1$ in $B \cap E$.
- If $⨍_B v \ge C$, we are done by Poincaré:

$$1 \le C ⨍_B v \le CR\Big(⨍_{2B} |\nabla v|^p \Big)^{1/p} \Rightarrow \int_{2B} |\nabla v|^p \ge CR^{n-p}$$

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w,R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \le v \le 1$ and $v \ge 1$ in $B \cap E$.
- If $\fint_B v \ge C$, we are done by Poincaré:

$$1 \le C \fint_B v \le CR \Big(\fint_{2B} |\nabla v|^p \Big)^{1/p} \;\Rightarrow\; \int_{2B} |\nabla v|^p \ge CR^{n-p}$$

- Otherwise $u = 1 - v$ must have values $\ge C_1$ in a large set $E \subset \frac{1}{4}B$; $|E| \ge C_2 |B|$. Moreover, $u = 0$ on $\Omega^c \cap B$.

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \le v \le 1$ and $v \ge 1$ in $B \cap E$.
- If $⨍_B v \ge C$, we are done by Poincaré:

$$1 \le C ⨍_B v \le CR\Big(⨍_{2B} |\nabla v|^p\Big)^{1/p} \;\Rightarrow\; \int_{2B} |\nabla v|^p \ge CR^{n-p}$$

- Otherwise $u = 1 - v$ must have values $\ge C_1$ in a large set $E \subset \frac{1}{4}B$; $|E| \ge C_2|B|$. Moreover, $u = 0$ on $\Omega^c \cap B$.
- $\Rightarrow$ we may use the pw p-Hardy on points $x \in E$; let r_x be the corresponding "almost" best radii $(0 < r_x < 2d_\Omega(x) < R/2)$.

From pointwise Hardy to fatness, pt. 2

How to prove [pointwise p-Hardy $\Rightarrow$ unif. p-fatness of Ω^c] ??
Main ideas:

- Fix $w \in \partial\Omega$, $R > 0$, let $B = B(w, R)$, and $v \in C_0^\infty(2B)$ s.t. $0 \le v \le 1$ and $v \ge 1$ in $B \cap E$.
- If $\fint_B v \ge C$, we are done by Poincaré:

$$1 \le C \fint_B v \le CR \Big(\fint_{2B} |\nabla v|^p \Big)^{1/p} \Rightarrow \int_{2B} |\nabla v|^p \ge CR^{n-p}$$

- Otherwise $u = 1 - v$ must have values $\ge C_1$ in a large set $E \subset \frac{1}{4}B$; $|E| \ge C_2 |B|$. Moreover, $u = 0$ on $\Omega^c \cap B$.
- $\Rightarrow$ we may use the pw p-Hardy on points $x \in E$; let r_x be the corresponding "almost" best radii $(0 < r_x < 2d_\Omega(x) < R/2)$.
- "$5r$"-covering thm. $\Rightarrow$ we find $x_i \in E$ s.t. $B_i = B(x_i, r_i)$ are pairwise disjoint but $E \subset \bigcup 5B_i$.

From pointwise Hardy to fatness, pt. 2...cont'd

- Thus $R^n \leq C|E| \leq C \sum {r_i}^n$

From pointwise Hardy to fatness, pt. 2...cont'd

- Thus $R^n \le C|E| \le C \sum {r_i}^n$
- On the other hand

$$C_1^p \le |u(x_i)|^p \le C d_\Omega(x_i)^p M_{2d_\Omega(x)} |\nabla u|^p(x) \le C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

From pointwise Hardy to fatness, pt. 2...cont'd

- Thus $R^n \leq C|E| \leq C \sum {r_i}^n$
- On the other hand

$$C_1^p \leq |u(x_i)|^p \leq C d_\Omega(x_i)^p M_{2 d_\Omega(x)} |\nabla u|^p(x) \leq C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

$$\Rightarrow \; {r_i}^n \leq C R^p \int_{B_i} |\nabla u|^p$$

From pointwise Hardy to fatness, pt. 2...cont'd

- Thus $R^n \leq C|E| \leq C \sum {r_i}^n$
- On the other hand

$$C_1^p \leq |u(x_i)|^p \leq C d_\Omega(x_i)^p M_{2d_\Omega(x)} |\nabla u|^p(x) \leq C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

$$\Rightarrow \ {r_i}^n \leq C R^p \int_{B_i} |\nabla u|^p$$

- Combining the above inequalities with the facts that $|\nabla u| = |\nabla v|$ in B and B_i's are pairwise disjoint, we get

$$R^n \leq C R^p \sum_{i=1}^{\infty} \int_{B_i} |\nabla u|^p \leq C R^p \int_{2B} |\nabla v|^p$$

From pointwise Hardy to fatness, pt. 2...cont'd

- Thus $R^n \leq C|E| \leq C \sum {r_i}^n$
- On the other hand

$$C_1^p \leq |u(x_i)|^p \leq C d_\Omega(x_i)^p M_{2d_\Omega(x)}|\nabla u|^p(x) \leq C R^p {r_i}^{-n} \int_{B_i} |\nabla u|^p$$

$$\Rightarrow \ {r_i}^n \leq C R^p \int_{B_i} |\nabla u|^p$$

- Combining the above inequalities with the facts that $|\nabla u| = |\nabla v|$ in B and B_i's are pairwise disjoint, we get

$$R^n \leq C R^p \sum_{i=1}^{\infty} \int_{B_i} |\nabla u|^p \leq C R^p \int_{2B} |\nabla v|^p$$

- Hence $\mathrm{cap}_p(\Omega^c \cap \overline{B}, 2B) \geq C R^{n-p}$, and so Ω^c is unif. p-fat.

Bibliography

A. Ancona, 'On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$', *J. London Math. Soc.* (2) 34 (1986), no. 2, 274–290.

P. Hajłasz, 'Pointwise Hardy inequalities', *Proc. Amer. Math. Soc.* 127 (1999), no. 2, 417–423.

G. H. Hardy, 'Notes on some points in the integral calculus (LX)', *Messenger of Math.* 54 (1925), 150–156.

J. Kinnunen and O. Martio, 'Hardy's inequalities for Sobolev functions', *Math. Res. Lett.* 4 (1997), no. 4, 489–500.

R. Korte, J. Lehrbäck and H. Tuominen, 'The equivalence between pointwise Hardy inequalities and uniform fatness', preprint 2009, arXiv:0906.2086v1

P. Koskela and X. Zhong, 'Hardy's inequality and the boundary size', *Proc. Amer. Math. Soc.* 131 (2003), no. 4, 1151–1158.

J. Lehrbäck, 'Pointwise Hardy inequalities and uniformly fat sets', *Proc. Amer. Math. Soc.* 136 (2008), no. 6, 2193–2200.

J. L. Lewis, 'Uniformly fat sets', *Trans. Amer. Math. Soc.* 308 (1988), no. 1, 177–196.

J. Nečas, 'Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle', *Ann. Scuola Norm. Sup. Pisa* (3) 16 (1962), 305–326.

A. Wannebo, 'Hardy inequalities', *Proc. Amer. Math. Soc.* 109 (1990), 85–95.

The End

Thank you for your attention