Dimensions, Whitney covers, and tubular neighborhoods

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Analysis seminar, 01.10.2012, Helsinki

1. Introduction

A question

For $E \subset \mathbb{R}^d$, the open *r-neighborhood* of E is

$$E_r = \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) < r\}.$$

(aka *tubular neighborhood* or *parallel set*).

Q: How is the 'size' (and 'geometry') of E related to the 'size' of

$$\partial E_r = \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) = r\} \text{ ?}$$

(In particular, what are the right ways to measure these 'sizes'?)

It appears that ∂E_r is always(?) $(d-1)$-dimensional, so one should find estimates for $\mathcal{H}^{d-1}(\partial E_r)$ in terms of E.

Bit of history

If $d \in \{2,3\}$ and $E \subset \mathbb{R}^d$ is compact, then ∂E_r is a $(d-1)$-Lipschitz manifold for $\mathcal{H}^1$-a.e. $r \in (0,\infty)$ [Brown 1972, $d=2$; Ferry 1975, $d=3$].

For $d \geq 4$ the above fails: there exists a compact set $E \subset \mathbb{R}^d$ such that ∂E_r, for $0 < r < 1$, is never a $(d-1)$-manifold. [Ferry 1975]

Oleksiv and Pesin gave in 1985 a general estimate for $\mathcal{H}^{d-1}(\partial E_r)$, when $E \subset \mathbb{R}^d$ is bounded:

$$\mathcal{H}^{d-1}(\partial E_r) \leq \begin{cases} C_1 r^{d-1}, & \text{for } r > d(E), \\ C_2 r^{-1}, & \text{for } 0 < r \leq d(E). \end{cases}$$

Here $C_1 = C_1(d) \geq 1$ and $C_2 = C_2(d, d(E)) \geq 1$, and the growth orders are sharp. In particular $\mathcal{H}^{d-1}(\partial E_r) < \infty$ for all $r > 0$.

Reliable sources rumour that related considerations have also taken place at HUMD during the 80's.

How to prove the estimate of Oleksiv and Pesin:

$$\mathcal{H}^{d-1}(\partial E_r) \le \begin{cases} C_1 r^{d-1}, & \text{for } r > d(E), \\ C_2 r^{-1}, & \text{for } 0 < r \le d(E). \end{cases}$$

- If $r > 2d(E)$, take a ball $B \supset E$ of radius $d(E)$ and project ∂E_r to ∂B. This projection is $C(r/d(E))$-bi-Lipschitz, and thus
$$\mathcal{H}^{d-1}(\partial E_r) \lesssim (r/d(E))^{d-1}\mathcal{H}^{d-1}(\partial B) \approx r^{d-1}$$
- For $r \le 2d(E)$ cover B using balls B_i with $d(B_i) \approx r/2$, and apply the previous case for $E_i = E \cap B_i$. As $\partial E_r \subset \bigcup_i \partial(E_i)_r$ and $\# B_i \lesssim (d(E)/r)^d$, we have
$$\mathcal{H}^{d-1}(\partial E_r) \lesssim \sum_i \mathcal{H}^{d-1}(\partial(E_i)_r) \lesssim (d(E)/r)^d r^{d-1} = C(d(E))r^{-1}$$
- Such ideas appear (at least) in [Brown 1972], [Oleksiv–Pesin 1985], and [Luukkainen 1998 (with a credit to Väisälä)]

A refinement of the above idea

Count only those balls B_i which really intersect E
(→ Minkowski content/dimension)!
Or better yet, look how much of ∂E_r there is at most/at least in *Whitney-type balls (or cubes)* B of $\mathbb{R}^d \setminus E$, with radii comparable to r, and then count the total number of such balls.

Before going to the details, we recall and introduce some preliminaries. At the end, we take a look at an example, which illustrates the sharpness of our results.

So here is the plan:

- Section 1: The Introduction
- Section 2: Preliminaries (on metric spaces and dimensions)
- Section 3: Whitney ball count and dimension
- Section 4: Tubular boundaries (and spherical dimension)
- Section 5: An example

Some more recent results

For more on the 'manifold problem', see e.g. [Gariepy and Pepe 1972, Fu 1985, Rataj and Zajíček 2012]

It is also true that for all but countably many $r > 0$

$$\tfrac{d}{dr}\mathcal{H}^d(E_r) = C\mathcal{H}^{d-1}(\partial E_r);$$

[Rataj and Winter 2010] based on [Stachó 1976], and that ∂E_r is $(d-1)$-rectifiable for all $r > 0$ [RW 2010].

These observations lead to a close connection between the asymptotics of

$$\frac{\mathcal{H}^{d-1}(\partial E_r)}{r^{d-1-\lambda}} \quad \text{and} \quad \frac{\mathcal{H}^d(E_r)}{r^{d-\lambda}} \quad \text{[RW 2010].}$$

Related results will be considered in Section 4, but from a purely 'geometrical' point of view.

2. Preliminaries

A metric space (X, d) is *doubling* if there is $N = N(X) \in \mathbb{N}$ so that any closed ball $B(x, r)$ of center x and radius $r > 0$ can be covered by at most N balls of radius $r/2$.

Iteration of this condition gives $C \geq 1$ and $s > 0$ such that each ball $B(x, R)$ can be covered by at most $C(r/R)^{-s}$ balls of radius r for all $0 < r < R < \operatorname{diam}(X)$.

The infimum of such exponents s is the *(upper) Assouad dimension* $\overline{\dim}_{\mathrm{A}}(X)$; we have the upper bound $\overline{\dim}_{\mathrm{A}}(X) \leq \log_2 N$. In particular:

Lemma

A metric space X is doubling if and only if $\overline{\dim}_{\mathrm{A}}(X) < \infty$.

the lower Assouad dimension

Conversely to the definition of the upper Assouad dimension, we may also consider all $t > 0$ for which there is a constant $c > 0$ so that if $0 < r < R < \operatorname{diam}(X)$, then for every $x \in X$ at least $c(r/R)^{-t}$ balls of radius r are needed to cover $B(x, R)$. We call the supremum of all such t the *lower Assouad dimension* of X.

The restriction metric is used to define the upper and lower Assouad dimensions of a subset $E \subset X$.

Recall that a metric space X is *uniformly perfect* if there exists a constant $C \geq 1$ so that for every $x \in X$ and $r > 0$ we have $B(x, r) \setminus B(x, r/C) \neq \emptyset$ whenever $X \setminus B(x, r) \neq \emptyset$.

Lemma

A metric space X is uniformly perfect if and only if $\underline{\dim}_{\mathrm{A}}(X) > 0$.

Some examples of Assouad dimensions

General idea: Assouad dimensions reflect the 'extreme' behaviour of sets and take into account all scales $0 < r < d(E)$.

- If $E = \{0\} \cup [1,2] \subset \mathbb{R}$, then $\underline{\dim}_A(E) = 0$ and $\overline{\dim}_A(E) = 1$.
- $\underline{\dim}_A(\mathbb{Z}) = 0$ and $\overline{\dim}_A(\mathbb{Z}) = 1$.
- If $S \subset \mathbb{R}^2$ is an infinite, locally rectifiable von Koch snowflake -type curve consisting of unit intervals, then $\underline{\dim}_A(S) = 1$ and $\overline{\dim}_A(E) = \log 4/\log 3$ (flat on small scales, fractal on large scales)
- If $S \subset \mathbb{R}^2$ consists of infinitely many copies of the usual (fractal) von Koch snowflake curve, laid side by side, then $\underline{\dim}_A(S) = 1$ and $\overline{\dim}_A(E) = \log 4/\log 3$ (fractal on small scales, flat on large scales).

Metric spaces: doubling measures I

A measure μ on X is *doubling* if there is $C \geq 1$ so that $0 < \mu(2B) \leq C\mu(B)$ for all closed balls $B \subset X$.

Iterating, we find $c > 0$ and $s \geq 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq c\Big(\frac{r}{R}\Big)^s \tag{1}$$

for all $y \in B(x,R)$ and $0 < r < R < d(X)$. The infimum of s satisfying (1) is called the *upper regularity dimension* of μ, $\overline{\dim}_{\mathrm{reg}}(\mu)$.

It is easy to see that $\overline{\dim}_{\mathrm{A}}(X) \leq \overline{\dim}_{\mathrm{reg}}(\mu)$ whenever μ is doubling on X. In particular, if X has a doubling measure, then X is doubling.

Conversely, if X is doubling and complete, then there is a doubling measure μ on X [Luukkainen and Saksman 1998; Vol'berg and Konyagin 1987 (for compact sets)].

Metric spaces: doubling measures II

If X is uniformly perfect and μ is doubling then there is a converse to (1): there are $t > 0$ and $C \geq 1$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq C\Big(\frac{r}{R}\Big)^t \tag{2}$$

whenever $0 < r < R < d(X)$ and $y \in B(x,R)$. The supremum of all t satisfying (2) is called the *lower regularity dimension* of μ, $\underline{\dim}_{\mathrm{reg}}(\mu)$.

Thus $\underline{\dim}_{\mathrm{reg}}(\mu) > 0$ if μ is doubling and X is uniformly perfect, and in fact $\underline{\dim}_{\mathrm{reg}}(\mu) \leq \underline{\dim}_{\mathrm{A}}(X)$. If X is not uniformly perfect, then it is natural to define $\underline{\dim}_{\mathrm{reg}}(\mu) = 0$.

Measure μ (or the space X) is called *(Ahlfors) s-regular*, if there is $C > 0$ such that

$$\tfrac{1}{C} r^s \leq \mu(B(x,r)) \leq Cr^s$$

for every $x \in X$ and all $0 < r < d(X)$. Then $\underline{\dim}_{\mathrm{reg}}(\mu) = \overline{\dim}_{\mathrm{reg}}(\mu) = s$.

Hausdorff and Minkowski contents

The *Hausdorff (r-)content* of dimension λ is

$$\mathcal{H}^\lambda_r(E) = \inf\Big\{ \sum_k r_k^\lambda : E \subset \bigcup_k B(x_k, r_k),\ x_k \in E,\ 0 < r_k \le r \Big\},$$

and the *Minkowski (r-)content* of dimension λ is

$$\mathcal{M}^\lambda_r(E) = \inf\Big\{ N r^\lambda : E \subset \bigcup_{k=1}^N B(x_k, r),\ x_k \in E \Big\}.$$

It is immediate that $\mathcal{H}^\lambda_r(E) \le \mathcal{M}^\lambda_r(E)$ for each compact $E \subset X$.

The λ-*Hausdorff measure* of E is $\mathcal{H}^\lambda(E) = \lim_{r\to 0} \mathcal{H}^\lambda_r(E)$.

Hausdorff and Minkowski dimensions

The *Hausdorff dimension* of $E \subset X$ is

$$\dim_{\mathrm{H}}(A) = \inf\{\lambda > 0 : \mathcal{H}^{\lambda}(A) = 0\}.$$

The *lower Minkowski dimension* of $E \subset X$ is

$$\underline{\dim}_{\mathrm{M}}(E) = \inf\left\{\lambda > 0 : \liminf_{r\to 0} \mathcal{M}_r^{\lambda}(E) = 0\right\}$$

and the *upper Minkowski dimension* of $E \subset X$ is

$$\overline{\dim}_{\mathrm{M}}(E) = \inf\left\{\lambda > 0 : \limsup_{r\to 0} \mathcal{M}_r^{\lambda}(E) = 0\right\}.$$

Notice that for each compact set $E \subset X$ we have

$$\dim_{\mathrm{H}}(E) \le \underline{\dim}_{\mathrm{M}}(E) \le \overline{\dim}_{\mathrm{M}}(E),$$

where all inequalities can be strict.

Lower Assouad and Hausdorff

Lemma

If X is complete and $E \subset X$ is closed, then $\underline{\dim}_{\mathrm{A}}(E) \le \dim_{\mathrm{H}}(E \cap B)$ for all balls B centered at E.

Proof. If $0 < t_0 < \underline{\dim}_{\mathrm{A}}(E)$, then

$$\mathcal{M}_r^{t_0}(E \cap B(R)) \ge c_0 R^{t_0} \text{ for all } 0 < r < R < \operatorname{diam}(E).$$

By iteration, we find for each $0 < t < t_0$ a Cantor-type set $C \subset E \cap B$, for which the above estimate holds with the exponent t, and thus also

$$\mathcal{H}_R^t\big(E \cap B(R)\big) \ge cR^t \text{ for all } 0 < r < R < \operatorname{diam}(E) \tag{3}$$

(see [L. 2009] for details). Therefore $\dim_{\mathrm{H}}(E \cap B) \ge \dim_{\mathrm{H}}(C) \ge t$ and the claim follows.

In fact, for compact $E \subset X$ we have $\underline{\dim}_A(E) = \inf\{t > 0 : (3) \text{ holds}\}$.

(Note however that e.g. $\underline{\dim}_{\mathrm{A}}(\mathbb{Q}) = 1$ but $\dim_{\mathrm{H}}(\mathbb{Q}) = 0$)

Geometric conditions

A metric space X is *q-quasiconvex* if there exists a constant $q \geq 1$ such that for every $x, y \in X$ there is a curve $\gamma\colon [0,1] \to X$ so that $x = \gamma(0)$, $y = \gamma(1)$, and $\text{length}(\gamma) \leq q d(x,y)$.

We say that a set $E \subset X$ is *(uniformly) ϱ-porous* (for $0 \leq \varrho \leq 1$), if for every $x \in E$ and all $0 < r < d(E)$ there exists a point $y \in X$ such that $B(y, \varrho r) \subset B(x,r) \setminus E$.

If X is s-regular and complete, then $E \subset X$ is porous if and only if there are $0 < t < s$ and a t-regular set $F \subset X$ so that $E \subset F$ [JJKRRS]. In addition,

Proposition (KLV)

If X is s-regular, then there is a constant $c > 0$ such that $\overline{\dim}_{\mathrm{A}}(E) \leq s - c\varrho^s$ for all ϱ-porous sets $E \subset X$.

Assouad dimensions and geometric conditions

- A set $E \subset X$ is doubling if and only if $\overline{\dim}_{\mathsf{A}}(E) < \infty$.
- A set $E \subset X$ is uniformly perfect if and only if $\underline{\dim}_{\mathsf{A}}(E) > 0$.
- Assume that X is s-regular.
  A set $E \subset X$ is porous if and only if $\overline{\dim}_{\mathsf{A}}(E) < s$.
- If μ is a doubling measure on X, then

$$\underline{\dim}_{\mathrm{reg}}(\mu) \le \underline{\dim}_{\mathsf{A}}(X) \le \overline{\dim}_{\mathsf{A}}(X) \le \overline{\dim}_{\mathrm{reg}}(\mu).$$

- If $E \subset X$ is compact, then

$$\underline{\dim}_{\mathsf{A}}(E) \le \dim_{\mathsf{H}}(E) \le \underline{\dim}_{\mathsf{M}}(E) \le \overline{\dim}_{\mathsf{M}}(E) \le \overline{\dim}_{\mathsf{A}}(E).$$

Whitney cover

If $\Omega \subset X$ is open, we can cover Ω with a countable collection $\mathcal{W}(\Omega)$ of closed balls $B_i = B(x_i, \frac{1}{8}\operatorname{dist}(x_i, X \setminus \Omega))$, $x_i \in \Omega$, such that the overlap of these balls is uniformly bounded.

For instance, we can use the $5r$-covering lemma for the sets

$$\{x \in \Omega : 2^{-k-1} \leq \operatorname{dist}(x, X \setminus \Omega) < 2^{-k}\}, \quad k \in \mathbb{Z}.$$

One can use any $0 < \delta \leq \frac{1}{2}$ instead of $\frac{1}{8}$ above, but for large δ some modifications in some of our results are necessary.

For $k \in \mathbb{Z}$ and $A \subset X$ we set

$$\mathcal{W}_k(\Omega; A) = \{B(x_i, r_i) \in \mathcal{W}(\Omega) : 2^{-k-1} < r_i \leq 2^{-k} \text{ and } A \cap B(x_i, r_i) \neq \emptyset\}$$

and $\mathcal{W}_k(\Omega) = \mathcal{W}_k(\Omega; X)$.

3. Whitney ball count and dimension

Background and motivating questions

In [Martio–Vuorinen 1987], the relation between upper Minkowski dimension and upper bounds for Whitney *cube* count was considered for compact $E \subset \mathbb{R}^d$. In particular, it was shown that if $\mathcal{H}^d(E) = 0$, then

$$\overline{\dim}_{\mathrm{M}}(E) = \inf\{\lambda \geq 0 : \#\mathcal{W}_k^C(\mathbb{R}^d \setminus E) \leq C2^{\lambda k} \text{ for all } k \geq k_0\},$$

or, equivalently, $\overline{\dim}_{\mathrm{M}}(E) = \limsup_{k\to\infty} \frac{1}{k} \log_2 \#\mathcal{W}_k^C(\mathbb{R}^d \setminus E)$.

The following questions are now relevant:

- Does this hold in metric spaces for Whitney balls?
- Does something similar hold for lower Minkowski dimension?
- Does something similar hold for Assouad dimensions? Local Whitney ball count?

From now on, X is a doubling metric space.

Upper bound for Whitney ball count..

Lemma

Let $E \subset X$ be closed set and fix $0 < \delta < 1$. If $B_0 = B(w, R)$ with $w \in E$, $0 < r < R$, and $\{B(w_j, r)\}_{j=1}^{N}$, $w_j \in E$, is a cover of $E \cap 2B_0$, then $\#\mathcal{W}_k(X \setminus E; B_0) \leq CN$ for all $\delta r \leq 2^{-k} \leq r$. (Here $C = C(X, \delta)$.)

Idea: If $B(x, r') \in \mathcal{W}_k(X \setminus E; B_0)$ then $B(x, r') \subset B(w_j, 10r)$ for some j. It follows (with a rather simple argument using doubling and the bounded overlap of $\mathcal{W}$-balls) that $\#\mathcal{W}_k\big(X \setminus E; B_0 \cap B(w_j, 10r)\big) \leq C\delta^{-s}$, where $s > \overline{\dim}_A(X)$.

Since each ball in $\mathcal{W}_k\big(X \setminus E; B_0\big)$ is in some $B(w_j, 10r)$, we conclude

$$\#\mathcal{W}_k(X \setminus E; B_0) \leq \sum_{j=1}^{N} \#\mathcal{W}_k\big(X \setminus E; B_0 \cap B(w_j, 10r)\big) \leq CN\delta^{-s}.$$

..and consequences of $\#\mathcal{W}_k(X \setminus E; B_0) \le CN$

- If $E \subset X$ is closed and $\overline{\dim}_{\mathrm{A}}(E) < \lambda$, then

$$\#\mathcal{W}_k(X \setminus E; B_0) \le C 2^{\lambda k} R^{\lambda}$$

for all $B_0 = B(w, R)$, with $0 < R < d(E)$ and $w \in E$, $k > -\log_2 R$.

- If $E \subset X$ is compact and $\overline{\dim}_{\mathrm{M}}(E) < \lambda$ (or $\limsup_{r \to 0} \mathcal{M}_r^{\lambda}(E) < \infty$) then $\#\mathcal{W}_k(X \setminus E) \le C 2^{\lambda k}$ for all $k \ge k_0$.
- If $E \subset X$ is closed and for each $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$

$$\#\mathcal{W}_k(X \setminus E; B_0) \ge c 2^{\lambda k} R^{\lambda}$$

for all $k \ge -\log_2 R + \ell$, then $\underline{\dim}_{\mathrm{A}}(E) \ge \lambda$.

- If $E \subset X$ is compact and $\#\mathcal{W}_k(X \setminus E) \ge c 2^{\lambda k}$ for all $k \ge k_0$, then $\underline{\dim}_{\mathrm{M}}(E) \ge \lambda$ (in fact $\liminf_{r \to 0} \mathcal{M}_r^{\lambda}(E) > 0$)

Lower bound for ball count..

Lemma

Assume that X is q-quasiconvex and $E \subset X$ is closed and ϱ-porous. Then there is $c > 0$ such that if $B_0 = B(w, R)$ with $0 < R < \operatorname{diam}(E)$ and $w \in E$, $0 < r < R/2q$, and $\{B(w_j, r/2)\}_{j=1}^N$, $w_j \in E$, is a maximal packing of $E \cap \frac{1}{2}B_0$, then $\#\mathcal{W}_k(X \setminus E; B_0) \geq cN$, where $k \in \mathbb{Z}$ is such that $\varrho r/10 < 2^{-k} \leq \varrho r/5$.

Idea: By porosity, there is $y_j \in B(w_j, r)$ satisfying $\operatorname{dist}(y_j, E) \geq \varrho r$. By quasiconvexity, there is $\gamma_j \colon [0,1] \to B(w_j, qr)$ connecting y_j and w_j. By continuity, find $x_j \in \gamma_j([0,1])$ with $\operatorname{dist}(x_j, E) = 5 \cdot 2^{-k} \leq \varrho r$. Then $x_j \in B(z_j, r_j) \in \mathcal{W}_k(X \setminus E; B_0)$, where $2^{-k-1} < r_j \leq 2^{-k}$. Since the balls $\{B(w_j, r/2)\}_{j=1}^N$ are pairwise disjoint, the overlap of the balls $\{B(w_j, qr + \varrho r)\}_{j=1}^N$ is uniformly bounded by M (by doubling). Since each ball $B(w_j, qr + \varrho r)$ contains a ball from $\mathcal{W}_k(X \setminus E; B_0)$, we conclude that $N \leq M\#\mathcal{W}_k(X \setminus E; B_0)$.

..and consequences of $\#\mathcal{W}_k(X \setminus E; B_0) \geq cN$

- If $E \subset X$ (here X is q-convex) is closed, porous, and $\underline{\dim}_{\mathrm{A}}(E) > \lambda$, then
$$\#\mathcal{W}_k(X \setminus E; B_0) \geq c 2^{\lambda k} R^{\lambda}$$
for all $B_0 = B(w, R)$, with $0 < R < d(E)$ and $w \in E$, and all $k > -\log_2 R + \ell$.
- If $E \subset X$ is compact, porous, and $\underline{\dim}_{\mathrm{M}}(E) > \lambda$ ($\liminf_{r \to 0} \mathcal{M}_r^{\lambda}(E) > 0$) then $\#\mathcal{W}_k(X \setminus E) \geq c 2^{\lambda k}$ for all $k \geq k_0$.
- If $E \subset X$ is closed, porous, and for all $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$
$$\#\mathcal{W}_k(X \setminus E; B_0) \leq C 2^{\lambda k} R^{\lambda},$$
and for all $k \geq -\log_2 R$, then $\overline{\dim}_{\mathrm{A}}(E) \leq \lambda$.
- If $E \subset X$ is compact and $\#\mathcal{W}_k(X \setminus E) \leq C 2^{\lambda k}$ for all $k \geq k_0$, then $\overline{\dim}_{\mathrm{M}}(E) \leq \lambda$ (in fact $\limsup_{r \to 0} \mathcal{M}_r^{\lambda}(E) < \infty$)

Characterization for Minkowski dimensions

If $E \subset X$ is compact (and X quasiconvex), then

- $\overline{\dim}_M(E) < \lambda \implies \#\mathcal{W}_k(X \setminus E) \leq C 2^{\lambda k}$ for all $k \geq k_0$.
- $\#\mathcal{W}_k(X \setminus E) \geq c 2^{\lambda k}$ for all $k \geq k_0 \implies \underline{\dim}_M(E) \geq \lambda$.
- $\underline{\dim}_M(E) > \lambda \implies \#\mathcal{W}_k(X \setminus E) \geq c 2^{\lambda k}$ for all $k \geq k_0$ if E is porous.
- $\#\mathcal{W}_k(X \setminus E) \leq C 2^{\lambda k}$ for all $k \geq k_0 \implies \overline{\dim}_M(E) \leq \lambda$ if E is porous.

In particular, if X is quasiconvex and $E \subset X$ is compact and porous, then

$$\overline{\dim}_M(E) = \limsup_{k\to\infty} \tfrac{1}{k} \log_2 \#\mathcal{W}_k(X \setminus E),$$
$$\underline{\dim}_M(E) = \liminf_{k\to\infty} \tfrac{1}{k} \log_2 \#\mathcal{W}_k(X \setminus E).$$

The porosity assumption is more or less crucial here (cf. the example of Section 5). However, if X is s-regular, then the characterization of the upper Minkowski dimension holds under weaker assumptions (we will get back to this soon).

Non-quasiconvex case and Euclidean Whitney balls

Quasiconvexity (as such) is not that essential in the previous results; in particular, the existence of rectifiable curves is not necessary. Even without any local connectivity properties, we have (for instance) the following:

If $E \subset X$ is compact and ϱ-porous, there is $\ell \in \mathbb{N}$ (depending on ϱ) such that if $\underline{\dim}_{\mathrm{M}}(E) > \lambda$, then

$$\sum_{j=k}^{k+\ell} \#\mathcal{W}_j(X \setminus E; B_0) \geq c2^{\lambda k} \text{ for all } k \geq k_0.$$

Actually, a similar modification is needed for the Euclidean Whitney cube decomposition $\mathcal{W}^C(\mathbb{R}^d \setminus E)$ (with $d(Q) \leq d(Q, \mathbb{R}^d \setminus E) \leq 4d(Q)$), where certain (but not two consecutive) generations of cubes may be 'missing'. For instance: if $E \subset \mathbb{R}^d$ is compact and porous, then

$$\underline{\dim}_{\mathrm{M}}(E) = \liminf_{k\to\infty} \tfrac{1}{k} \log_2 \#\Big(\mathcal{W}_k^C(\mathbb{R}^d \setminus E) \cup \mathcal{W}_{k+1}^C(\mathbb{R}^d \setminus E)\Big).$$

Upper dimensions in s-regular space

Under the existence of an s-regular measure μ on X, we can slightly improve the previous results:

- If $E \subset X$ is closed, $\mu(E) = 0$, and for all $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$ we have $\#\mathcal{W}_k(X \setminus E; B_0) \le C2^{\lambda k} R^{\lambda}$ for all $k \ge -\log_2 R$, then $\overline{\dim}_{\mathrm{A}}(E) \le \lambda$.
- If $E \subset X$ is compact, $\mu(E) = 0$, and $\#\mathcal{W}_k(X \setminus E) \le C2^{\lambda k}$ for all $k \ge k_0$, then $\overline{\dim}_{\mathrm{M}}(E) \le \lambda$.

The condition $\mu(E) = 0$ can not be omitted; consider $B(0,1) \subset \mathbb{R}^n$.

Since always $\overline{\dim}_{\mathrm{M}}(E) \ge \limsup_{k\to\infty} \frac{1}{k} \log_2 \#\mathcal{W}_k(X \setminus E)$, we conclude that if $\mu(E) = 0$, then $\overline{\dim}_{\mathrm{M}}(E) = \limsup_{k\to\infty} \frac{1}{k} \log_2 \#\mathcal{W}_k(X \setminus E)$. (In $\mathbb{R}^d$, this follows from [MV 1987].)

If μ is a non-regular (but doubling) measure on X, then we obtain a weaker result for Minkowski and Assouad *codimensions*.

Proof of the s-regular case

Fix a ball $B_0 = B(w, R)$ with $0 < R < d(E)$ and $w \in E$, and take $k_1 \in \mathbb{Z}$ such that $2^{-k_1} \le r < 2^{-k_1+1}$. Since $E_r \cap B_0 \subset E \cup \bigcup_{k=k_1}^{\infty} \mathcal{W}_k(X \setminus E; B_0)$ and $\mu(B) \approx 2^{-sk}$ for $B \in \mathcal{W}_k(X \setminus E; B_0)$, we obtain

$$
\begin{aligned}
\mu(E_r \cap B_0) &\le \mu(E) + C \sum_{k=k_1}^{\infty} \# \mathcal{W}_k(X \setminus E; B_0) 2^{-sk} \\
&\le C \sum_{k=k_1}^{\infty} 2^{(\lambda - s)k} R^{\lambda} \le C 2^{-k_1(s-\lambda)} R^{\lambda} \approx r^{s-\lambda} R^{\lambda}
\end{aligned}
$$

(we may assume $\lambda < s$). Using the s-regularity and considering maximal packings, it follows that $E \cap B_0$ can be covered by $C(r/R)^{-\lambda}$ balls of radius r, and thus $\overline{\dim}_{\mathrm{A}}(E) \le \lambda$.

For the upper Minkowski dimension, the claim follows with a similar computation.

4. Tubular boundaries and spherical dimension

Minkowski dimension in $\mathbb{R}^d$

In $\mathbb{R}^d$ (or in fact in any d-regular space) the Minkowski dimensions of a compact $E \subset \mathbb{R}^d$ can be defined equivalently as

$$\underline{\dim}_{\mathrm{M}}(E) = \inf \left\{ \lambda \geq 0 : \liminf_{r \downarrow 0} \frac{\mathcal{H}^d(E_r)}{r^{d-\lambda}} = 0 \right\}$$

and

$$\overline{\dim}_{\mathrm{M}}(E) = \inf \left\{ \lambda \geq 0 : \limsup_{r \downarrow 0} \frac{\mathcal{H}^d(E_r)}{r^{d-\lambda}} = 0 \right\}.$$

If $\mathcal{H}^d(E) = 0$, then for $\overline{\dim}_{\mathrm{M}}(E)$ we can replace E_r by $E_{2r} \setminus E_r$; for $\underline{\dim}_{\mathrm{M}}(E)$ we need in addition that E is porous.

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But what happens if we replace $E_{2r} \setminus E_r$ by ∂E_r ?

Spherical dimension

Rataj and Winter defined the *lower spherical dimension* of a compact $E \subset \mathbb{R}^d$ as

$$\underline{\dim}_{\mathrm{S}}(E) = \inf\{\lambda \geq 0 : \liminf_{r \downarrow 0} \frac{\mathcal{H}^{d-1}(\partial E_r)}{r^{d-1-\lambda}} = 0\}$$

and the *upper spherical dimension* as

$$\overline{\dim}_{\mathrm{S}}(E) = \inf\{\lambda \geq 0 : \limsup_{r \downarrow 0} \frac{\mathcal{H}^{d-1}(\partial E_r)}{r^{d-1-\lambda}} = 0\}.$$

If $\mathcal{H}^d(E) = 0$, then actually $\overline{\dim}_{\mathrm{S}}(E) = \overline{\dim}_{\mathrm{M}}(E)$, but

$$\tfrac{d-1}{d}\,\underline{\dim}_{\mathrm{M}}(E) \leq \underline{\dim}_{\mathrm{S}}(E) \leq \underline{\dim}_{\mathrm{M}}(E), \tag{4}$$

where the bounds are sharp (Winter: '$<$' sharp in the lower bound; KLV: can have '$=$' in the lower bound)

Spherical dimension: our contribution

Theorem

If $E \subset \mathbb{R}^d$ is a compact set, then

$$\underline{\dim}_S(E) = \liminf_{k\to\infty} \tfrac{1}{k}\log_2 \#\mathcal{W}_k(\mathbb{R}^d \setminus E)$$

$$\overline{\dim}_S(E) = \limsup_{k\to\infty} \tfrac{1}{k}\log_2 \#\mathcal{W}_k(\mathbb{R}^d \setminus E).$$

Corollary

If $E \subset \mathbb{R}^d$ is compact and porous, then $\underline{\dim}_S(E) = \underline{\dim}_M(E)$ (and if $\mathcal{H}^d(E) = 0$, then $\overline{\dim}_S(E) = \overline{\dim}_M(E)$ [RW]).

Proposition

For each $d \in \mathbb{N}$ there exists a compact set $E \subset \mathbb{R}^d$ with $\mathcal{H}^d(E) = 0$, $\underline{\dim}_M(E) = d$, and $\underline{\dim}_S(E) = d - 1$.

Main geometric lemmas

Lemma (1)

If $E \subset \mathbb{R}^d$ *is a closed set,* $k \in \mathbb{Z}$*, and* $B \in \mathcal{W}_k(\mathbb{R}^d \setminus E)$*, then*

$$\mathcal{H}^{d-1}(\partial E_r \cap B) \leq C 2^{-k(d-1)}$$

for all $r > 0$*, where* $C \geq 1$ *depends only on* d*.*

Lemma (2)

If $E \subset \mathbb{R}^d$ *is a closed set,* $k \in \mathbb{Z}$*, and* $B \in \mathcal{W}_k(\mathbb{R}^d \setminus E)$*, then*

$$\mathcal{H}^{d-1}(\partial E_r \cap 8B) \geq c r^{d-1}$$

for all $2^{-k-1} \leq r \leq 2^{-k}$*, where* $c > 0$ *depends only on* d*.*

Main estimates for $\mathcal{H}^{d-1}(\partial E_r)$

Let $E \subset \mathbb{R}^d$ be a closed set, and let B_0 be a closed ball centered at E. If $k \in \mathbb{Z}$, and $2^{-(k+1)} < r \leq 2^{-k}$, then

$$\mathcal{H}^{d-1}(\partial E_r \cap B_0) \leq C r^{d-1} \sum_{j=k+2}^{k+4} \# \mathcal{W}_j(\mathbb{R}^d \setminus E; B_0),$$

and

$$\mathcal{H}^{d-1}(\partial E_r \cap 3B_0) \geq c r^{d-1} \# \mathcal{W}_k(\mathbb{R}^d \setminus E; B_0),$$

where $C \geq 1$ and $c > 0$ depend only on d.

In particular, for each compact set $E \subset \mathbb{R}^d$

$$c r^{d-1} \# \mathcal{W}_k(\mathbb{R}^d \setminus E) \leq \mathcal{H}^{d-1}(\partial E_r) \leq C r^{d-1} \sum_{j=k+2}^{k+4} \# \mathcal{W}_j(\mathbb{R}^d \setminus E),$$

where $2^{-(k+1)} < r \leq 2^{-k}$, and the constants $c, C \geq 0$ depend only on the dimension d. The characterizations of spherical dimensions follow.

Proofs of the main estimates

(1) Let $k \in \mathbb{Z}$ and $2^{-(k+1)} < r \leq 2^{-k}$. If $B = B(x, r_0) \in \mathcal{W}_j(\mathbb{R}^d \setminus E; B_0)$ and $\partial E_r \cap B \neq \emptyset$, then $2^{-j-1} < r_0 \leq r/7 < 2^{-k-2}$ and $2^{-k-5} < r/9 \leq r_0 \leq 2^{-j}$. Thus

$$\partial E_r \cap B_0 \subset \bigcup_{j=k+2}^{k+4} \mathcal{W}_j(\mathbb{R}^d \setminus E; B_0)$$

and, consequently, by Lemma (1),

$$\begin{aligned}\mathcal{H}^{d-1}(\partial E_r \cap B_0) &\leq \sum_{j=k+2}^{k+4} \sum_{B \in \mathcal{W}_j(\mathbb{R}^d \setminus E; B_0)} \mathcal{H}^{d-1}(\partial E_r \cap B) \\ &\leq C \sum_{j=k+2}^{k+4} \#\mathcal{W}_j(\mathbb{R}^d \setminus E; B_0) 2^{-j(d-1)}.\end{aligned}$$

(2) Let $k \in \mathbb{N}$, and $2^{-k-1} < r \leq 2^{-k}$. The overlap of the balls $8B$, for $B \in \mathcal{W}_k(\mathbb{R}^d \setminus E; B_0)$, is uniformly bounded by a constant $C_1 \geq 1$. Moreover, we have for these balls that $8B \subset 3B_0$. Thus Lemma (2) yields that

$$\begin{aligned} \mathcal{H}^{d-1}(\partial E_r \cap 3B_0) &\geq C \sum_{B \in \mathcal{W}_k(\mathbb{R}^d \setminus E; B_0)} \mathcal{H}^{d-1}(\partial E_r \cap 8B) \\ &\geq C r^{d-1} \# \mathcal{W}_k(\mathbb{R}^d \setminus E; B_0), \end{aligned}$$

as desired.

Conclusion for Minkowski contents

Proposition

(1) If $E \subset \mathbb{R}^d$ *is compact and* $\lambda \geq 0$*, then for all* $r > 0$

$$\mathcal{H}^{d-1}(\partial E_r) \leq C r^{d-1-\lambda} \mathcal{M}_r^{\lambda}(E)$$

(2) If $E \subset \mathbb{R}^d$ *is compact and* ϱ*-porous, and* $\lambda \geq 0$*, then for all* $0 < r < \varrho \operatorname{diam}(E)/5$

$$\mathcal{H}^{d-1}(\partial E_r) \geq c r^{d-1-\lambda} \mathcal{M}_{10r/\varrho}^{\lambda}(E)$$

Corollary

If $E \subset \mathbb{R}^d$ *is compact and* s*-regular for* $0 < s < d$*, then*

$$c r^{d-1-s} \leq \mathcal{H}^{d-1}(\partial E_r) \leq C r^{d-1-s} \qquad \text{for all } 0 < r < r_0.$$

Conclusion for Assouad dimensions

Here $E \subset \mathbb{R}^d$ is closed and $B_0 = B(w, R)$, with $0 < R < d(E)$ and $w \in E$.

Corollary

(1) $\overline{\dim}_A(E) < \lambda$

$\implies \quad \mathcal{H}^{d-1}(\partial E_r \cap B_0) \le C r^{d-1}(r/R)^{-\lambda}$ *for all* B_0, $0 < r < R$.

(2) $\mathcal{H}^{d-1}(\partial E_r \cap B_0) \ge c r^{d-1}(r/R)^{-\lambda}$ *for all* B_0, $0 < r < \delta R$

$\implies \underline{\dim}_A(E) \ge \lambda$.

(3) If $\mathcal{H}^d(E) = 0$, *then*

$\mathcal{H}^{d-1}(\partial E_r \cap B_0) \le C r^{d-1}(r/R)^{-\lambda}$ *for all* B_0, $0 < r < R$

$\implies \quad \overline{\dim}_A(E) \le \lambda$.

(4) If E *is porous, then* $\underline{\dim}_A(E) > \lambda$

$\implies \quad \mathcal{H}^{d-1}(\partial E_r \cap B_0) \ge c r^{d-1}(r/R)^{-\lambda}$ *for all* B_0, $0 < r < \delta R$.

5. An example

The goal and the idea of the construction

We construct a set $E \subset \mathbb{R}^2$ with $\mathcal{H}^2(E) = 0$ and $\dim_{\mathrm{H}}(E) = \underline{\dim}_{\mathrm{M}}(E) = 2$, but $\underline{\dim}_{\mathrm{S}}(E) = 1$. This example can be easily generalized to all $\mathbb{R}^d$, $d \geq 1$, with dimensions $\underline{\dim}_{\mathrm{M}}(E) = d$ and $\underline{\dim}_{\mathrm{S}}(E) = d - 1$.

(Such E is necessarily non-porous).

The idea is to use a typical 'alternating' Cantor-type construction, where we have

(a) 'thick' generations of squares which guarantee the loss of porosity and give Minkowski dimension 2 for the resulting set E

and

(b) 'thin' generations which make E to be of zero measure (but not too thin so that $\dim_{\mathrm{H}}(E) = 2$).

Details I

We use the following λ-operation:

(λ) If $\mathcal{Q}$ is a collection of rectangles, we replace each $Q \in \mathcal{Q}$ by four rectangles of side-length $\lambda\ell(Q)$ placed in the corners of Q.

Let $\Lambda = (\lambda_j)_{j=1}^{\infty}$, with $\lambda_j = \frac{1}{2}$ for odd j and $\frac{1}{4} \le \lambda_j = (\frac{1}{2})^{1+1/j} < \frac{1}{2}$ for even j. Let $(s_j)_{j=1}^{\infty}$ be such that $s_j > 1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} s_j = 1$. We choose $(n_j)_{j=1}^{\infty}$, $n_j \in \mathbb{N}$, to be such that n_{j+1} is much bigger than $\sum_{i=1}^{j} n_i$.

Set $\mathcal{Q}_0 = \{[0,1]^2\}$ and for each $j \in \mathbb{N}$ construct $\mathcal{Q}_j$ recursively from $\mathcal{Q}_{j-1}$ by applying the λ_j-operation n_j times. Then $\bigcup_{Q\in\mathcal{Q}_j} Q = \bigcup_{Q\in\mathcal{Q}_{j-1}} Q$, but $\#\mathcal{Q}_j = 4^{n_j}\#\mathcal{Q}_{j-1}$ for all odd j. Define $E = \bigcap_{j=1}^{\infty}\bigcup_{Q\in\mathcal{Q}_j} Q$.

For odd j the λ_j-construction would produce a 2-dimensional set and for even j a Cantor set of dimension $\nu_j \nearrow 2$. Thus, if n_j is chosen large enough (depending on Λ and $n_1, \dots, n_{j-1}$), it should be clear that $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{M}}(E) = 2$.

Details II

When j is even, then the distance between two cubes in $\mathcal{Q}_j$ is at least $D_j = \lambda_j^{-1}\ell_j - 2\ell_j = \ell_j(\lambda_j^{-1} - 2) > 0$. Choose $d_j = \min\{D_j/3, (\#\mathcal{Q}_j\ell_j)^{-1/(s_j-1)}\} > 0$. If we take n_{j+1} (depending on Λ, (s_j), and $n_1, \dots, n_j$) to be large enough, the ratio ℓ_{j+1}/d_j is as small as we wish. Thus we have for all $d_j/2 < r < d_j$

$$\frac{\mathcal{H}^1(\partial E_r)}{r^{2-1-s_j}} \approx \#\mathcal{Q}_j\ell_j d_j^{s_j-1} \leq 1,$$

and so the desired estimate $\underline{\dim}_{\mathsf{S}}(E) \leq s_j \to 1$ follows.

Finally, $\mathcal{H}^2(E) = 0$, since for even j $\mathcal{H}^2(E) \leq \sum_{Q\in\mathcal{Q}_j} \ell(Q)^2 = \left(\prod_{i=1}^{j-1}(4\lambda_i^2)^{n_i}\right)(4\lambda_j^2)^{n_j} \leq (4\lambda_j^2)^{n_j}$, and here $4\lambda_j^2 < 1$ and n_j can be chosen as large as we want.

Some questions of Winter

In Remark 2.4 of [Winter 2011] the following questions were asked/indicated:,

- Is there $E \subset \mathbb{R}^d$ with $\mathcal{H}^d(E) = 0$ and $\underline{\dim}_{\mathrm{S}}(E) = \frac{d-1}{d}\,\underline{\dim}_{\mathrm{M}}(E)$?
- If $\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E)$, is $\underline{\dim}_{\mathrm{S}}(E) = \dim_{\mathrm{M}}(E)$?
- If $\underline{\dim}_{\mathrm{M}}(E) = \underline{\dim}_{\mathrm{S}}(E)$, is $\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E)$?

Some questions of Winter

In Remark 2.4 of [Winter 2011] the following questions were asked/indicated:,

- Is there $E \subset \mathbb{R}^d$ with $\mathcal{H}^d(E) = 0$ and $\underline{\dim}_S(E) = \frac{d-1}{d}\,\underline{\dim}_M(E)$?
  Yes! by our example; here $\underline{\dim}_S(E) = d - 1$ and $\underline{\dim}_M(E) = d$
- If $\underline{\dim}_M(E) = \overline{\dim}_M(E)$, is $\underline{\dim}_S(E) = \dim_M(E)$?
  No! by our example. Here $\dim_M(E) = d$. Are there examples with $\dim_M(E) < d$? In a recent preprint, Rataj and Winter show that if $0 < \liminf_{r\to 0} \mathcal{M}_r^\lambda(E) \le \limsup_{r\to 0} \mathcal{M}_r^\lambda(E) < \infty$, then $\underline{\dim}_S(E) = \dim_M(E)\,(= \overline{\dim}_S(E)\,) = \lambda$.
- If $\underline{\dim}_M(E) = \underline{\dim}_S(E)$, is $\underline{\dim}_M(E) = \overline{\dim}_M(E)$?
  No! Construct a compact and porous set E with $\underline{\dim}_M(E) < \overline{\dim}_M(E)$. Then $\underline{\dim}_S(E) = \underline{\dim}_M(E) < \overline{\dim}_M(E) = \overline{\dim}_S(E)$.

What happens at $d-1$?

In all the known examples of $E \subset \mathbb{R}^d$ with $\underline{\dim}_S(E) < \underline{\dim}_M(E)$, we have $\underline{\dim}_S(E) \geq d - 1$.

Is this essential? (I claim that it is.)

But why? And what really happens below $d - 1$?

Please tell me, if you have an idea.

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