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Asymptotic behavior of viscosity solutions for a degenerate parabolic equation associated with the infinity-Laplacian

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Abstract. The asymptotic behavior of viscosity solutions to the Cauchy-Dirichlet problem for the degenerate parabolic equation $u_t = \Delta_\infty u$ in $\Omega \times (0, \infty)$, where Δ_∞ stands for the so-called infinity-Laplacian, is studied in three cases: (i) $\Omega = \mathbb{R}^N$ and the initial data has a compact support; (ii) Ω is bounded and the boundary condition is zero; (iii) Ω is bounded and the boundary condition is non-zero. Our method of proof is based on the comparison principle and barrier function arguments. Explicit representations of separable type and self-similar type of solutions are also established. Moreover, in case (iii), we propose another type of barrier function deeply related to a solution of $\Delta_\infty \phi = 0$.

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1. Introduction

In [2], Aronsson introduced a nonlinear elliptic differential operator Δ_∞ defined by

$$\Delta_\infty u(x) := \left\langle D^2u(x)Du(x), Du(x)\right\rangle = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x).$$

This operator, nowadays known as the *infinity-Laplacian*, was proposed to describe the Euler equation of an L^∞-variational problem related to minimal Lipschitz extensions of functions defined on the boundary $\partial\Omega$ of a domain Ω in $\mathbb{R}^N$ (see also [4]). More precisely, a function $\phi \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ is called an *absolutely minimizing Lipschitz extension* (AMLE for short) of a function $\varphi : \partial\Omega \to \mathbb{R}$ into Ω, if $\phi = \varphi$ on $\partial\Omega$ and

$$|D\phi|_{L^\infty(U)} \le |Dw|_{L^\infty(U)} \tag{1}$$

for every open subset U of Ω and $w \in W^{1,\infty}(U) \cap C(\overline{U})$ satisfying $w = \phi$ on ∂U. Here (1) is regarded as a variational problem in L^∞. Aronsson [2] proposed the following Dirichlet problem:

$$\Delta_\infty \phi = 0 \text{ in } \Omega, \tag{2}$$

$$\phi = \varphi \text{ on } \partial\Omega \tag{3}$$

as an Euler equation for the variational problem. He also proved the equivalence between smooth AMLEs of φ into Ω and classical solutions of (2), (3).

In [3], it is proved that if ϕ is a non-constant classical solution of (2), (3), then $|D\phi| > 0$ in Ω for the case of $N = 2$. Furthermore, this fact yields a simple counter-example to the existence of classical solutions of (2) (see p.55 of [11]). Jensen [11] introduced a weak formulation of the Dirichlet problem (2), (3) by using the notion of viscosity solutions and also proved:

Theorem 1 (Jensen [11]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $\varphi \in C(\partial\Omega)$. Then there exists a unique viscosity solution $\phi \in C(\overline{\Omega})$ of* (2), (3)*. Moreover, if φ is Lipschitz continuous on $\partial\Omega$, then $\phi \in W^{1,\infty}(\Omega)$.*

The equivalence between AMLEs of φ into Ω and viscosity solutions of (2), (3) is also proved in [11], provided that φ is Lipschitz continuous on $\partial\Omega$. Moreover, many authors have studied the elliptic problem (2) from various points of view (see, e.g., [4] and the references therein).

On the other hand, only a few authors have dealt with parabolic equations involving the infinity-Laplacian (see [1], [13]). Akagi-Suzuki [1] proved the Lipschitz regularity of viscosity solutions $u = u(x,t)$ as well as the well-posedness for the following Cauchy-Dirichlet problem:

$$u_t = \Delta_\infty u \text{ in } Q := \Omega \times (0,\infty), \tag{4}$$

$$u = \varphi \text{ on } \partial\Omega \times (0,\infty), \tag{5}$$

$$u = u_0 \text{ on } \Omega \times \{0\}, \tag{6}$$

where $u_t = \partial u/\partial t$. If $\Omega = \mathbb{R}^N$, we ignore (5). Throughout this paper, we assume that $u_0 \in C(\overline{\Omega})$, $\varphi \in C(\partial\Omega)$ and $u_0 = \varphi$ on $\partial\Omega$ to let a solution belong to $C(\overline{\Omega} \times [0,\infty))$.

The aim of this paper is to reveal the asymptotic behavior of viscosity solutions for (4)–(6) as $t \to \infty$ in the following three cases: (i) $\Omega = \mathbb{R}^N$ and u_0 has a compact support; (ii) Ω is a bounded domain and $\varphi \equiv 0$ (homogeneous Dirichlet case); (iii) Ω is a bounded domain and $\varphi \not\equiv 0$ (inhomogeneous Dirichlet case); notice that we have assumed φ to be independent of the time variable.

Our method of proof relies on barrier function arguments. For parabolic equations involving elliptic operators in divergence form such as p-Laplace operator $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$, the energy method is an effective tool in analyzing the asymptotic behavior of solutions. However, this method can not be directly applied to our problem, since the infinity-Laplacian is not in divergence form. Hence we employ a barrier function argument instead of the usual energy method.

In case (i), where $\Omega = \mathbb{R}^N$ and $u_0 \in C_0(\mathbb{R}^N)$, we propose a self-similar viscosity solution $B(x,t)$ of (4) in $\mathbb{R}^N \times (0,\infty)$ and derive an optimal decay rate ($= O(t^{-1/6})$) in supremum norm for every bounded viscosity solution $u(x,t)$ as $t \to \infty$ by using $B(x,t)$ and the comparison principle.

In case (ii), where Ω is bounded and $\varphi \equiv 0$, we show the explicit representation of viscosity solutions for (4) in the separable form: $V(x,t) = \rho(t)\psi(|x|)$, where ψ is a solution of

$$\frac{d}{d\xi}(\psi'(\xi)^3) + \frac{3}{2}\psi(\xi) = 0, \qquad \xi \in \mathbb{R}.$$

Using $V(x,t)$ instead of $B(x,t)$, we derive an optimal decay rate ($= O(t^{-1/2})$) for every viscosity solution $u(x,t)$ as $t \to \infty$ in case Ω is bounded. Moreover, we verify that the same conclusion remains valid even if Ω is unbounded but bounded in at least one direction.

We emphasize that these optimal decay rates are independent of the dimension N. This fact stems from the severe degeneracy of the infinity-Laplacian. More precisely, diffusion occurs only in the direction of Du, because the rank of the diffusion matrix $Du \otimes Du$ is equal to 1 even if $Du \neq 0$, and its eigenvectors of the largest eigenvalue $|Du|^2$ are parallel to Du. Thus the diffusion described by (4) seems one-dimensional (see also Remark 2). Recall that, in contrast, the decay rate for the Gauss kernel in $\mathbb{R}^N$ is $O(t^{-N/2})$, and for the p-Laplace parabolic equation $u_t = \operatorname{div}(|Du|^{p-2}Du)$, the so-called Barenblatt solution in $\mathbb{R}^N$ decays to zero in supremum norm at the explicit rate $O(t^{-N/(Np-2N+p)})$ as $t \to \infty$.

The case (iii), where Ω is bounded and $\varphi \not\equiv 0$, is naturally motivated by the connection between the AMLE of φ and the solution of the Dirichlet problem (2), (3), and moreover, by the fact that the main interest of finding AMLEs is in this case, because the AMLE of $\varphi \equiv 0$ is trivial. We prove that the unique solution $u(x,t)$ of the Cauchy-Dirichlet problem (4)–(6) converges to the solution ϕ of (2), (3), i.e., the AMLE of the boundary value φ into Ω, uniformly on $\overline{\Omega}$ as $t \to \infty$ with estimates for the convergence rate. This result provides a continuous deformation $(\phi_t)_{t\in[0,1]}$ between an arbitrary Lipschitz continuous function u_0 on $\overline{\Omega}$ and the AMLE of $\varphi = u_0|_{\partial\Omega}$ such that $t \mapsto |D\phi_t|_{L^\infty(\Omega)}$ is non-increasing (see also [1]) and the error can be estimated. Furthermore, evolution equations involving the infinity-Laplacian have appeared in numerical schemes designed for computing AMLEs (see [5]). The estimates for the rate of convergence yield an estimate for the number of steps needed in the scheme.

We first prove that a viscosity solution $u(\cdot,t)$ of (4)–(6) exponentially converges to a solution ϕ of the stationary problem (2), (3) as $t \to \infty$, provided that $\inf_{x\in\Omega} |D\phi(x)| > 0$ in the viscosity sense. We also establish a lower estimate for the convergence rate of $u(\cdot,t)$ as $t \to \infty$ in a special setting. Finally, the convergence of $u(\cdot,t)$ is verified at the rate of $O(t^{-1/p})$ with any $p > 4$ without imposing the positivity of the infimum of $|D\phi|$.

Here we note that the barrier functions employed in the cases (i), (ii) are no longer useful in the case (iii), because $\varphi \not\equiv 0$. As for linear or semilinear problems, one can often follow the usual strategy of substituting $u(x,t) = \phi(x) + v(x,t)$ into equations and investigating the asymptotic behavior of $v(x,t)$, which satisfies the homogeneous Dirichlet boundary condition, as $t \to \infty$. However, this method does not seem to be effective for (4) due to the strong nonlinearity of the infinity-Laplacian. In case $\inf_{x\in\Omega} |D\phi(x)| > 0$, we overcome these difficulties by proposing super- and subsolutions of (4) deeply depending on ϕ and carrying out a barrier argument with them. Moreover, for a general ϕ, this strategy is combined with the approximation of ϕ by particular super- and subsolutions of (2) whose gradients do not vanish.

Juutinen [12] also studied the asymptotic behavior of viscosity solutions of the Cauchy-Dirichlet problems for a parabolic equation involving the singular infinity-Laplacian, i.e., $u_t = \Delta_\infty u/|Du|^2$ in $\Omega \times (0,\infty)$, for the case (ii) mentioned above (i.e., Ω is bounded and $\varphi \equiv 0$). He proved the comparison principle and the existence of viscosity solutions for the following eigenvalue problem $-\Delta_\infty v/|Dv|^2 = \lambda v$ in Ω, $v = 0$ on $\partial\Omega$, and applied these results to an analysis of the asymptotic behavior.

This paper consists of six sections. In Section 2, the definition of viscosity solutions is given and the invariance of solutions for (4) is discussed. Moreover, the comparison principle and the existence of viscosity solutions for (4) are reviewed as well. Section 3 provides explicit representations of viscosity solutions for (4) in two different forms. Furthermore, Sections 4, 5 and 6 are devoted to the cases (i), (ii) and (iii) mentioned above respectively.

Notation. We denote by $\mathbf{B}(x_0; r)$ the open ball $\{x \in \mathbb{R}^N; |x - x_0| < r\}$ of radius $r > 0$ with center at $x_0 \in \mathbb{R}^N$. We write $\mathbb{R}^+ = (0, \infty)$ and denote by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^N$. Moreover, we also use the notation:

$$D_i = \frac{\partial}{\partial x_i}, \quad D = (D_1, D_2, \ldots, D_N), \quad D_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$$

and D^2 denotes the $N \times N$ matrix whose (i,j)-th element is D_{ij}^2. We simply denote by $|\cdot|_\infty$ the sup-norm in the corresponding space if no confusion arises. Furthermore, let U be a subset of $\mathbb{R}^N$ and let f be a function from U into $\mathbb{R}$. Then $\mathrm{Lip}_f(U)$ denotes the infimum of Lipschitz constants for f on U.

2. Viscosity solutions

The notion of viscosity solutions is often employed to deal with nonlinear PDEs involving elliptic operators not in divergence form such as the infinity-Laplacian. In this section we review the definition of viscosity solutions for (4)–(6) and their existence and uniqueness. We first recall the definition of parabolic super- and subjets.

Definition 1. *Let Q be an open subset of $\mathbb{R}^{N+1}$. Then the parabolic superjet $\mathcal{P}^{2,+}u(x_0, t_0)$ and subjet $\mathcal{P}^{2,-}u(x_0, t_0)$ of a function $u : Q \to \mathbb{R}$ at $(x_0, t_0) \in Q$ are defined as follows:*

$$\begin{aligned}\mathcal{P}^{2,+}u(x_0,t_0) := \Bigg\{ &(s,p,X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N; u(x,t) \le u(x_0,t_0) + s(t-t_0) \\ &+\langle p, x - x_0\rangle + \frac{1}{2}\langle X(x-x_0), x - x_0\rangle \\ &+o(|x-x_0|^2 + |t-t_0|) \quad \textit{as} \quad (x,t) \to (x_0,t_0) \Bigg\},\end{aligned}$$

$$\mathcal{P}^{2,-}u(x_0,t_0) := -\mathcal{P}^{2,+}(-u)(x_0,t_0).$$

Here $\mathbb{S}^N$ denotes the set of all symmetric $N \times N$ matrices.

Now viscosity solutions of (4) are defined as follows (see also [1], [8]):

Definition 2. *Let Q be an open subset of $\mathbb{R}^{N+1}$. We denote the set of all upper semicontinuous (respectively, lower semicontinuous) functions from Q into $\mathbb{R}$ by $USC(Q)$ (respectively, $LSC(Q)$). A function $u \in USC(Q)$ is said to be a viscosity subsolution of (4) in Q if*

$$s - \langle Xp, p\rangle \leq 0 \tag{7}$$

for $(s,p,X) \in \mathcal{P}^{2,+}u(x_0,t_0)$ and $(x_0,t_0) \in Q$. A function $u \in LSC(Q)$ is said to be a viscosity supersolution of (4) in Q if

$$s - \langle Xp, p\rangle \geq 0 \tag{8}$$

for $(s,p,X) \in \mathcal{P}^{2,-}u(x_0,t_0)$ and $(x_0,t_0) \in Q$. Moreover, $u \in C(Q)$ is said to be a viscosity solution of (4) in Q if it is both a viscosity subsolution and a viscosity supersolution of (4) in Q.

Remark 1. By [14], the parabolic superjet can be written as follows.

$$\mathcal{P}^{2,+}u(x_0,t_0) = \Big\{ (\varphi_t(x_0,t_0), D\varphi(x_0,t_0), D^2\varphi(x_0,t_0));\ \varphi \in C^2(Q) \\ \text{and } u - \varphi \text{ attains its global maximum at } (x_0,t_0) \Big\}.$$

Hence one can easily check that $u \in USC(Q)$ is a viscosity subsolution of (4) in Q if and only if

$$\varphi_t(x_0,t_0) - \Delta_\infty \varphi(x_0,t_0) \leq 0$$

whenever $u - \varphi$ attains its global maximum, zero, at (x_0,t_0), for all $(x_0,t_0) \in Q$ and $\varphi \in C^2(Q)$. This fact will be employed in §6.

Furthermore, viscosity solutions of the Cauchy-Dirichlet problem (4)–(6) are defined as follows:

Definition 3. *Let Ω be an open subset of $\mathbb{R}^N$ and $T \in (0,\infty]$. Set $Q = \Omega \times (0,T)$. A function $u \in USC(\overline{Q})$ (respectively, $u \in LSC(\overline{Q})$) is said to be a viscosity subsolution (respectively, supersolution) of (4)–(6) in Q if u is a viscosity subsolution (respectively, supersolution) of (4) in Q, $u \leq \varphi$ (respectively, $u \geq \varphi$) on $\partial\Omega \times [0,T)$ and $u \leq u_0$ (respectively, $u \geq u_0$) on $\Omega \times \{0\}$. Furthermore, $u \in C(\overline{Q})$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution of (4)–(6) in Q.*

Concerning the comparison and uniqueness of viscosity solutions for (4)–(6), we have the following theorem.

Theorem 2 (Akagi-Suzuki [1]). *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$ be a viscosity subsolution and a viscosity supersolution in $Q = \Omega \times \mathbb{R}^+$ of (4) respectively. If $u \le v$ on $\partial_p Q$, then $u \le v$ in Q. Here $\partial_p Q$ denotes the parabolic boundary of Q, that is,*

$$\partial_p Q := \partial\Omega \times [0,\infty) \cup \Omega \times \{0\}.$$

In particular, if $u_1, u_2 \in C(\overline{Q})$ are viscosity solutions of (4), then

$$\sup_{(x,t)\in Q} |u_1(x,t) - u_2(x,t)| = \sup_{(x,t)\in\partial_p Q} |u_1(x,t) - u_2(x,t)|, \tag{9}$$

which also guarantees the uniqueness of solutions for (4)–(6).

For the existence of viscosity solutions, we recall the following

Theorem 3 (Akagi-Suzuki [1]). *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that for all $x_0 \in \partial\Omega$, there exists $y_0 \in \mathbb{R}^N$ such that $|x_0 - y_0| = R$ and $\{x \in \mathbb{R}^N; |x - y_0| < R\} \cap \Omega = \emptyset$ for some positive constant R independent of x_0. Then for any $u_0 \in C(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$ satisfying $u_0 = \varphi$ on $\partial\Omega$, the problem (4)–(6) admits a viscosity solution $u \in C(\overline{Q})$.*

Equation (4) has the invariance under the following change of variables:

$$v(x,t) = \mu u(\lambda U(x + x_*), \lambda^4\mu^2(t + t_*)) \tag{10}$$

with $\lambda, \mu > 0$, $x_* \in \mathbb{R}^N$, $t_* \in \mathbb{R}$ and U an orthogonal matrix. That is, if u is a solution of (4), so is v. This property remains valid for viscosity solutions. Indeed, we have the next proposition.

Proposition 1. *Let Ω be a domain in $\mathbb{R}^N$, $T \in (0,\infty]$ and $Q = \Omega \times (0,T)$. For each $\lambda, \mu > 0$, $(x_*, t_*) \in \mathbb{R}^N \times \mathbb{R}$ and an $N \times N$ orthogonal matrix U, we put*

$$Q^{\lambda,\mu}_{x_*,t_*,U} := \{(x,t) \in \mathbb{R}^N \times \mathbb{R};\ (\lambda U(x + x_*), \lambda^4\mu^2(t + t_*)) \in Q\}.$$

If u is a viscosity subsolution (respectively, supersolution) of (4) in Q, then $v(x,t)$ defined by (10) becomes a viscosity subsolution (respectively, supersolution) of (4) in $Q^{\lambda,\mu}_{x_*,t_*,U}$.

Proof. By the definitions of super- and subjets, we have, for $(x_0, t_0) \in Q^{\lambda,\mu}_{x_*,t_*,U}$,

$$\begin{aligned}\mathcal{P}^{2,\pm} v(x_0,t_0) = \{(&\lambda^4\mu^3 s, \lambda\mu U^t p, \lambda^2\mu U^t X U);\\ &(s,p,X) \in \mathcal{P}^{2,\pm} u(\lambda U(x_0 + x_*), \lambda^4\mu^2(t_0 + t_*))\},\end{aligned}$$

where U^t denotes the transpose of U. In the above relation, we put

$$\tau = \lambda^4\mu^3 s, \quad q = \lambda\mu U^t p, \quad Y = \lambda^2\mu U^t X U.$$

Let u be a viscosity subsolution. Since U is an orthogonal matrix, we get

$$\tau - \langle Yq, q\rangle = \lambda^4\mu^3(s - \langle Xp, p\rangle) \le 0.$$

Thus v is a viscosity subsolution of (4) in $Q^{\lambda,\mu}_{x_*,t_*,U}$. This method of proof is valid for viscosity supersolutions as well.

3. Exact solutions

In this section we provide explicit viscosity solutions of separable type and self-similar type for (4) in $\mathbb{R}^N \times \mathbb{R}^+$. They will play important roles in Sections 4 and 5.

Remark 2. In case of radially symmetric solutions $u(x,t) = U(|x|,t)$ for some $U = U(r,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, we can derive formally from (4) that $U_t = U_{rr}U_r^2$, which is the same form as in (4) with $N = 1$, i.e., $u_t = \Delta_\infty u = u_{xx}u_x^2$. This fact stems from the strong degeneracy of the infinity-Laplacian and will be used in the construction of exact solutions.

3.1. Separable type

This subsection is devoted to the construction of radially symmetric exact solutions to (4) in $\mathbb{R}^N \times \mathbb{R}^+$ by using the technique of separation of variables. We first deal with the case of $N = 1$, i.e.,

$$u_t = \Delta_\infty u = (u_\xi)^2 u_{\xi\xi} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \tag{11}$$

where u_ξ and $u_{\xi\xi}$ denote $\partial u/\partial\xi$ and $\partial^2 u/\partial\xi^2$ respectively. We seek a solution in the form of the separation of variables, i.e., $u(\xi,t) = \rho(t)\psi(\xi)$ with some functions ρ and ψ. Then (11) decomposes into the following two ordinary differential equations:

$$\dot\rho(t) = -\mu\rho(t)^3, \quad t > 0 \tag{12}$$

and

$$\psi'(\xi)^2\psi''(\xi) = -\mu\psi(\xi), \quad \xi \in \mathbb{R} \tag{13}$$

with an arbitrary constant $\mu \ge 0$. If $\mu = 0$, we obtain

$$\rho(t) = C_1 \quad \text{and} \quad \psi(\xi) = C_2\xi + C_3$$

with constants $C_1, C_2, C_3 \in \mathbb{R}$, so the solution $u(\xi,t) = \rho(t)\psi(\xi)$ corresponds to a stationary solution of (11). If $\mu > 0$, positive solutions of (12) are given as follows:

$$\rho(t) = (C + 2\mu t)^{-1/2} \qquad \text{with} \ \ C \geq 0.$$

For simplicity, we put $\mu = 1/2$. Then $\rho(t) = (C+t)^{-1/2}$ and (13) is rewritten as

$$\frac{d}{d\xi}(\psi'(\xi)^3) + \frac{3}{2}\psi(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{14}$$

By a *solution* of (14) we mean a function $\psi(\xi)$ for which $\psi(\xi), \psi'(\xi)^3 \in C^1(\mathbb{R})$ and (14) holds.

To give an explicit form of solutions for (14), we define the function $\xi : [-\pi/2, \pi/2] \to \mathbb{R}$ by

$$\xi(y) = \int_0^y \sqrt{\cos t}\, dt. \tag{15}$$

Since ξ is strictly increasing on $[-\pi/2, \pi/2]$, we can define the inverse function $y := \xi^{-1} : [-T, T] \to [-\pi/2, \pi/2]$ of ξ, where

$$T := \int_0^{\pi/2} \sqrt{\cos t}\, dt.$$

Moreover, the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined below will become a solution of (14) in $\mathbb{R}$.

Definition 4. *Define the function $\Phi_1 : [-T, T] \to \mathbb{R}$ by*

$$\Phi_1(\xi) = \sin y(\xi) \qquad \textit{for} \ \ \xi \in [-T, T],$$

and the function $\Phi_2 : [-T, 3T] \to \mathbb{R}$ by

$$\Phi_2(\xi) = \begin{cases} \Phi_1(\xi) & \textit{if} \ \ \xi \in [-T, T], \\ \Phi_1(2T - \xi) & \textit{if} \ \ \xi \in [T, 3T], \end{cases}$$

and the function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\xi) = \Phi_2(\xi - 4kT) \qquad \textit{if} \ \ \xi \in [-T + 4kT, 3T + 4kT]$$

for each $k \in \mathbb{Z}$.

Remark 3. The function ξ can be written as the incomplete elliptic integral $E(\sqrt{2},\cdot)$ of the second kind with modulus $\sqrt{2}$ in Jacobi's form. Indeed, substituting $t = 2s$ into (15), we find

$$\xi(y) = 2\int_0^{y/2} \sqrt{\cos 2s}\; ds = 2\int_0^{y/2} \sqrt{1-2\sin^2 s}\; ds = 2E(\sqrt{2}, y/2).$$

The constant T is the complete elliptic integral of the second kind. Moreover, Φ_1 satisfies

$$2E(\sqrt{2}, (1/2)\sin^{-1}\Phi_1(\xi)) = \xi \quad \text{for all } \ \xi \in [-T, T].$$

Then we see the following properties of Φ.

Lemma 1. *It follows that*

(i) Φ *is* $4T$*-periodic, i.e.,* $\Phi(\xi + 4T) = \Phi(\xi)$*, and the range of* Φ *is* $[-1, 1]$.
(ii) $\Phi \in C^\infty(\mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}}\{(2k-1)T\})$.
(iii) $\Phi \in C^1(\mathbb{R})$ *and* $\Phi'((2k-1)T) = 0$ *for all* $k \in \mathbb{Z}$.
(iv) $\Phi'(\cdot)^3 \in C^1(\mathbb{R})$ *and* Φ *is a solution of* (14).

Proof. First, the assertion (i) follows immediately from the definition of Φ. To prove (ii), it is enough to show $\Phi \in C^\infty(-T, T)$ because of the definition of Φ. Since $\xi(y) \in C^\infty(-\pi/2, \pi/2)$ and $d\xi/dy = \sqrt{\cos y} > 0$ for $y \in (-\pi/2, \pi/2)$, the inverse function theorem guarantees that $y(\xi)$ belongs to $C^\infty(-T, T)$. Then $\Phi(\xi) = \sin y(\xi) \in C^\infty(-T, T)$.

Now, we prove (iii). Note that

$$\Phi_1'(\xi) = \cos y(\xi)\frac{dy}{d\xi} = \sqrt{\cos y(\xi)} > 0 \quad \text{for } \ \xi \in (-T, T), \tag{16}$$

which also implies $\Phi'(\xi) = -\Phi_1'(2T - \xi) = -\sqrt{\cos y(2T - \xi)} < 0$ for $\xi \in (T, 3T)$. Then $\Phi'(\xi)$ converges to zero as $\xi \to T$, and therefore Φ belongs to the C^1-class in a neighborhood of $\xi = T$ and $\Phi'(T) = 0$. This method is valid at $\xi = -T$ also. From the periodicity of Φ, it follows that $\Phi \in C^1(\mathbb{R})$ and $\Phi'((2k-1)T) = 0$ for $k \in \mathbb{Z}$.

Finally, we give a proof of (iv). Since

$$\Phi''(\xi) = -\frac{1}{2}\tan y(\xi) \quad \text{for } \ \xi \in (-T, T), \tag{17}$$

it follows that

$$\frac{d}{d\xi}(\Phi'(\xi)^3) = -\frac{3}{2}\sin y(\xi) = -\frac{3}{2}\Phi(\xi) \quad \text{for } \ \xi \in (-T, T). \tag{18}$$

This equation is valid for all $\xi \in \mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}}\{(2k-1)T\}$. Since the right-hand side of (18), i.e., $(-3/2)\Phi(\xi)$, is continuous on $\mathbb{R}$, $\Phi'(\xi)^3$ belongs to $C^1(\mathbb{R})$. Therefore (18) holds for all $\xi \in \mathbb{R}$ and Φ becomes a solution of (14).

Remark 4. (i) The function Φ is not of class C^2 at $\xi = T$. Indeed, it follows from (17) that $\Phi''(\xi) \to -\infty$ as $\xi \to T - 0$.

(ii) We can prove that $V(\xi,t) := (C+t)^{-1/2}\Phi(\xi)$ becomes a viscosity solution of (11) in $\mathbb{R} \times (-C,\infty)$ as in the proof of Proposition 2, the next proposition. However, this fact is not obvious, because Φ is not of class C^2 at $\xi = (2k-1)T$ with $k \in \mathbb{N}$.

We next proceed to construct exact solutions of (4) in $\mathbb{R}^N \times \mathbb{R}^+$.

Proposition 2. *Let Φ be the function given in Definition 4. Then for every $\alpha > 0$ and $\beta \geq 0$, the function*

$$V(x,t;\alpha,\beta) := \alpha^2(t+\beta)^{-1/2}\Phi(\alpha^{-1}|x| + T) \tag{19}$$

becomes a viscosity solution of (4) in $\mathbb{R}^N \times \mathbb{R}^+$.

Proof. Set $Q = \mathbb{R}^N \times \mathbb{R}^+$. By (ii) and (iv) of Lemma 1, $V(x,t;\alpha,\beta)$ is a classical solution of (4) in $Q \setminus Q_0$, where

$$Q_0 := \{(x,t) \in Q;\ |x| = 2\alpha(k-1)T \text{ for some } k \in \mathbb{N}\}.$$

To prove this proposition, it suffices to show that $V(x,t;\alpha,\beta)$ satisfies (7) and (8) at $(x_0,t_0) \in Q_0$.

Here note that $V(x,t;\alpha,\beta) \in C^1(Q)$ (see (iii) of Lemma 1) and

$$DV(x,t;\alpha,\beta) = \begin{cases} \alpha(t+\beta)^{-1/2}\Phi'(\alpha^{-1}|x|+T)\dfrac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$V_t(x,t;\alpha,\beta) = -\frac{\alpha^2}{2}(t+\beta)^{-3/2}\Phi(\alpha^{-1}|x|+T) \quad \text{for } (x,t) \in Q.$$

Let $(x_0,t_0) \in Q_0$ be fixed. Then we have $|x_0| = 2\alpha(k-1)T$ for some $k \in \mathbb{N}$. We distinguish two cases: k is odd or even.

Let k be odd, that is, $k = 2m-1$ with some $m \in \mathbb{N}$. Then we claim that $\mathcal{P}^{2,-}V(x_0,t_0;\alpha,\beta) = \emptyset$. Suppose on the contrary that there exists $(s_0,p_0,X_0) \in \mathcal{P}^{2,-}V(x_0,t_0;\alpha,\beta)$. Then

$$\begin{aligned} &V(x,t;\alpha,\beta) - V(x_0,t_0;\alpha,\beta) \\ &\quad \geq s_0(t-t_0) + \langle p_0, x-x_0\rangle + \frac{1}{2}\langle X_0(x-x_0), x-x_0\rangle \\ &\quad\quad + o(|x-x_0|^2 + |t-t_0|) \end{aligned} \tag{20}$$

as $(x,t) \to (x_0,t_0)$. Since $V \in C^1(Q)$, we have $p_0 = DV(x_0,t_0;\alpha,\beta) = 0$, $\Phi(\alpha^{-1}|x_0| + T) = \Phi_1(T) = 1$. Substituting these equalities into (20) and putting $t = t_0$, we have

$$\begin{aligned} &\alpha^2(t_0+\beta)^{-1/2}(\Phi(\alpha^{-1}|x|+T) - 1) \\ &\quad \geq \frac{1}{2}\langle X_0(x-x_0), x-x_0\rangle + o(|x-x_0|^2) \end{aligned} \tag{21}$$

as $x \to x_0$. In particular, put $x = x_0 + \alpha\zeta e$, where $\zeta > 0$ and

$$e := \begin{cases} \dfrac{x_0}{|x_0|} & \text{if } \ x_0 \neq 0, \\ \text{(an arbitrary unit vector in } \mathbb{R}^N) & \text{if } \ x_0 = 0. \end{cases}$$

Then $\alpha^{-1}|x| + T = (2k-1)T + \zeta = (4m-3)T + \zeta$. Since Φ is $4T$-periodic and symmetric about $t = T$, we have

$$\Phi(\alpha^{-1}|x| + T) = \Phi((4m-3)T + \zeta) = \Phi(T - \zeta).$$

Substituting this relation into (21) and dividing both sides by ζ^2 ,we get

$$\alpha^2(t_0 + \beta)^{-1/2} \frac{\Phi(T-\zeta) - 1}{\zeta^2} \geq \frac{\alpha^2}{2}\langle Xe, e\rangle + o(1) \tag{22}$$

as $\zeta \to +0$. Using L'Hospital's rule twice with (17), we obtain

$$\lim_{\zeta \to +0} \frac{\Phi(T-\zeta) - 1}{\zeta^2} = \lim_{\zeta \to +0} \frac{\Phi''(T-\zeta)}{2} = -\infty,$$

which contradicts (22). Thus we deduce that $\mathcal{P}^{2,-}V(x_0, t_0; \alpha, \beta) = \emptyset$. Now, we show (7). Let $(s, p, X) \in \mathcal{P}^{2,+}V(x_0, t_0; \alpha, \beta)$. Since $V \in C^1(Q)$, we have

$$s = V_t(x_0, t_0; \alpha, \beta) = -\frac{\alpha^2}{2}(t_0 + \beta)^{-3/2}, \quad p = DV(x_0, t_0; \alpha, \beta) = 0.$$

Therefore we obtain

$$s - \langle Xp, p\rangle = -\frac{\alpha^2}{2}(t_0 + \beta)^{-3/2} < 0.$$

Thus (7) holds.

We consider the case where k is even, that is, $k = 2m$ with some $m \in \mathbb{N}$. Then we prove $\mathcal{P}^{2,+}V(x_0, t_0; \alpha, \beta) = \emptyset$ by contradiction. Assume that there exists $(s_0, p_0, X_0) \in \mathcal{P}^{2,+}V(x_0, t_0; \alpha, \beta)$. It then follows that $p_0 = 0$, $\Phi(\alpha^{-1}|x_0| + T) = \Phi_1(-T) = -1$. Thus repeating the same argument as in case k is odd, we have

$$\alpha^2(t_0 + \beta)^{-1/2} \frac{\Phi((4m-1)T + \zeta) + 1}{\zeta^2} \leq \frac{\alpha^2}{2}\langle Xe, e\rangle + o(1) \tag{23}$$

as $\zeta \to +0$. Since

$$\frac{\Phi((4m-1)T + \zeta) + 1}{\zeta^2} = \frac{\Phi(-T + \zeta) + 1}{\zeta^2} \to \infty \quad \text{as } \ \zeta \to +0,$$

we obtain a contradiction to (23). Therefore $\mathcal{P}^{2,+}V(x_0,t_0;\alpha,\beta)=\emptyset$. Furthermore, we get, for every $(s,p,X)\in\mathcal{P}^{2,-}V(x_0,t_0;\alpha,\beta)$,

$$s-\langle Xp,p\rangle=\frac{\alpha^2}{2}(t_0+\beta)^{-3/2}>0.$$

Consequently, $V(x,t;\alpha,\beta)$ becomes a viscosity solution of (4) in Q.

Remark 5. From Proposition 2, we see that Equation (4) has another invariance besides (10): if $u(\xi,t)$ is a solution of (4) with $N=1$, then $u(|x|,t)$ becomes a radially symmetric solution of (4) in $(\mathbb{R}^N\setminus\{0\})\times(0,\infty)$ for any $N\in\mathbb{N}$. In case $u\in C^{2,1}((0,\infty)\times(0,\infty))$, this claim is easily confirmed. However, in case u is not smooth (indeed, $\Phi\notin C^2$ in Proposition 2), it is not evident that $u(|x|,t)$ becomes a viscosity solution. We verified this claim for $V(x,t;\alpha,\beta)$ in Proposition 2 (see also §3.3 in [15] for another case).

3.2. Self-similar type

In this subsection we present an explicit form of a viscosity solution of self-similar type of (4) in $\mathbb{R}^N\times\mathbb{R}^+$ by employing the so-called Barenblatt solution $\mathcal{B}(\xi,t)$ of the equation:

$$u_t=(u_\xi)^2u_{\xi\xi}=\frac{1}{3}\frac{\partial}{\partial\xi}(u_\xi^3)\quad\text{in }\ \mathbb{R}\times\mathbb{R}^+. \tag{24}$$

The Barenblatt solution (or ZKB solution) is a self-similar solution having a Dirac delta as initial data and has a compact support expanding at finite velocity. This type of solution was first supplied by Zel'dovich and Kompaneets and investigated in detail by Barenblatt in the study of the porous medium equation, $u_t=\Delta(|u|^{m-2}u)$ with $m>2$, and it has been provided for other degenerate parabolic equations, in particular, (24). More precisely, $\mathcal{B}(\xi,t)$ can be written as follows (see, e.g., [17, p. 191]):

$$\mathcal{B}(\xi,t)=\frac{1}{4}t^{-1/6}(1-|\xi|^{4/3}t^{-2/9})_+^{3/2}\quad\text{for }\ \xi\in\mathbb{R}\ \text{ and }\ t\in\mathbb{R}^+,$$

where $(x)_+=\max\{x,0\}$. We observe that $\mathcal{B}\in C^1(\mathbb{R}\times\mathbb{R}^+)\cap C^\infty(\Lambda)$, where $\Lambda:=\{(\xi,t)\in\mathbb{R}\times\mathbb{R}^+;\xi\neq0\ \text{ and }\ 1-|\xi|^{4/3}t^{-2/9}\neq0\}$. Furthermore, we have the following

Proposition 3. *Define $B(x,t):=\mathcal{B}(|x|,t)$. Then B is a bounded viscosity solution of (4) in $Q=\mathbb{R}^N\times\mathbb{R}^+$.*

Proof. We put

$$\eta(\xi,t) := (1-\xi^{4/3}t^{-2/9})_+,$$

$$Q_0 := \{(x,t) \in Q;\ x=0 \ \text{ or } \ |x| = t^{1/6}\}.$$

Then we have, for $\xi, t \in \mathbb{R}^+$,

$$\begin{aligned}
\mathcal{B}_t(\xi,t) &= -\frac{1}{24}t^{-7/6}\eta^{3/2} + \frac{1}{12}t^{-25/18}\xi^{4/3}\eta^{1/2},\\
\mathcal{B}_\xi(\xi,t) &= -\frac{1}{2}t^{-7/18}\xi^{1/3}\eta^{1/2},\\
\mathcal{B}_{\xi\xi}(\xi,t) &= -\frac{1}{6}t^{-7/18}\xi^{-2/3}\eta^{1/2} + \frac{1}{3}t^{-11/18}\xi^{2/3}\eta^{-1/2} \quad (\eta \neq 0).
\end{aligned}$$

Since $\eta = 0$ for $\xi = t^{1/6}$, we see that $B \in C^1(Q) \cap C^\infty(Q \setminus Q_0)$, $DB(x,t) = 0$ for $(x,t) \in Q_0$ and B becomes a classical solution of (4) in $Q \setminus Q_0$.

Let $(x_0,t_0) \in Q_0$. Then $x_0 = 0$ or $\eta(|x_0|,t_0) = 0$. For $(s,p,X) \in \mathcal{P}^{2,\pm}B(x_0,t_0)$, we have $p = DB(x_0,t_0) = 0$ and

$$s = B_t(x_0,t_0) = \begin{cases} -t_0^{-7/6}/24 & \text{if } x_0 = 0,\\ 0 & \text{if } \eta(|x_0|,t_0) = 0. \end{cases} \tag{25}$$

In both cases $x_0 = 0$ and $\eta(|x_0|,t_0) = 0$, it holds that $p = 0$ and $s \le 0$. Therefore for $(s,p,X) \in \mathcal{P}^{2,+}B(x_0,t_0)$, we have

$$s - \langle Xp, p\rangle = s \le 0,$$

which shows that B is a viscosity subsolution.

We next deal with $\mathcal{P}^{2,-}B(x_0,t_0)$. For the case where $x_0 = 0$, we claim that $\mathcal{P}^{2,-}B(0,t_0) = \emptyset$. Suppose on the contrary that there exists $(s,p,X) \in \mathcal{P}^{2,-}B(0,t_0)$. It then follows that

$$\begin{aligned}
\frac{1}{4}t^{-1/6}\eta(|x|,t)^{3/2} - \frac{1}{4}t_0^{-1/6} &= B(x,t) - B(0,t_0)\\
&\ge s(t-t_0) + \frac{1}{2}\langle Xx, x\rangle + o(|x|^2 + |t-t_0|)
\end{aligned}$$

as $(x,t) \to (0,t_0)$. Put $x = \zeta e$, where e is a unit vector in $\mathbb{R}^N$ and $\zeta > 0$. Letting $t \to t_0$, we have

$$\frac{1}{4}t_0^{-1/6}\left\{\left(1-\zeta^{4/3}t_0^{-2/9}\right)^{3/2} - 1\right\} \ge \frac{\zeta^2}{2}\langle Xe, e\rangle + o(\zeta^2), \tag{26}$$

as $\zeta \to 0$. Dividing both sides by ζ^2 and noting that

$$\lim_{\zeta\to+0} \frac{\left(1-\zeta^{4/3}t_0^{-2/9}\right)^{3/2} - 1}{\zeta^2} = -\infty,$$

we derive a contradiction to (26). Thus $\mathcal{P}^{2,-}B(0,t_0) = \emptyset$. For the case where $\eta(|x_0|,t_0) = 0$, by (25), every $(s,p,X) \in \mathcal{P}^{2,-}B(x_0,t_0)$ satisfies that $s = 0$ and $p = 0$, and therefore $s - \langle Xp, p\rangle = 0$. Hence B is a viscosity supersolution, and consequently, it is a viscosity solution.

Furthermore, by Proposition 1, we have the following corollary, which will be used in Section 4.

Corollary 1. *Let $Q = \mathbb{R}^N \times \mathbb{R}^+$. For $\alpha > 0$, we define*

$$B(x,t;\alpha) := \frac{1}{4}\alpha^3(t+1)^{-1/6}(1-\alpha^{-2}|x|^{4/3}(t+1)^{-2/9})_+^{3/2}.$$

Then $B(x,t;\alpha)$ is a viscosity solution of (4) *in Q.*

Proof. For $\alpha > 0$, we put $\mu = \alpha^{9/2}$. Then it follows that

$$\begin{aligned} B(x,t;\alpha) &= \frac{1}{4}\mu^{2/3}(t+1)^{-1/6}(1-\mu^{-4/9}|x|^{4/3}(t+1)^{-2/9})_+^{3/2} \\ &= \mu B(x, \mu^2(t+1)). \end{aligned}$$

Hence, by Propositions 1 and 3, $B(x,t;\alpha)$ becomes a viscosity solution of (4) in Q.

Remark 6. By virtue of Proposition 1 with $\mu = \lambda$, if u solves (4) in the viscosity sense, then we can obtain a one-parameter family $\{v_\lambda\}_{\lambda>0}$ of viscosity solutions to (4) defined by the scaling

$$v_\lambda(x,t) := \lambda u(\lambda x, \lambda^6 t). \tag{27}$$

In particular, from the definition of $\mathcal{B}$, we find a self-similarity of B, i.e.,

$$\lambda B(\lambda x, \lambda^6 t) = B(x,t) \quad \text{for } \lambda > 0 \text{ and } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Furthermore, the L^N-norm of $B(\cdot,t)$ in $\mathbb{R}^N$ is invariant under the scaling (27).

4. Optimal decay rate of viscosity solutions in $\mathbb{R}^N$

In this section we investigate the optimal decay rate of bounded viscosity solutions for the Cauchy problem (4), (6) where $\Omega = \mathbb{R}^N$ and the initial data u_0 has a compact support. Our result reads:

Theorem 4. *Let $Q = \mathbb{R}^N \times \mathbb{R}^+$ and let $u_0 \in C_0(\mathbb{R}^N)$. Then the unique bounded viscosity solution u of* (4), (6) *satisfies*

$$|u(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \leq C(t+1)^{-1/6} \quad \text{for} \ \ t > 0 \tag{28}$$

with some $C > 0$ independent of x and t. In addition, if $u_0 \geq 0$ in $\mathbb{R}^N$ and $u_0 \not\equiv 0$, then there is $c > 0$ independent of x and t such that

$$c(t+1)^{-1/6} \leq |u(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \quad \text{for} \ \ t > 0. \tag{29}$$

Therefore $(t+1)^{-1/6}$ is the optimal decay rate for the supremum norm of bounded solutions in $\Omega = \mathbb{R}^N$.

Proof. To prove the theorem, we recall the explicit representation of the viscosity solutions $B(x,t;\alpha)$ of self-similar type given in Corollary 1. Let $v(x,t) = B(x,t;\alpha)$ where $\alpha > 0$ will be determined later. First, since u_0 has a compact support (denoted by $\operatorname{supp} u_0$), we take $R > 0$ so large that $\operatorname{supp} u_0 \subset \mathbf{B}(0;R)$. Next, we note that $\operatorname{supp} v(\cdot,0) = \overline{\mathbf{B}(0;\alpha^{3/2})}$. Then we choose an $\alpha > 0$ so large that

$$\operatorname{supp} u_0 \subset \mathbf{B}(0;R) \subset \operatorname{supp} v(\cdot,0),$$

$$v(x,0) \geq (\alpha^3/4)(1 - \alpha^{-2}R^{4/3})^{3/2} \geq |u_0|_\infty \quad \text{for } x \in \mathbf{B}(0;R).$$

Hence $v(x,0) \geq u_0(x)$ in $\mathbb{R}^N$. From the comparison theorem (see Theorem 2.1 of [9]), it follows that $u \leq v$ in Q. Moreover, a similar argument also implies $-v \leq u$ in Q. Thus we have (28).

We next derive a decay estimate from below. Let $u_0 \in C_0(\mathbb{R}^N)$ be such that $u_0 \geq 0$ in $\mathbb{R}^N$ and $u_0 \not\equiv 0$. Then we can take $x_0 \in \mathbb{R}^N$ such that $u_0(x_0) > 0$. Thanks to Proposition 1, we can assume $x_0 = 0$ without any loss of generality. Hence there exists $\varepsilon > 0$ such that $u_0(x) \geq u_0(0)/2 > 0$ in $\mathbf{B}(0;\varepsilon)$. We put $w(x,t) := B(x,t;\beta)$ with $\beta > 0$ so small that

$$\operatorname{supp} w(\cdot,0) \subset \mathbf{B}(0;\varepsilon) \subset \operatorname{supp} u_0,$$

$$w(x,0) \leq \beta^3/4 \leq u_0(0)/2 \leq u_0(x) \quad \text{in } \mathbf{B}(0;\varepsilon).$$

Then $w(x,0) \leq u_0(x)$ in $\mathbb{R}^N$, and the comparison principle yields $w(x,t) \leq u(x,t)$ in Q. We have, in particular,

$$\frac{1}{4}\beta^3(t+1)^{-1/6} = w(0,t) \leq u(0,t) \leq |u(\cdot,t)|_{L^\infty(\mathbb{R}^N)}.$$

This completes our proof.

We can also obtain the optimal decay rate for the L^p-norm of bounded viscosity solutions as $t \to \infty$ in the next corollary. It is noteworthy that the L^p-norm of $u(\cdot,t)$ decays at the rate of $O(t^{(N-p)/(6p)})$ if $p > N$, and it goes to $+\infty$ if $p < N$, $u_0 \geq 0$ and $u_0 \not\equiv 0$.

Corollary 2. *Let $Q = \mathbb{R}^N \times \mathbb{R}^+$ and let $u_0 \in C_0(\mathbb{R}^N)$. Let u be a unique bounded viscosity solution of the Cauchy problem (4), (6). Then for $p \in [1, \infty)$, there exists a constant $C_p > 0$ independent of x and t such that*

$$|u(\cdot, t)|_{L^p(\mathbb{R}^N)} \le C_p(t+1)^{(N-p)/(6p)} \quad \textit{for} \;\; t > 0. \tag{30}$$

In addition, if $u_0 \ge 0$ and $u_0 \not\equiv 0$, then

$$c_p(t+1)^{(N-p)/(6p)} \le |u(\cdot, t)|_{L^p(\mathbb{R}^N)} \quad \textit{for} \;\; t > 0 \tag{31}$$

with some positive constant $c_p \le C_p$ independent of x and t.

Proof. From the definition of $B(x, t; \alpha)$, it follows that

$$\int_{\mathbb{R}^N} |B(x, t; \alpha)|^p \, dx = \left(\frac{\alpha^3}{4}\right)^p (t+1)^{(N-p)/6} \int_{\mathbb{R}^N} \left(1 - \alpha^{-2}|\xi|^{4/3}\right)_+^{3p/2} d\xi,$$

where $\xi := (t+1)^{-1/6}x$. Hence we have

$$|B(\cdot, t; \alpha)|_{L^p(\mathbb{R}^N)} = C_{p,\alpha}(t+1)^{(N-p)/(6p)} \tag{32}$$

with

$$C_{p,\alpha} := |B(\cdot, 0; \alpha)|_{L^p(\mathbb{R}^N)} = \frac{\alpha^3}{4} \left[\int_{\mathbb{R}^N} \left(1 - \alpha^{-2}|\xi|^{4/3}\right)_+^{3p/2} d\xi \right]^{1/p} > 0.$$

In the proof of Theorem 4, we have proved that

$$|u(x, t)| \le B(x, t; \alpha) \quad \text{for} \;\; (x, t) \in Q$$

with some constant $\alpha > 0$. Hence by (32),

$$|u(\cdot, t)|_{L^p(\mathbb{R}^N)} \le C_{p,\alpha}(t+1)^{(N-p)/(6p)} \quad \text{for} \;\; t > 0.$$

Moreover, in case $u_0 \ge 0$, we also obtained

$$0 \le B(x, t; \beta) \le u(x, t) \quad \text{for} \;\; (x, t) \in Q$$

for some $\beta > 0$. Therefore we conclude that

$$C_{p,\beta}(t+1)^{(N-p)/(6p)} \le |u(\cdot, t)|_{L^p(\mathbb{R}^N)} \quad \text{for} \;\; t > 0,$$

which proves this corollary.

5. Optimal decay rate of viscosity solutions for homogeneous Dirichlet case

In this section, we establish the optimal decay rate of viscosity solutions in Q of (4)–(6) with the homogeneous Dirichlet boundary condition, i.e., $\varphi \equiv 0$. The main result of this section is stated as follows:

Theorem 5. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $Q := \Omega \times \mathbb{R}^+$. Moreover, let $u_0 \in C(\overline{\Omega})$ be such that $u_0 = 0$ on $\partial\Omega$. Then the unique viscosity solution u of (4)–(6) with $\varphi \equiv 0$ satisfies the following:*

$$|u(\cdot,t)|_{L^\infty(\Omega)} \leq C(t+1)^{-1/2} \quad \textit{for} \ \ t > 0 \tag{33}$$

with some positive constant C independent of x and t. In addition, if $u_0 \geq 0$ and $u_0 \not\equiv 0$, then there is $c > 0$ independent of x and t such that

$$c(t+1)^{-1/2} \leq |u(\cdot,t)|_{L^\infty(\Omega)} \quad \textit{for} \ \ t > 0. \tag{34}$$

Therefore $(t+1)^{-1/2}$ is the optimal decay rate of solutions for a bounded domain Ω.

Proof. Choose $R > 0$ so large that $\overline{\Omega} \subset \mathbf{B}(0;R)$ and put

$$v(x,t) = V(x,t;\alpha_1,\beta_1) = \alpha_1^2(t+\beta_1)^{-1/2}\Phi(\alpha_1^{-1}|x| + T).$$

We determine α_1 and β_1 in the following. First, fix α_1 in $(R/T, \infty)$. Then $\Phi(\alpha_1^{-1}|x|+T) \geq \Phi(\alpha_1^{-1}R+T) > 0$ in $\mathbf{B}(0;R)$, and v is a viscosity solution in $\mathbf{B}(0;R) \times \mathbb{R}^+$ because of Proposition 2. Next, choose $\beta_1 > 0$ so small that

$$\min_{x\in\overline{\Omega}} v(x,0) \geq \alpha_1^2\beta_1^{-1/2}\Phi(\alpha_1^{-1}R + T) \geq |u_0|_\infty.$$

Then we deduce from the comparison principle that $u \leq v$ in Q. Furthermore, we can similarly verify that $-v \leq u$ in Q as well. Hence there exists a constant $C \geq 0$ depending on R and $|u_0|_\infty$ but independent of x and t such that

$$|u(\cdot,t)|_{L^\infty(\Omega)} \leq |v(\cdot,t)|_{L^\infty(\Omega)} \leq C(t+1)^{-1/2}.$$

We next derive (34). Since $u_0 \geq 0$ and $u_0 \not\equiv 0$, we take $x_0 \in \Omega$ such that $u_0(x_0) > 0$. By virtue of Proposition 1, we can assume that $x_0 = 0$. Since $u_0 \in C(\overline{\Omega})$, there exists $\varepsilon > 0$ such that $u_0(x) \geq u_0(0)/2 > 0$ in $\mathbf{B}(0;\varepsilon)$. Fix $\alpha_2 \in (0, \varepsilon/T)$ and define $w(x,t)$ by

$$w(x,t) := \begin{cases} V(x,t;\alpha_2,\beta_2) & \text{if } \ |x| < \alpha_2 T, \\ 0 & \text{if } \ |x| \geq \alpha_2 T, \end{cases}$$

where $\beta_2 > 0$ is chosen to be so large that

$$\max_{x \in \overline{\Omega}} w(x, 0) = \alpha_2^2 \beta_2^{-1/2} \leq u_0(0)/2.$$

We then observe that $w(x, 0) \leq u_0(x)$ in Ω and $w(x, t) = u(x, t) = 0$ on $\partial\Omega \times \mathbb{R}^+$. Moreover, by Propositions 1 and 2, w is a viscosity subsolution of (4) in Q (see Proposition 5.1 of [10]). Hence by the comparison principle, we conclude that $w \leq u$ in Q, which yields

$$\alpha_2^2 (t + \beta_2)^{-1/2} = w(0, t) \leq u(0, t) \leq |u(\cdot, t)|_\infty \quad \text{for} \quad t > 0.$$

This completes our proof.

Repeating the same argument as in the proof of Corollary 2 with $B(x, t; \alpha)$ by $V(x, t; \alpha, \beta)$, we can also verify the following corollary.

Corollary 3. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $Q = \Omega \times \mathbb{R}^+$. Moreover, let $u_0 \in C(\Omega)$ be such that $u_0 = 0$ on $\partial\Omega$. Let u be a unique viscosity solution of the Cauchy-Dirichlet problem* (4)–(6)*. Then for $p \in [1, \infty)$, there exists a constant $C_p > 0$ independent of x and t such that*

$$|u(\cdot, t)|_{L^p(\Omega)} \leq C_p (t + 1)^{-1/2} \quad \textit{for} \quad t > 0. \tag{35}$$

In addition, if $u_0 \geq 0$ and $u_0 \not\equiv 0$, then

$$c_p (t + 1)^{-1/2} \leq |u(\cdot, t)|_{L^p(\Omega)} \quad \textit{for} \quad t > 0 \tag{36}$$

with some positive constant $c_p \leq C_p$ independent of x and t.

Here we note that the optimal decay rate $O(t^{-1/2})$ for the L^p-norm of solutions is independent of N and p (cf. Corollary 2) since the variables x, t of the barrier function $V(x, t; \alpha, \beta)$ are separable.

Even if Ω is unbounded but bounded in at least one direction, Theorem 5 is still valid. We show this in the next theorem.

Theorem 6. *Let Ω be a (possibly unbounded) domain in $\mathbb{R}^N$ which lies between two parallel hyperplanes at a distance $d > 0$ apart. Moreover, let $u_0 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ be such that $u_0 = 0$ on $\partial\Omega$. Then the assertions of Theorem 5 remain valid for all bounded viscosity solutions of* (4)–(6)*.*

Proof. Thanks to Proposition 1, by translating and rotating the coordinate system, we can assume

$$\Omega \subset D := \{(x_1, x_2, \ldots, x_N);\ |x_1| < d/2,\ x_2, \ldots, x_N \in \mathbb{R}\}.$$

Define $v(x, t)$ on $\overline{Q}$ by

$$v(x, t) := \alpha^2 (t + \beta)^{-1/2} \Phi(\alpha^{-1} x_1 + T).$$

Then it becomes a viscosity solution of (4) in Q. Indeed, recalling of (ii)-(iv) of Lemma 1, we deduce that $v \in C^1(Q) \cap C^\infty(Q \setminus Q_0)$ and v is a classical solution in $Q \setminus Q_0$, where

$$Q_0 := \{(x,t) \in Q;\ x_1 = 2\alpha(k-1)T \ \text{ with some } \ k \in \mathbb{Z}\}.$$

Now, let $(x_0,t_0) \in Q_0$ be fixed. Repeating the same argument as in the proof of Proposition 2 with obvious modifications, we can verify (7) and (8) with u replaced by v at (x_0,t_0). Hence v also becomes a viscosity solution of (4) in Q.

Set

$$\alpha = d/T \ \text{ and } \ \beta = \alpha^4 c_0^2 |u_0|_\infty^{-2} = (d/T)^4 c_0^2 |u_0|_\infty^{-2}$$

with

$$c_0 := \min_{|s| \le T/2} \Phi(s+T) = \Phi(T/2) > 0.$$

We then find that

$$v(x,0) \ge \alpha^2 \beta^{-1/2} c_0 = |u_0|_\infty \quad \text{for } \ x \in \Omega.$$

Furthermore, it holds that $u(x,t) = 0 \le v(x,t)$ on $\partial\Omega \times \mathbb{R}^+$. By the comparison principle, we can deduce that $u \le v$ in Q. Moreover, we can also derive $-v \le u$ in Q. Thus we have

$$|u(\cdot,t)|_\infty \le C(t+1)^{-1/2} \quad \text{for } \ t > 0$$

with some positive constant C independent of x and t. Inequality (34) can be also proved as in the proof of Theorem 5.

Remark 7. For a general domain Ω in $\mathbb{R}^N$, which is possibly unbounded in all directions, repeating the same argument with the functions v and w as in the proofs of Theorems 4 and 5, respectively, one can ensure at least the following fact: for each $u_0 \in C_0(\Omega)$, there exists a positive constant C independent of x and t such that the unique bounded viscosity solution u of (4)–(6) satisfies

$$|u(\cdot,t)|_\infty \le C(t+1)^{-1/6} \quad \text{for } \ t > 0.$$

Moreover, if $u_0 \ge 0$ and $u_0 \not\equiv 0$, then there is a constant $c > 0$ independent of x and t such that

$$c(t+1)^{-1/2} \le |u(\cdot,t)|_\infty. \quad \text{for } \ t > 0.$$

6. Asymptotic behavior of viscosity solutions for inhomogeneous Dirichlet case

In this section, we investigate the asymptotic behavior of viscosity solutions of (4)–(6) with $\varphi \not\equiv 0$. In order to do so, we direct our attention to the stationary solution ϕ of (4)–(5), which is the unique viscosity solution of the Dirichlet problem (2), (3) (see, e.g., [11] for the definition of viscosity solutions to (2), (3)). Then one can verify that every viscosity solution $u(\cdot,t)$ of (4)–(6) converges to the unique stationary solution ϕ as $t \to \infty$. In §6.1 we impose an additional assumption $\inf_{x\in\Omega} |D\phi(x)| > 0$ on ϕ, and derive the convergence of $u(\cdot,t)$ as $t \to \infty$ at an exponential rate. To prove this, we present barrier functions deeply related to ϕ. In §6.2 we also establish a lower estimate for the convergence rate in a special setting, where $\partial\Omega$ is composed of two disjoint closed subsets of $\mathbb{R}^N$ and φ takes two different constant values on $\partial\Omega$. Finally in §6.3 we deal with the general case where ϕ may not satisfy the assumption used in §6.1. More precisely, we obtain the convergence at the rate of $O(t^{-1/p})$ for any $p > 4$, by combining the method of proof employed in §6.1 with some approximations of ϕ recently developed in [7].

6.1. Exponential convergence

Our result of this subsection is the following:

Theorem 7. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $Q = \Omega \times \mathbb{R}^+$. Let $\varphi \in C(\partial\Omega)$ and let $\phi \in C(\overline{\Omega})$ be the unique viscosity solution of the elliptic problem* (2), (3) *in Ω. Suppose that there exists a constant $\delta > 0$ such that*

$$|D\theta(x)| \geq \delta \quad \text{if} \quad \phi - \theta \text{ attains its maximum or minimum at } x \\ \text{for all } x \in \Omega \text{ and } \theta \in C^2(\Omega). \tag{37}$$

Let $u \in C(\overline{Q})$ be a viscosity solution in Q of (4)–(6) *with an initial data $u_0 \in C(\overline{\Omega})$ satisfying $u_0 = \varphi$ on $\partial\Omega$. Then there exist positive constants $\lambda_0 = \lambda_0(|\phi|_\infty, \delta)$ and $C_0 = C_0(|\phi|_\infty, |u_0|_\infty)$ independent of x and t such that*

$$\sup_{x\in\Omega} |u(x,t) - \phi(x)| \leq C_0 e^{-\lambda_0 t} \quad \text{for all} \quad t > 0. \tag{38}$$

Remark 8. (i) If $\phi \in C^1(\Omega)$, then (37) can be simply written as

$$\inf_{x\in\Omega} |D\phi(x)| \geq \delta. \tag{39}$$

It is also known that if $N = 2$ then every viscosity solution ϕ of (2) belongs to $C^1(\Omega)$ (see [16]).

(ii) We can easily give an example of Ω and φ which satisfy all the assumptions of Theorem 7. Indeed, set $p(x) := \langle a, x\rangle + c_0$ for $x \in \mathbb{R}^N$, where $a \in \mathbb{R}^N \setminus \{0\}$ and $c_0 \in \mathbb{R}$. Moreover, let Ω be a bounded domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$, and put $\varphi := p|_{\partial\Omega}$. Then $\phi := p|_{\overline{\Omega}}$ becomes the unique classical solution of (2), (3), and furthermore, $|D\phi(x)| = |a| > 0$ for all $x \in \Omega$. Hence by Theorem 7, for any initial data $u_0 \in C(\overline{\Omega})$ satisfying $u_0 = \varphi$ on $\partial\Omega$, every viscosity solution $u(\cdot, t)$ of (4)–(6) converges to ϕ as $t \to \infty$ at an exponential rate.

(iii) Aronsson [3] proved that (2), (3) does not admit a non-constant classical solution $\phi \in C^2(\Omega)$ for which $D\phi(x_0) = 0$ at some $x_0 \in \Omega$ if $N = 2$ (see also [11]). Moreover, Yu [18] also obtained the same conclusion for C^2 infinity-harmonic functions with general N.

Theorem 8 (Yu [18]). *Let $\phi \in C^2(\Omega)$ be a solution of* (2) *in Ω. If $D\phi(x_0) = 0$ for some $x_0 \in \Omega$, then $\phi(x)$ is constant in Ω.*

(iv) If $\phi \in C^2(\Omega)$ and (39) is not satisfied, that is, $\inf_{x\in\Omega} |D\phi(x)| = 0$, then there exist a sequence $\{x_n\}$ in Ω and $x_0 \in \overline{\Omega}$ such that $x_n \to x_0$ and $D\phi(x_n) \to 0$ in $\mathbb{R}^N$. The case where $x_0 \in \Omega$ can be reduced to the homogeneous Dirichlet case, i.e., $\varphi \equiv 0$ on $\partial\Omega$, since Theorem 8 implies that ϕ is constant. The case where $x_0 \in \partial\Omega$ still remains for §6.3.

(v) In case Ω is a bounded smooth domain in $\mathbb{R}^2$, every non-constant solution $\phi \in C^2(\Omega)$ satisfies $\inf_{x\in\Omega} |D\phi(x)| > 0$ (see [3]). Hence we can ensure the same conclusion as in Theorem 7 by assuming that $\phi \in C^2(\Omega)$ instead of (37) with $\delta > 0$.

Now, we proceed to prove Theorem 7.

Proof (Proof of Theorem 7). Define the barrier function $v^+ : \overline{Q} \to \mathbb{R}$ by

$$v^+(x, t) := \phi(x) + Ce^{-\lambda t} \left\{\phi(x) + |\phi|_\infty + 1\right\}^\alpha,$$

where the constants $\lambda > 0$ and $\alpha \in (0, 1)$ will be determined later and $C := \sup_{x\in\Omega} |u_0(x) - \phi(x)|$. If $x \in \partial\Omega$, then

$$v^+(x, t) \ge \phi(x) = \varphi(x) \quad \text{for all} \;\; t > 0.$$

Moreover, we observe that

$$\begin{aligned} v^+(x, 0) &= \phi(x) + C\{\phi(x) + |\phi|_\infty + 1\}^\alpha \\ &\ge \phi(x) + C \ge u_0(x) \quad \text{for all} \;\; x \in \Omega. \end{aligned}$$

Now we shall determine $\lambda > 0$ so that v^+ becomes a viscosity supersolution of (4) in Q. To this end, let $(x_0, t_0) \in Q$ and $\psi \in C^2(Q)$ be such that

$$\min_{(x,t)\in Q} (v^+ - \psi)(x, t) = (v^+ - \psi)(x_0, t_0) = 0. \tag{40}$$

Then the function $f^+(s) := s + Ce^{-\lambda t_0}\{s + |\phi|_\infty + 1\}^\alpha$ is smooth and its derivative $(f^+)'$ is positive in $(-|\phi|_\infty - 1, \infty)$. Put

$$F^+(x, s) := \psi(x, t_0) - f^+(s).$$

Then F^+ is of class C^2 in $\Omega \times (-|\phi|_\infty - 1, \infty)$. We also note that

$$F^+(x_0, \phi(x_0)) = 0, \quad \frac{\partial F^+}{\partial s}(x_0, \phi(x_0)) = -(f^+)'(\phi(x_0)) \neq 0.$$

Hence due to the implicit function theorem, there exist a neighborhood U of x_0 and $\theta \in C^2(U)$ such that $\theta(x_0) = \phi(x_0)$ and $F^+(x, \theta(x)) = 0$ for $x \in U$, i.e.,

$$\psi(x, t_0) = f^+(\theta(x)) = \theta(x) + Ce^{-\lambda t_0}\left\{\theta(x) + |\phi|_\infty + 1\right\}^\alpha. \tag{41}$$

Moreover, compare (41) with the fact that $v^+(x, t_0) = f^+(\phi(x))$ by the definition of v^+, and recall the inequality $\psi \leq v^+$ in Q by (40). Then since f^+ is strictly increasing, we have $\theta \leq \phi$ in U. Therefore $\phi - \theta$ attains its minimum, zero, at x_0. Since ϕ solves (2) in the viscosity sense, we have $-\Delta_\infty\theta(x_0) \geq 0$. Furthermore, it follows from (37) that $|D\theta(x_0)| \geq \delta > 0$.

For simplicity of computation, we put

$$A := \phi(x_0) + |\phi|_\infty + 1 = \theta(x_0) + |\phi|_\infty + 1.$$

Since $\psi_t(x_0, t_0) = v_t^+(x_0, t_0)$ by (40), we see that

$$\psi_t(x_0, t_0) = v_t^+(x_0, t_0) = -\lambda Ce^{-\lambda t_0}A^\alpha.$$

Moreover, we differentiate (41) to get

$$\begin{aligned} D_i\psi(x_0, t_0) &= D_i\theta(x_0) + \alpha Ce^{-\lambda t_0}A^{\alpha-1}D_i\theta(x_0) \\ &= \left[1 + \alpha Ce^{-\lambda t_0}A^{\alpha-1}\right] D_i\theta(x_0), \\ D_{ij}^2\psi(x_0, t_0) &= \alpha(\alpha-1)Ce^{-\lambda t_0}A^{\alpha-2}D_i\theta(x_0)D_j\theta(x_0) \\ &\quad + \left[1 + \alpha Ce^{-\lambda t_0}A^{\alpha-1}\right] D_{ij}^2\theta(x_0). \end{aligned}$$

Since $\Delta_\infty\theta(x_0) \leq 0$, it follows that

$$\begin{aligned} \Delta_\infty\psi(x_0, t_0) &= \alpha(\alpha-1)Ce^{-\lambda t_0}A^{\alpha-2} \\ &\qquad \times \left[1 + \alpha Ce^{-\lambda t_0}A^{\alpha-1}\right]^2 |D\theta(x_0)|^4 \\ &\qquad\quad + \left[1 + \alpha Ce^{-\lambda t_0}A^{\alpha-1}\right]^3 \Delta_\infty\theta(x_0) \\ &\leq \alpha(\alpha-1)Ce^{-\lambda t_0}A^{\alpha-2} \\ &\qquad \times \left[1 + \alpha Ce^{-\lambda t_0}A^{\alpha-1}\right]^2 |D\theta(x_0)|^4 \\ &\leq \alpha(\alpha-1)Ce^{-\lambda t_0}A^{\alpha-2}\delta^4, \end{aligned}$$

where we have used the fact that $\alpha - 1 < 0$. Therefore we obtain

$$\begin{aligned}
&\psi_t(x_0,t_0) - \Delta_\infty \psi(x_0,t_0) \\
&\geq -\lambda C e^{-\lambda t_0} A^\alpha + \alpha(1-\alpha) C e^{-\lambda t_0} A^{\alpha-2} \delta^4 \\
&\geq C e^{-\lambda t_0} A^\alpha \left[-\lambda + \alpha(1-\alpha) \left(2|\phi|_\infty + 1\right)^{-2} \delta^4 \right] \geq 0,
\end{aligned}$$

by choosing $\lambda > 0$ so small that

$$-\lambda + \alpha(1-\alpha) \left(2|\phi|_\infty + 1\right)^{-2} \delta^4 \geq 0.$$

Hence by Remark 1, we deduce from the arbitrariness of (x_0,t_0) and ψ that v^+ becomes a viscosity supersolution of (4) in Q. Moreover, the estimate for the size of λ above is at its best if we take $\alpha = 1/2$.

Next we apply the same reasoning to functions $-u$ and $-\phi$ that are solutions to (4)–(6) and (2), (3), respectively, with φ and u_0 replaced by $-\varphi$ and $-u_0$. This yields that

$$w(x,t) = -\phi(x) + C e^{-\lambda t} \left\{ -\phi(x) + |-\phi|_\infty + 1 \right\}^\alpha$$

is a viscosity supersolution of (4) and, in particular, that $w \geq -u$ in Q. Here $\alpha = 1/2$, and λ and C are the constants chosen above. Thus if we set

$$v^-(x,t) = -w(x,t) = \phi(x) - C e^{-\lambda t} \left\{ -\phi(x) + |\phi|_\infty + 1 \right\}^\alpha,$$

then we have $u \geq v^-$ in Q. Putting these things together, by the comparison principle, we obtain

$$\sup_{x \in \Omega} |u(x,t) - \phi(x)| \leq C e^{-\lambda t} \left(2|\phi|_\infty + 1\right) \qquad \text{for all} \;\; t > 0,$$

which proves the claim.

Remark 9. The method of proof for Theorem 7 could be also applied to other degenerate parabolic equations such as

$$u_t = \Delta_p u = (p-2)|Du|^{p-4} \Delta_\infty u + |Du|^{p-2} \Delta u \tag{42}$$

with $p \geq 2$ for the inhomogeneous Dirichlet case. More precisely, we can prove that $u(\cdot,t)$ converges to a stationary solution ϕ as $t \to \infty$ at an exponential rate under the assumption (37) with $\delta > 0$.

6.2. Lower estimate for the convergence rate

We can also establish a lower estimate for the distance between $u(\cdot,t)$ and ϕ in a special setting.

Proposition 4. *Let Ω be a (possibly) unbounded domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$, which is composed of two disjoint closed subsets Γ_1, Γ_2 of $\mathbb{R}^N$ (that is, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$) satisfying $\rho(\Gamma_1,\Gamma_2) := \inf\{|x-y|;\ x \in \Gamma_1,\ y \in \Gamma_2\} > 0$, and let $Q = \Omega \times \mathbb{R}^+$. Let $\varphi \in C(\partial\Omega)$ be such that*

$$\varphi(x) = \begin{cases} a & \text{if } x \in \Gamma_1, \\ b & \text{if } x \in \Gamma_2 \end{cases} \tag{43}$$

with two different numbers $a, b \in \mathbb{R}$. Let ϕ be the unique bounded viscosity solution of (2), (3) in Ω. Then there exist $u_0 \in C(\overline{\Omega})$ satisfying $u_0 = \varphi$ on $\partial\Omega$ and a bounded viscosity solution $u \in C(\overline{Q})$ of (4)–(6) with the data φ and u_0 such that

$$\sup_{x\in\Omega} |u(x,t) - \phi(x)| \geq C_0 e^{-\lambda t} \quad \text{for } t > 0, \tag{44}$$

where λ and C_0 are positive constants independent of x and t.

Remark 10. As simple examples of Ω, φ and ϕ satisfying all the assumptions of Proposition 4, we give the following.

(i) $\phi(x) = |x|$, $\Omega = \{x \in \mathbb{R}^N; 1 < |x| < 2\}$, $\Gamma_i = \{x \in \mathbb{R}^N; |x| = i\}$ for $i = 1, 2$, $a = 1$ and $b = 2$.
(ii) $\phi(x) = \langle \alpha, x\rangle + c_0$ with $\alpha \in \mathbb{R}^N \setminus \{0\}$ and $c_0 \in \mathbb{R}$, $\Omega = \{x \in \mathbb{R}^N; 1 < \phi(x) < 2\}$, $\Gamma_i = \{x \in \mathbb{R}^N; \phi(x) = i\}$ for $i = 1, 2$, $a = 1$ and $b = 2$.

We observe that $\inf_{x\in\Omega} |D\phi(x)| = 1$ in (i); $\inf_{x\in\Omega} |D\phi(x)| = |\alpha| > 0$ in (ii). Thus, in view of Proposition 4, the order of convergence obtained in Theorem 7 is optimal in the following sense

$$C_1 e^{-\lambda_1 t} \leq \sup_{x\in\Omega} |u(x,t) - \phi(x)| \leq C_2 e^{-\lambda_2 t} \quad \text{for } t > 0$$

with some positive constants $C_1, C_2, \lambda_1, \lambda_2$ such that $C_1 \leq C_2$ and $\lambda_2 \leq \lambda_1$. Hence $u(\cdot,t)$ converges to ϕ uniformly in Ω at an exponential rate as $t \to \infty$; however, our proofs of Theorem 7 and Proposition 4 do not derive $\lambda_1 = \lambda_2$ in general; indeed, the exponents λ_1 and λ_2 were taken very large and small respectively.

Remark 11. If Ω is unbounded, then the Dirichlet problem (2), (3) may have more than one solution. However, if φ is bounded, then by the results in [7] there exists a unique *bounded* solution ϕ to (2), (3) that satisfies

$\inf \varphi \leq \phi \leq \sup \varphi$. Moreover, if φ is given by (43), such a unique bounded solution ϕ satisfies

$$|D\phi|_\infty \leq \mathrm{Lip}_\phi(\overline{\Omega}) = \mathrm{Lip}_\varphi(\partial\Omega) = \frac{|a-b|}{\rho(\Gamma_1, \Gamma_2)} \tag{45}$$

(see, e.g., Remark 3.4 of [4]).

To prove Proposition 4, we use the following well-known fact.

Lemma 2. *Let Ω and ϕ be as in Proposition 4. Then ϕ belongs to $W^{1,\infty}(\Omega)$. Moreover, for all $x_0 \in \Omega$ and $\theta \in C^1(\Omega)$, it follows that*

$$|D\theta(x_0)| \leq |D\phi|_\infty, \tag{46}$$

provided that $\phi - \theta$ attains its maximum at x_0.

Proof. By Remark 11, ϕ belongs to $W^{1,\infty}(\Omega)$. Moreover, let $x_0 \in \Omega$ and $\theta \in C^1(\Omega)$ satisfy the assumption of this lemma. Then we can assume that $\phi(x_0) = \theta(x_0)$ without any loss of generality, by replacing the function $\theta(x)$ by $\theta(x) + \phi(x_0) - \theta(x_0)$ if necessary. Let n be an arbitrary unit vector in $\mathbb{R}^N$. We calculate the directional derivative of $\theta(x)$ at x_0 in the direction n. Choose a positive constant ε so small that $\mathbf{B}(x_0; \varepsilon) \subset \Omega$. Put $u(t) = \phi(x_0 + tn)$, $v(t) = \theta(x_0 + tn)$ and $L = |D\phi|_\infty$. Then u is Lipschitz continuous with the constant L, and v is of class C^1 in $(-\varepsilon, \varepsilon)$ since $\theta \in C^1(\Omega)$. Moreover, by assumption, it holds that $u(t) \leq v(t)$ for $t \in (-\varepsilon, \varepsilon)$ and $u(0) = v(0)$. Therefore we see

$$-L \leq \frac{u(t) - u(0)}{t} \leq \frac{v(t) - v(0)}{t} \quad \text{for} \quad 0 < t < \varepsilon,$$
$$\frac{v(t) - v(0)}{t} \leq \frac{u(t) - u(0)}{t} \leq L \quad \text{for} \quad -\varepsilon < t < 0.$$

Letting $t \to 0$, we have $|v'(0)| \leq L$, i.e., $|\langle D\theta(x_0), n\rangle| \leq L$. Since n is an arbitrary unit vector, we obtain (46).

Proof (Proof of Proposition 4). Let us assume that $a < b$. From Remark 11 with (43), it follows that

$$a \leq \phi(x) \leq b \quad \text{for} \quad x \in \Omega. \tag{47}$$

We set

$$u_0(x) := \phi(x) + h(\phi(x)), \quad h(s) := \frac{b-a}{2\pi} \sin\left(\frac{\pi}{b-a}(s-a)\right).$$

Then, since $h(a) = h(b) = 0$, we can easily check $u_0 = \varphi$ on $\partial\Omega$. Moreover, we put

$$v(x,t) := \phi(x) + e^{-\lambda t} h(\phi(x))$$

and claim that v becomes a viscosity subsolution of (4)–(6), provided that λ is large enough. It follows clearly that $v(\cdot,t)=\varphi$ on $\partial\Omega$ for $t>0$ and $v(\cdot,0)=u_0$ in Ω. To prove that v is a viscosity subsolution of (4) in Q, let $(x_0,t_0)\in Q$ and $\psi\in C^2(Q)$ be such that

$$\max_{(x,t)\in Q}(v-\psi)(x,t)=(v-\psi)(x_0,t_0)=0. \tag{48}$$

We put $f(s):=s+e^{-\lambda t_0}h(s)$, which has a derivative of the form

$$f'(s)=1+\frac{e^{-\lambda t_0}}{2}\cos\left(\frac{\pi}{b-a}(s-a)\right)\geq\frac{1}{2}\quad\text{for}\quad s\in\mathbb{R}. \tag{49}$$

Therefore f is C^∞-diffeomorphic in $\mathbb{R}$. Define $\theta(x):=f^{-1}(\psi(x,t_0))\in C^2(\Omega)$, which is rewritten into

$$\psi(x,t_0)=f(\theta(x))=\theta(x)+e^{-\lambda t_0}h(\theta(x)). \tag{50}$$

The definition of v means

$$v(x,t_0)=f(\phi(x))=\phi(x)+e^{-\lambda t_0}h(\phi(x)).$$

Hence comparing two relations above and using (48), we find $\theta(x_0)=\phi(x_0)$. Note that $v\leq\psi$ in Q by (48). As f is increasing, it follows that $\phi\leq\theta$ in Ω. Hence $\phi-\theta$ attains its maximum, zero, at x_0. Since ϕ solves (2) in the viscosity sense, we get $-\Delta_\infty\theta(x_0)\leq 0$. Moreover, Lemma 2 together with (45) gives $|D\theta(x_0)|\leq C$ with a constant C independent of θ and x_0.

Since $\psi_t(x_0,t_0)=v_t(x_0,t_0)$ by (48), we have

$$\psi_t(x_0,t_0)=-\lambda e^{-\lambda t_0}h(\phi(x_0))=-\lambda e^{-\lambda t_0}h(\theta(x_0)).$$

Differentiating (50), we obtain

$$D_i\psi(x_0,t_0)=\left[1+e^{-\lambda t_0}h'(\theta(x_0))\right]D_i\theta(x_0)$$

and

$$\begin{aligned}D^2_{ij}\psi(x_0,t_0)&=e^{-\lambda t_0}h''(\theta(x_0))D_i\theta(x_0)D_j\theta(x_0)\\&\quad+\left[1+e^{-\lambda t_0}h'(\theta(x_0))\right]D^2_{ij}\theta(x_0).\end{aligned}$$

Since $\Delta_\infty\theta(x_0)\geq 0$ and $1+e^{-\lambda t_0}h'(\theta(x_0))=f'(\theta(x_0))\geq 0$ by (49), it follows that

$$\begin{aligned}\Delta_\infty\psi(x_0,t_0)&=e^{-\lambda t_0}h''(\theta(x_0))\left[1+e^{-\lambda t_0}h'(\theta(x_0))\right]^2|D\theta(x_0)|^4\\&\quad+\left[1+e^{-\lambda t_0}h'(\theta(x_0))\right]^3\Delta_\infty\theta(x_0)\\&\geq e^{-\lambda t_0}h''(\theta(x_0))\left[1+e^{-\lambda t_0}h'(\theta(x_0))\right]^2|D\theta(x_0)|^4.\end{aligned}$$

Here combining the fact that $\theta(x_0) = \phi(x_0)$ with (47), we have

$$h(\theta(x_0)) \geq 0, \quad h''(\theta(x_0)) = -\left(\frac{\pi}{b-a}\right)^2 h(\theta(x_0)) \leq 0,$$

which implies

$$\begin{aligned}\Delta_\infty \psi(x_0, t_0) \geq & -\left(\frac{\pi}{b-a}\right)^2 e^{-\lambda t_0} h(\theta(x_0)) \\ & \times \left[1 + e^{-\lambda t_0} h'(\theta(x_0))\right]^2 |D\theta(x_0)|^4 \\ \geq & -\frac{9}{4} C^4 \left(\frac{\pi}{b-a}\right)^2 e^{-\lambda t_0} h(\theta(x_0)).\end{aligned}$$

In the last inequality, we have used the fact that $|D\theta(x_0)| \leq C$ due to Lemma 2 and

$$|1 + e^{-\lambda t_0} h'(\theta(x_0))| = \left|1 + \frac{e^{-\lambda t_0}}{2} \cos\left(\frac{\pi}{b-a}(s-a)\right)\right| \leq \frac{3}{2}.$$

Therefore

$$\begin{aligned}& \psi_t(x_0, t_0) - \Delta_\infty \psi(x_0, t_0) \\ & \leq e^{-\lambda t_0} h(\theta(x_0)) \left[-\lambda + \frac{9}{4} C^4 \left(\frac{\pi}{b-a}\right)^2\right] \leq 0,\end{aligned}$$

provided that λ is so large that $-\lambda + (9/4)C^4 \pi^2 (b-a)^{-2} \leq 0$. Consequently, v is a viscosity subsolution of (4)–(6) in Q.

Moreover, since u_0 is bounded and Lipschitz continuous on $\overline{\Omega}$ and $u_0 = \varphi$ on $\partial\Omega$, there exists a bounded viscosity solution u to (4)–(6) in Q. Hence the comparison principle (see [9]) yields

$$u(x,t) \geq v(x,t) = \phi(x) + e^{-\lambda t} h(\phi(x)) \quad \text{for } x \in \Omega \text{ and } t > 0.$$

Since $\phi \in C(\overline{\Omega})$, there is $x_* \in \Omega$ by (43) such that $\phi(x_*) = (a+b)/2$. Then we have

$$u(x_*, t) - \phi(x_*) \geq \frac{b-a}{2\pi} e^{-\lambda t} \quad \text{for } t > 0,$$

which implies (44).

6.3. Convergence to general stationary solutions

We finally discuss the case where the stationary solution ϕ may not satisfy (37). An example of such a solution is the famous explicit solution of Aronsson, $\phi(x) = x_1^{4/3} - x_2^{4/3}$ for $x = (x_1, x_2) \in \Omega := \mathbf{B}(0;1)$ with $N = 2$.

Theorem 9. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $Q = \Omega \times \mathbb{R}^+$. Let $\varphi \in C(\partial\Omega)$ and let $\phi \in C(\overline{\Omega})$ be the unique viscosity solution of the elliptic problem* (2), (3) *in Ω. Let $u \in C(\overline{Q})$ be a viscosity solution in Q to* (4)–(6) *with an initial data $u_0 \in C(\overline{\Omega})$ satisfying $u_0 = \varphi$ on $\partial\Omega$. Then for any $p > 4$, there exists a positive constant C_p independent of x and t such that*

$$\sup_{x\in\Omega} |u(x,t) - \phi(x)| \leq C_p(t+1)^{-1/p} \quad \textit{for} \ \ t \geq 0. \tag{51}$$

In our proof for Theorem 9, we employ some approximations of the stationary solution ϕ recently developed in [7]. Before going to details, we recall the definition of the *local Lipschitz constant* of a function $f : \Omega \to \mathbb{R}$ at $x \in \Omega$. We denote

$$L(f,x) := \lim_{r\to+0} \mathrm{Lip}_f(\mathbf{B}(x;r)).$$

We can now formulate our lemma on the approximations of ϕ.

Lemma 3. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let $\phi \in C(\overline{\Omega})$ be a viscosity solution of* (2) *in Ω. For $\varepsilon > 0$, put $V_\varepsilon := \{x \in \Omega\colon L(\phi,x) < \varepsilon\}$. Then there exist $\phi_\varepsilon^+, \phi_\varepsilon^- \in C(\overline{\Omega})$ such that*

(i) *ϕ_ε^+ and ϕ_ε^- are a viscosity supersolution and a subsolution of* (2) *in Ω, respectively*;
(ii) *$\phi_\varepsilon^+ = \phi_\varepsilon^- = \phi$ on $\overline{\Omega} \setminus V_\varepsilon$ and $\phi_\varepsilon^- \leq \phi \leq \phi_\varepsilon^+$ on $\overline{\Omega}$*;
(iii) *$L(\phi_\varepsilon^\pm, x) \geq \varepsilon$ for $x \in \Omega$*;
(iv) *it follows that*

$$\sup_{x\in\Omega} |\phi(x) - \phi_\varepsilon^\pm(x)| \leq 2\,\mathrm{diam}(\Omega)\varepsilon, \tag{52}$$

where $\mathrm{diam}(\Omega)$ *is the diameter of Ω.*

Remark 12. In the case where ϕ is a viscosity subsolution (respectively, supersolution) of (2) in Ω, one can still ensure the existence of the function ϕ_ε^- (respectively, ϕ_ε^+) satisfying (i)–(iv).

Proof. The assertions (i)–(iii) have already been proved in [7], where the convergence of $\phi_\varepsilon^\pm$ as $\varepsilon \to +0$ is also obtained without any explicit estimate such as (52). Thus it suffices to establish (52). To this end, let us recall the construction of ϕ_ε^- performed in [7]. Since ϕ is a solution of (2), the function $x \mapsto L(\phi, x)$ is upper-semicontinuous, and thus the set V_ε is open in $\mathbb{R}^N$. Then one can define a function $w_\varepsilon^- : V_\varepsilon \to \mathbb{R}$ by solving

$$\varepsilon - |Dw_\varepsilon^-| = 0 \quad \text{in } V_\varepsilon, \quad w_\varepsilon^- = \phi \quad \text{on } \partial V_\varepsilon$$

in the viscosity sense (the explicit form of w_ε^- will be given later). Moreover, $w_\varepsilon^- \le \phi$ in V_ε and w_ε^- is a viscosity subsolution of (2) in V_ε. The function $\phi_\varepsilon^- \in C(\overline{\Omega})$ is given by $\phi_\varepsilon^-(x) = \phi(x)$ if $x \in \overline{\Omega} \setminus V_\varepsilon$; $\phi_\varepsilon^-(x) = w_\varepsilon^-(x)$ if $x \in V_\varepsilon$. Then ϕ_ε^- enjoys the properties (i)–(iii) (see Theorem 2.1 of [7] for more details), and moreover,

$$0 \le \sup_{x \in \Omega} \left(\phi(x) - \phi_\varepsilon^-(x) \right) = \sup_{x \in V_\varepsilon} \left(\phi(x) - w_\varepsilon^-(x) \right). \tag{53}$$

Let $x \in V_\varepsilon$ be arbitrarily given and let U_ε be a connected component of V_ε such that $x \in U_\varepsilon$. In [7], the function w_ε^- is explicitly given as follows:

$$w_\varepsilon^-(x) := \varepsilon \sup_{y \in \partial U_\varepsilon} \left(\frac{\phi(y)}{\varepsilon} - d_{U_\varepsilon}(x,y) \right) = \sup_{y \in \partial U_\varepsilon} \left(\phi(y) - \varepsilon d_{U_\varepsilon}(x,y) \right),$$

where $d_{U_\varepsilon}(x,y)$ stands for the distance between x and y in U_ε (see [7] for its precise definition), and $d_{U_\varepsilon}(x,y)$ coincides with $|x-y|$, if the line segment $[x,y] := \{(1-\theta)x + \theta y; \theta \in [0,1]\}$ is included in $\overline{U}_\varepsilon$.

Since U_ε is open in $\mathbb{R}^N$, we can take the largest ball $\mathbf{B}(x; r_\varepsilon)$ included in U_ε, where r_ε is the distance between x and ∂U_ε, and choose $z_\varepsilon \in \partial U_\varepsilon \cap \partial \mathbf{B}(x; r_\varepsilon)$. By Remark 2.16 of [4], we note that

$$|\phi(x) - \phi(y)| \le \left(\sup_{x' \in U_\varepsilon} L(\phi, x') \right) |x - y| \le \varepsilon |x - y|$$

for all $y \in U_\varepsilon$ such that $[x,y] \subset U_\varepsilon$, because $\phi \in C(\overline{\Omega})$ and $U_\varepsilon \subset V_\varepsilon$. Choosing $y = (1-\theta)x + \theta z_\varepsilon$ with $\theta \in (0,1)$ above and then letting $\theta \to 1$, we obtain

$$|\phi(x) - \phi(z_\varepsilon)| \le \varepsilon |x - z_\varepsilon|.$$

Moreover, we have $d_{U_\varepsilon}(x, z_\varepsilon) = |x - z_\varepsilon|$ from the fact that $[x, z_\varepsilon] \subset \overline{U}_\varepsilon$. Therefore

$$\begin{aligned} 0 \le \phi(x) - w_\varepsilon^-(x) &\le \phi(x) - \phi(z_\varepsilon) + \varepsilon d_{U_\varepsilon}(x, z_\varepsilon) \\ &\le 2\varepsilon |x - z_\varepsilon| \le 2 \mathrm{diam}(\Omega) \varepsilon. \end{aligned}$$

Thus recalling (53), we conclude that

$$0 \le \sup_{x \in \Omega} \left(\phi(x) - \phi_\varepsilon^-(x) \right) \le 2 \, \mathrm{diam}(\Omega) \varepsilon.$$

Furthermore, we can also construct a viscosity supersolution ϕ_ε^+ of (2) in Ω and verify our desired results, in particular, (52), by repeating the argument above with obvious modifications.

We also prepare the following lemma, which gives an interpretation of $L(\phi, x)$ in the viscosity sense.

Lemma 4. *Let Ω be an open set in $\mathbb{R}^N$ and let ϕ be a viscosity subsolution (respectively, supersolution) of (2) in Ω. Let $x_0 \in \Omega$ and $\theta \in C^1(\Omega)$ be such that $\phi - \theta$ attains its local maximum (respectively, minimum) at x_0. Then it holds that*

$$L(\phi, x_0) \le |D\theta(x_0)|. \tag{54}$$

Proof. We first treat the case where ϕ is a subsolution. From the assumptions on x_0 and θ, we have $\phi - \theta \le (\phi - \theta)(x_0)$ in a neighborhood U of x_0, which gives

$$\phi(x) - \phi(x_0) \le \theta(x) - \theta(x_0) \quad \text{for } \ x \in U.$$

Since $-\Delta_\infty \phi \le 0$ in the viscosity sense, we have

$$L(\phi, x_0) = \lim_{r \to +0} \max \left\{ \frac{\phi(x) - \phi(x_0)}{r};\ x \in \partial \mathbf{B}(x_0; r) \right\}$$

(see Lemma 4.6 of [6]). On the other hand, let x_r be a maximum point of θ on $\partial \mathbf{B}(x_0; r)$. Then, since $\theta \in C^1(\Omega)$, it follows that

$$\frac{\theta(x_r) - \theta(x_0)}{r} \to |D\theta(x_0)| \quad \text{as } \ r \to +0.$$

Therefore we conclude that $L(\phi, x_0) \le |D\theta(x_0)|$. As for the case where ϕ is a supersolution, we can also derive (54), since $-\phi$ becomes a subsolution in Ω and $L(\phi, x_0) = L(-\phi, x_0)$.

We proceed to give a proof of Theorem 9.

Proof (Proof of Theorem 9). For $\varepsilon > 0$, let ϕ_ε^+ and ϕ_ε^- be the functions provided by Lemma 3 and define

$$v^+(x,t) := \phi_\varepsilon^+(x) + Ce^{-\lambda t} \left\{ \phi_\varepsilon^+(x) + |\phi|_\infty + 2 \right\}^{1/2}$$

with $C := \sup_{x \in \Omega} |u_0(x) - \phi(x)| + 1$ and a constant $\lambda > 0$ which will be determined later. Put $C_1 := 2\,\mathrm{diam}(\Omega)$. By (iv) of Lemma 3, we note that

$$\phi_\varepsilon^+(x) \ge -\sup_{x \in \Omega} |\phi_\varepsilon^+(x) - \phi(x)| + \phi(x) \ge -C_1\varepsilon - |\phi|_\infty \ge -1 - |\phi|_\infty$$

for $\varepsilon \in (0, 1/C_1)$. Hence $v^+(\cdot, t) \ge \phi_\varepsilon^+ = \varphi$ on $\partial\Omega$. Moreover

$$v^+(x, 0) \ge \phi_\varepsilon^+(x) + C \ge u_0(x) \quad \text{for } \ x \in \Omega.$$

We next prove that v^+ is a viscosity supersolution of (4) in Q. To do so, let $(x_0, t_0) \in Q$ and $\psi \in C^2(Q)$ be such that

$$\min_{(x,t)\in Q} (v^+ - \psi)(x,t) = (v^+ - \psi)(x_0, t_0) = 0. \tag{55}$$

Then by repeating the same argument as in the proof of Theorem 7, we can obtain a neighborhood U_ε of x_0 and a function $\theta_\varepsilon \in C^2(U_\varepsilon)$ such that $\theta_\varepsilon(x_0) = \phi_\varepsilon^+(x_0)$ and

$$\psi(x, t_0) = f^+(\theta_\varepsilon(x)) := \theta_\varepsilon(x) + Ce^{-\lambda t_0} \left\{\theta_\varepsilon(x) + |\phi|_\infty + 2\right\}^{1/2}. \tag{56}$$

Moreover, $\phi_\varepsilon^+ - \theta_\varepsilon$ attains its minimum, zero, at x_0. Hence we obtain $-\Delta_\infty \theta_\varepsilon(x_0) \geq 0$ from the fact that $-\Delta_\infty \phi_\varepsilon^+ \geq 0$ in the viscosity sense.

For simplicity of computation, we put

$$A_\varepsilon := \phi_\varepsilon^+(x_0) + |\phi|_\infty + 2 = \theta_\varepsilon(x_0) + |\phi|_\infty + 2.$$

As in the proof of Theorem 7, it follows from (55) that

$$\psi_t(x_0, t_0) = v_t^+(x_0, t_0) = -\lambda C e^{-\lambda t_0} A_\varepsilon^{1/2},$$

and moreover, by (iii) of Lemma 3 and Lemma 4, we have

$$\begin{aligned}
\Delta_\infty \psi(x_0, t_0) &= -\frac{C}{4} e^{-\lambda t_0} A_\varepsilon^{-3/2} \left[1 + \frac{C}{2} e^{-\lambda t_0} A_\varepsilon^{-1/2}\right]^2 |D\theta_\varepsilon(x_0)|^4 \\
&\quad + \left[1 + \frac{C}{2} e^{-\lambda t_0} A_\varepsilon^{-1/2}\right]^3 \Delta_\infty \theta_\varepsilon(x_0) \\
&\leq -\frac{C}{4} e^{-\lambda t_0} A_\varepsilon^{-3/2} \varepsilon^4,
\end{aligned}$$

since $\Delta_\infty \theta_\varepsilon(x_0) \leq 0$. Thus we obtain

$$\begin{aligned}
&\psi_t(x_0, t_0) - \Delta_\infty \psi(x_0, t_0) \\
&\geq -\lambda C e^{-\lambda t_0} A_\varepsilon^{1/2} + \frac{C}{4} e^{-\lambda t_0} A_\varepsilon^{-3/2} \varepsilon^4 \\
&\geq C e^{-\lambda t_0} A_\varepsilon^{1/2} \left[-\lambda + \frac{1}{4} (2|\phi|_\infty + 3)^{-2} \varepsilon^4\right] = 0,
\end{aligned}$$

by choosing

$$\lambda = \lambda_\varepsilon := \frac{1}{4} (2|\phi|_\infty + 3)^{-2} \varepsilon^4 > 0.$$

Hence from the arbitrariness of (x_0, t_0) and ψ, we deduce that v^+ becomes a viscosity supersolution of (4) in Q. Therefore the comparison principle and (iv) of Lemma 3 yield

$$\begin{aligned}
u(x,t) &\leq \phi_\varepsilon^+(x) + Ce^{-\lambda_\varepsilon t} (2|\phi|_\infty + 3)^{1/2} \\
&\leq \phi(x) + C_1 \varepsilon + Ce^{-\lambda_\varepsilon t} (2|\phi|_\infty + 3)^{1/2}.
\end{aligned} \tag{57}$$

By applying the same argument to $-\phi$ and $-u$, and noting that $(-\phi)_\varepsilon^+ = -\phi_\varepsilon^-$, we obtain

$$\begin{aligned} u(x,t) &\geq \phi_\varepsilon^-(x) - Ce^{-\lambda_\varepsilon t}\left(2|\phi|_\infty + 3\right)^{1/2} \\ &\geq \phi(x) - C_1\varepsilon - Ce^{-\lambda_\varepsilon t}\left(2|\phi|_\infty + 3\right)^{1/2}. \end{aligned} \tag{58}$$

Therefore combining (57) with (58), we deduce that

$$\sup_{x\in\Omega} |u(x,t) - \phi(x)| \leq C_1\varepsilon + C_2 \exp\left(-C_3\varepsilon^4 t\right) \tag{59}$$

for all $t > 0$ and $\varepsilon \in (0, 1/C_1)$ with positive constants C_2, C_3. Now, put $\varepsilon(t) := t^{-1/p}$ with $p > 4$. Then we can take a constant $t_* > 0$ such that $\varepsilon(t) < 1/C_1$ for all $t \geq t_*$. Hence it follows from (59) with $\varepsilon = \varepsilon(t)$ that

$$\sup_{x\in\Omega} |u(x,t) - \phi(x)| \leq C_1 t^{-1/p} + C_2 \exp\left(-C_3 t^{(p-4)/p}\right) \quad \text{for } \ t \geq t_*.$$

Since the exponential part decays faster than $t^{-1/p}$ as $t \to \infty$, we finally obtain (51).

References

1. Akagi, G., Suzuki, K.: Existence and uniqueness of viscosity solutions for a degenerate nonlinear parabolic equation associated with the infinity-Laplacian. Calculus of Variations and Partial Differential Equations, **31**, 457–471 (2008)
2. Aronsson, G.: Extension of functions satisfying Lipschitz conditions. Ark. Mat., **6**, 551–561 (1967)
3. Aronsson, G.: On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. Ark. Mat., **7**, 395–425 (1968)
4. Aronsson, G., Crandall, M., Juutinen, P.: A tour of the theory of absolutely minimizing functions. Bull. Amer. Math. Soc., **41**, 439–505 (2004)
5. Caselles, V., Morel, J.-M., Sbert, C.: An axiomatic approach to image interpolation, IEEE Transactions on Image Processing, **7** (1998), 376–386.
6. Crandall, M.G.: A visit with the ∞-Laplace equation. In: Calculus of Variations and Nonlinear Partial Differential Equations, Lecture Notes in Mathematics, **1927**, pp. 75–122. Springer Berlin/Heidelberg (2008)
7. Crandall, M.G., Gunnarsson, G., Wang, P.Y.: Uniqueness of infinity harmonic functions and the eikonal equation. Comm. Partial Differential Equations, **32**, 1587–1615 (2007)
8. Crandall, M.G., Ishii, H., Lions, P.L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc., **27**, 1–67 (1992)
9. Giga, Y., Goto, S., Ishii, H, Sato, M.-H: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J., **40**, 443–470 (1991)
10. Ishii, H.: Viscosity solutions of nonlinear second-order partial differential equations in Hilbert spaces. Comm. Partial Differential Equations, **18**, 601–650 (1993)

11. Jensen, R.: Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. Arch. Rational Mech. Anal., **123**, 51–74 (1993)
12. Juutinen, P.: Principal eigenvalue of a very badly degenerate operator and applications. J. Differential Equations, **236**, 532–550 (2007)
13. Juutinen, P., Kawohl, B.: On the evolution governed by the infinity Laplacian. Math. Ann., **335**, 819–851 (2006)
14. Lions, P.L.: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness. Comm. Partial Differential Equations, **8**, 1229–1276 (1983)
15. Maruo, K., Tomita, Y.: Radial viscosity solutions of the Dirichlet problem for semilinear degenerate elliptic equations. Osaka J. Math., **38**, 737–757 (2001)
16. Savin, O.: C^1 regularity for infinity harmonic functions in two dimensions. Arch. Rational Mech. Anal., **176**, 351–361 (2005)
17. Vázquez, J.L.: Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type. Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford (2006)
18. Yu, Y.: A remark on C^2 infinity-harmonic functions. Electronic J. Differential Equations, **2006**, 1–4 (2006)