ON THE HIGHER EIGENVALUES FOR THE ∞-EIGENVALUE PROBLEM

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ABSTRACT. We study the higher eigenvalues and eigenfunctions for the so-called ∞-eigenvalue problem. The problem arises as an asymptotic limit of the nonlinear eigenvalue problems for the p-Laplace operators and is very closely related to the geometry of the underlying domain. We are able to prove several properties that are known in the linear case $p = 2$ of the Laplacian, but are unknown for other values of p. In particular, we establish the validity of the Payne-Pólya-Weinberger conjecture regarding the ratio of the first two eigenvalues and the Payne nodal conjecture, which deals with the zero set of a second eigenfunction. The limit problem also exhibits phenomena that are not encountered for any $1 < p < \infty$.

1. INTRODUCTION

Let Ω be a given bounded domain in the Euclidean space $\mathbb{R}^n$. The minimization of the so-called Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \qquad 1 < p < \infty \tag{1.1}$$

among all nonzero functions in the Sobolev space $W_0^{1,p}(\Omega)$ leads to a nonlinear eigenvalue problem. The corresponding Euler-Lagrange equation is

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0, \tag{1.2}$$

interpreted in the usual weak form with test-functions under the integral sign. The objective of our work is the asymptotic case $p = \infty$, the so-called ∞-eigenvalue problem.

Notice that when $p = 2$, we have the familiar linear equation

$$\Delta u + \lambda u = 0, \tag{1.3}$$

a solution of which describes the shape of an eigenvibration, of frequency $\sqrt{\lambda}$, of a homogeneous membrane stretched in the frame Ω, see [9]. It is well-known that in this case the spectrum is discrete and eigenfunctions form an orthonormal basis for $L^2(\Omega)$. For any $1 < p < \infty$, the first eigenvalue, say $\lambda_1 = \lambda_1(p)$, is the minimum of the Rayleigh quotient (1.1) and it is simple and isolated. For a new and direct proof of the simplicity we refer to [6]. It is also known that the second eigenvalue, say $\lambda_2 = \lambda_2(p)$, is well-defined and has a "variational characterization", see [2]. However, surprisingly little is known about the higher eigenvalues and eigenfunctions when

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$p \neq 2$. To the best of our knowledge, it has not even been rigorously proved that it is impossible that every $\lambda \geq \lambda_2$ is an eigenvalue. Neither is anything known about the multiplicity of the higher eigenvalues. It does not seem to help to assume that Ω is a ball or some other appealing geometric object, when it comes to the characterization of *all* the higher eigenvalues. One can produce infinitely many eigenvalues using various methods but none of them seems to guarantee that the whole spectrum is exhausted.

In order to shed some light on this problem it seems to be well motivated to study the extreme cases $p = 1$ and $p = \infty$. We are here interested in the case $p = \infty$ and continue our studies in [27]. For the first eigenfunction, or the "ground state", the correct limit equation of (1.2) is

$$\max\{\Lambda_1 - |\nabla \log u|,\ \Delta_\infty u\} = 0, \tag{1.4}$$

where Λ_1 is the reciprocal number of the radius of the largest ball that can be inscribed in the domain Ω, and

$$\Delta_\infty u := \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{1.5}$$

is the nowadays very popular infinity Laplace operator. The notation in (1.4) means that at each point $x \in \Omega$ the larger of the two quantities is zero. Moreover, the above equation has to be interpreted in the viscosity sense. The ground state is in $W_0^{1,\infty}(\Omega)$. Every first eigenfunction is a minimizer of

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}}, \tag{1.6}$$

where $\|v\|_{\infty,\Omega} = \operatorname{ess\,sup}_{x\in\Omega} |v(x)|$. However, the converse is not true, as (1.6) has minimizers that do not satisfy (1.4). For all this we refer to [27]. The equation is, indeed, somewhat complicated for $p = \infty$ but it has at least one advantage: the value of Λ_1 can immediately be read off from the geometry of the underlying domain Ω. No such property is known even for $p = 2$. In passing, we mention that if Λ_1 is replaced by a number $\Lambda \neq \Lambda_1$ in equation (1.4), then there is no other solution in $W_0^{1,\infty}(\Omega)$ than $u \equiv 0$, which of course, does not qualify as an eigenfunction.

The equation for the higher eigenvalues, when $p = \infty$, is slightly more involved than (1.4), and is given below in (2.2). We are able to prove several properties in the case $p = \infty$ that are known in the linear case $p = 2$ but are unknown for $2 < p < \infty$. First, the Payne-Pólya-Weinberger conjecture about the ratio of the first two eigenvalues holds. This was proved by Ashbaugh and Benguria in [4], [5] when $p = 2$ and is, as it were, open for other finite p's. Second, the Payne nodal conjecture about the second eigenfunction holds in convex domains. This was proved by Melas [33] and Alessandrini [1] when $p = 2$. Third, the second eigenvalue has a variational characterization. All these results rely on a geometric characterization of the second eigenvalue for our problem, viz.

$$\frac{1}{\Lambda_2(\Omega)} = \sup\big\{r : \text{there are disjoint balls } B_1, B_2 \subset \Omega \text{ of radius } r\big\}, \tag{1.7}$$

and are quite easily proved once this characterization has been obtained. In fact, we give three different characterizations of Λ_2.

Given the geometric characterizations of the first and the second eigenvalue, it is natural to ask whether the entire spectrum could be obtained as a sequence of optimal values for certain sphere packing problems. At this moment, we are unable to give a definite answer, but is seems that the situation is not quite that simple. Nevertheless, the sphere packing problems are clearly related to the higher eigenvalues and can be used as an aid in the study of the spectrum with $\Lambda \geq \Lambda_2$. We show that there exists an unbounded sequence of eigenvalues for every domain Ω, and establish some other results that parallel those known in the case $1 < p < \infty$. However, not all the features of the problem (1.2) survive the passage to the limit. For example, there are domains for which the first eigenvalue Λ_1 is not isolated.

Our work [27] with Manfredi was devoted to the ground state in the case $p = \infty$. An anisotropic version has been studied in [7] by Belloni and Kawohl. The other extreme situation, the case $p = 1$, is studied in [16] by Fridman and Kawohl, see also the references therein.

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2. Preliminaries and Definitions

In order to write our equation, let us define the function $F_\Lambda : \mathbb{R} \times \mathbb{R}^n \times S_{n\times n} \to \mathbb{R}$ by

$$F_\Lambda(s, \xi, X) = \begin{cases} \min\{|\xi| - \Lambda s, -X\xi \cdot \xi\} & \text{if } s > 0, \\ -X\xi \cdot \xi & \text{if } s = 0, \\ \max\{-\Lambda s - |\xi|, -X\xi \cdot \xi\} & \text{if } s < 0, \end{cases} \tag{2.1}$$

where $X\xi \cdot \xi = \sum X_{ij}\xi_i\xi_j$. By $S_{n\times n}$ we mean the space of real, symmetric $n \times n$ matrices. Observe that F_Λ is not continuous at $s = 0$.

Definition 2.1. We say that a function $u \in C(\overline{\Omega})$, $u|_{\partial\Omega} = 0$, $u \not\equiv 0$, is an ∞-*eigenfunction*, if there exists $\Lambda \in \mathbb{R}$ such that

$$F_\Lambda(u, \nabla u, D^2 u) = 0 \qquad \text{in } \Omega \tag{2.2}$$

in the viscosity sense. The number Λ is called an ∞-*eigenvalue.*

The equation (2.2) is derived as a limit equation of (1.2) as $p \to \infty$ in Section 4 below. Notice that it has the property that a constant multiple of a solution is also a solution. Moreover, it follows from Harnack's inequality (see the appendix and [30]) that an ∞-eigenfunction is in $W_0^{1,\infty}(\Omega)$[1]. Better global regularity than Lipschitz continuity cannot be obtained, as an ∞-eigenfunction is never differentiable at its maximum and minimum points, cf. [26]. The regularity outside these points is an open problem.

For readers not familiar with the theory of viscosity solutions, we recommend [10] and [8]. A very thorough account of the general theory can be found in [11].

[1]By $W_0^{1,\infty}(\Omega)$ we mean the functions that are Lipschitz continuous in $\overline{\Omega}$ and vanish on the boundary $\partial\Omega$.

It is a common feature of eigenvalue problems that many important properties of the higher eigenvalues and eigenfunctions are derived from the fact that any eigenfunction v is a first eigenfunction of each of its *nodal domains*, that is, connected components of the set where $v \neq 0$. This is true for our problem as well, see Theorem 8.1 in the Appendix, and combined with the geometric characterization of the first eigenvalue it turns out be a valuable tool in the analysis of the rest of the spectrum. For the reader's convenience, we collect relevant facts about the first eigenvalue and eigenfunctions below. For the proof, see [27].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists a positive viscosity solution $u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ to the problem*

$$\begin{cases} \min\{|\nabla u| - \Lambda_1 u, -\Delta_\infty u\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

where

$$\Lambda_1 = \Lambda_1(\Omega) = \frac{1}{\max\limits_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)}. \tag{2.4}$$

Moreover, any positive solution u to (2.3) is a minimizer of the ∞-Rayleigh quotient:

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}} = \Lambda_1 = \inf\Big\{ \frac{\|\nabla v\|_{\infty,\Omega}}{\|v\|_{\infty,\Omega}} : v \in W_0^{1,\infty}(\Omega) \Big\}.$$

Such a u can be constructed as a cluster point, as $p \to \infty$, of a properly normalized sequence of the first eigenfunctions in (1.2). Furthermore,

$$\Lambda_1 = \lim_{p\to\infty} \lambda_1(p)^{1/p},$$

where $\lambda_1(p)$ denotes the first eigenvalue in (1.2). An alternative proof for the existence of the first eigenfunction can be found in [26]. As far as we know, the simplicity of Λ_1 is an open problem. In fact, it is not even proven that two different subsequences of the aforementioned sequence of first eigenfunctions have the same limit. However, a local uniqueness result has been obtained in [27].

We have already referred to Λ_1 as the first ∞-eigenvalue. Let us now justify this designation by showing that for any domain Ω, it is the least number Λ for which (2.2) has a nontrivial solution with zero boundary values. Indeed, if we have $F_\Lambda(u, \nabla u, D^2u) = 0$ in Ω, then u is a solution to (2.3) in a connected component N of $\{x \in \Omega : u(x) > 0\}$ (we may assume that this set is nonempty) and consequently,

$$\Lambda = \frac{\|\nabla u\|_{\infty,N}}{\|u\|_{\infty,N}} = \frac{\|\nabla(u\chi_N)\|_{\infty,\Omega}}{\|u\chi_N\|_{\infty,\Omega}} \geq \inf\Big\{ \frac{\|\nabla v\|_{\infty,\Omega}}{\|v\|_{\infty,\Omega}} : v \in W_0^{1,\infty}(\Omega) \Big\} = \Lambda_1(\Omega)$$

by Theorem 8.1 in the appendix and Theorem 2.2. Here χ_N denotes the characteristic function of the set N.

The fact that the first ∞-eigenvalue Λ_1 can be read off from the geometry of the underlying domain, as indicated by (2.4), has some immediate consequences. For example, one has the *Faber-Krahn inequality*

$$\Lambda_1(B_\omega) \leq \Lambda_1(\Omega) \qquad \text{if } B_\omega \text{ is a ball such that } |\Omega| = |B_\omega|. \tag{2.5}$$

In fact, it is elementary to show the (sharp) estimate

$$\Lambda_1(\Omega) \geq \frac{\Lambda_1(B_\omega)}{(1-\alpha(\Omega))^{1/n}}, \tag{2.6}$$

where the asymmetry of Ω,

$$\alpha(\Omega) := \inf_{x\in\mathbb{R}^n} \frac{|\Omega \setminus B_\omega(x)|}{|\Omega|}$$

see e.g. [21], [22], measures how close a set is to being a ball. Moreover, due to (2.4) it is easy to analyze the stability of Λ_1 under domain variations.

3. Eigenvalues of the p-Laplacian operator

In this section, we briefly review some known facts about the spectrum and the eigenfunctions of the p-Laplacian operator,

$$\Delta_p v := \operatorname{div}(|\nabla v|^{p-2}\nabla v), \qquad 1 < p < \infty. \tag{3.1}$$

Their usage is twofold. On one hand, some of these results are needed in the passage to the limit as $p \to \infty$. On the other hand, it is interesting to compare various features of the ∞-eigenvalue problem to the corresponding features in the theory of the p-Laplacian.

Definition 3.1. We say that a function $u \in W_0^{1,p}(\Omega) \cap C(\Omega)$, $u \not\equiv 0$, is a *p-eigenfunction*, if there exists a $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = \lambda \int_\Omega |u|^{p-2} u\phi \, dx \tag{3.2}$$

for all $\phi \in W_0^{1,p}(\Omega)$. The associated number λ is called a *p-eigenvalue*.

By the elliptic regularity theory, see e.g. [15], p-eigenfunctions are actually in $C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha > 0$. Moreover, one can show that, at least for $p \geq 2$, a function $u \in W_0^{1,p}(\Omega) \cap C(\Omega)$ is a p-eigenfunction if and only if it is a viscosity solution to $-\Delta_p u = \lambda |u|^{p-2}u$. In fact, we only need to know that any p-eigenfunction satisfies the equation in the viscosity sense. The proof is written down e.g. in [27] or [26].

We begin by describing the construction of an infinite sequence of eigenvalues. In order to generalize the idea of the Courant-Fischer minimax principle to this nonlinear setting, a suitable device for measuring the size of sets is the genus of Krasnoselskii.

Definition 3.2. Let E be a real Banach space and let $A \subset E$ be any closed symmetric set ($v \in A$ implies $-v \in A$.). The *genus* $\gamma(A)$ of the set A is defined to be the smallest integer m for which there exists a continuous odd mapping $\varphi : A \to \mathbb{R}^m\setminus\{0\}$. If no such integer exists, then we set $\gamma(A) = \infty$.

Observe, in particular, that if $0 \in A$, then $\gamma(A) = \infty$, because $\varphi(0) = 0$ for any odd mapping. In what follows, we need to compute the genus of some relatively simple subsets of $W_0^{1,p}(\Omega)$. The lemma below will be enough for our purposes. For the proof, see e.g. [38], [39].

Lemma 3.3. *Let $A \subset W_0^{1,p}(\Omega)$. If the bounded neighborhood $U \subset \mathbb{R}^k$ of 0 is such that there exists an odd homeomorphism $h : A \to \partial U$, then $\gamma(A) = k$.*

Let us now turn to the construction of the eigenvalues. Let Σ_k, $k = 1, 2, \dots$, denote the collection of all symmetric subsets A of $W_0^{1,p}(\Omega)$ such that $\gamma(A) \geq k$ and the set $\{v \in A : \|v\|_{p,\Omega} = 1\}$ is compact. Then it is known that the numbers

$$\lambda_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}$$

are p-eigenvalues and that they form an increasing sequence tending to infinity as $k \to \infty$, see e.g. [19], [37]. For $p \neq 2$ it is not known whether this sequence contains all the p-eigenvalues, but the method at least gives the correct λ_1 and λ_2. Observe that

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}$$

because $\gamma(\{u, -u\}) = 1$ for any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. It is clear from (3.2) that λ_1 above is the least possible eigenvalue (just choose $\phi = u$). Another known result, due to Anane and Tsouli [2] (see also [13]), says that between λ_1 and λ_2, which are always distinct, there are no eigenvalues. Thus the second p-eigenvalue is well-defined and equal to λ_2. For the reader's convenience, we give a simple proof for this fact.

Theorem 3.4. *Let Ω be a bounded domain and $\lambda_2 = \lambda_2(p)$ be as above. Then*

$$\begin{aligned} \lambda_2 &= \min\{\lambda : \lambda \text{ is a } p\text{-eigenvalue that admits a sign changing eigenfunction}\} \\ &= \min\{\lambda : \lambda \text{ is a } p\text{-eigenvalue and } \lambda > \lambda_1\}. \end{aligned}$$

Proof. It is enough to prove the first equality, see the discussion after the proof. Let $u \in W_0^{1,p}(\Omega)$ be any p-eigenfunction with eigenvalue λ having at least two nodal domains N_j, $j = 1, 2$. We claim that $\lambda_2 \leq \lambda$. Define

$$v_j = u\chi_{N_j} \in W_0^{1,p}(N_j),$$

where χ_{N_j} is the characteristic function of the set N_j. Then

$$\begin{aligned} \lambda \int_{N_j} |u|^p \, dx &= \lambda \int_\Omega |u|^{p-2} u v_j \, dx \\ &= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v_j \, dx = \int_{N_j} |\nabla u|^p \, dx, \end{aligned}$$

that is,

$$\lambda = \frac{\int_{N_j} |\nabla u|^p \, dx}{\int_{N_j} |u|^p \, dx} = \frac{\int_{N_j} |\nabla v_j|^p \, dx}{\int_{N_j} |v_j|^p \, dx} \tag{3.3}$$

for $j = 1, 2$. Let

$$A = \operatorname{Span}\{v_1, v_2\} \cap \{v \in W_0^{1,p}(\Omega) : \int_\Omega |v|^p \, dx = 1\}.$$

Using Lemma 3.3 it is easily seen that $\gamma(A) = 2$, and thus

$$\lambda_2 \leq \sup_{v \in A} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx}.$$

To conclude the proof, we notice that

$$\frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx} = \lambda \qquad \text{for all } v \in A.$$

Indeed, if $v = \alpha_1 v_1 + \alpha_2 v_2$, then by (3.3)

$$\int_\Omega |\nabla v|^p \, dx = \sum_{j=1}^{2} |\alpha_j|^p \int_{N_j} |\nabla v_j|^p \, dx = \lambda \Big(\sum_{j=1}^{2} |\alpha_j|^p \int_{N_j} |v_j|^p \, dx \Big) = \lambda \int_\Omega |v|^p \, dx.$$

□

As in the case of the Laplacian, the first p-eigenvalue λ_1 and the eigenfunctions associated to it have many special properties. First of all, λ_1 is simple, which means that the first eigenfunction is unique up to multiplication by constants. Second, a first eigenfunction does not change its sign, whereas all higher eigenfunctions necessarily do so. The conclusion that λ_1 is isolated can be drawn from these facts. Another immediate and useful consequence is that an eigenfunction restricted to a nodal domain is a first eigenfunction of that nodal domain. In literature, there are several proofs for these results. We refer the reader to [29] and to [6], which contains a particularly elegant proof for the simplicity of λ_1.

4. The second ∞-eigenvalue

Just as in the case of the p-Laplacian, we are able to prove much more about the second ∞-eigenvalue than about the other higher eigenvalues. This is essentially due to the fact that the second eigenvalue is the smallest eigenvalue admitting a sign changing eigenfunction.

The second ∞-eigenvalue will turn out to have a geometric characterization analogous to that of Λ_1. Let

$$r_2 = \sup\{r > 0 : \text{there exist disjoint open balls } B_1, B_2 \subset \Omega \text{ of radius } r\},$$

and define

$$\Lambda_2 = \frac{1}{r_2}.$$

The notation anticipates that this is the second ∞-eigenvalue. Obviously $0 < \Lambda_1 \leq \Lambda_2 < \infty$, and $\Lambda_2 > (\frac{2\omega_n}{|\Omega|})^{\frac{1}{n}}$, where $|\Omega|$ denotes the Lebesgue measure of Ω and ω_n is the measure of the unit ball in $\mathbb{R}^n$. Observe that for some domains $\Lambda_1 = \Lambda_2$. We return to this matter in Section 7.

Theorem 4.1. *Let $\lambda_2(p)$ be the second p-eigenvalue in Ω. Then $\lambda_2(p)^{1/p} \to \Lambda_2$ as $p \to \infty$ and Λ_2 is the second ∞-eigenvalue.*

Before continuing any further we had better explain precisely what we mean by the phrase "second eigenvalue" in this context. For a finite exponent p, we saw in Section 3 that

$$\begin{aligned}\lambda_2 &= \min\{\lambda : \lambda \text{ is a } p\text{-eigenvalue and } \lambda > \lambda_1\} \\ &= \min\{\lambda : \lambda \text{ is a } p\text{-eigenvalue with a sign changing eigenfunction}\}.\end{aligned}$$

We will show below that the second characterization is valid for Λ_2 in any domain, while the first one holds if $\Lambda_1(\Omega) < \Lambda_2(\Omega)$. In particular, it is proven that Λ_2 actually is an ∞-eigenvalue and that there are no ∞-eigenvalues in between Λ_1 and Λ_2.

We have divided the proof of Theorem 4.1 into three lemmas. The first one shows that the sequence $\{\lambda_2(p)^{1/p}\}$ remains stable as $p \to \infty$.

Lemma 4.2. *For any bounded domain $\Omega \subset \mathbb{R}^n$,*

$$\limsup_{p\to\infty} \lambda_2(p)^{1/p} \le \Lambda_2.$$

Proof. Let us fix two disjoint balls $B_{r_2}(x_1)$ and $B_{r_2}(x_2)$ contained in Ω. We define the truncated cone functions C_1 and C_2 in Ω by setting

$$C_1(x) = (r_2 - |x - x_1|)^+, \qquad C_2(x) = (r_2 - |x - x_2|)^+,$$

where f^+ is the positive part of a function f. Define

$$A_0 = \text{Span}\{C_1, C_2\} \cap \{v \in W_0^{1,\infty}(\Omega) : \|v\|_{\infty,\Omega} = 1\}.$$

Then A_0 is symmetric and closed in $W_0^{1,p}(\Omega)$ for each $1 < p \le \infty$, and its genus is 2 by Lemma 3.3. Thus $A_0 \in \Sigma_2$, and we obtain

$$\lambda_2(p)^{1/p} \le \sup_{v\in A_0} \frac{\left(\int_\Omega |\nabla v|^p\,dx\right)^{1/p}}{\left(\int_\Omega |v|^p\,dx\right)^{1/p}}.$$

Now if $v = \alpha C_1 + \beta C_2$ for some $\alpha, \beta \in \mathbb{R}$, then

$$\int_\Omega |v|^p\,dx = (|\alpha|^p + |\beta|^p) \int_{B_{r_2}(0)} (r_2 - |x|)^p\,dx$$

and

$$\int_\Omega |\nabla v|^p\,dx = (|\alpha|^p + |\beta|^p)|B_{r_2}|.$$

Hence

$$\limsup_{p\to\infty} \lambda_2(p)^{1/p} \le \limsup_{p\to\infty} \frac{|B_{r_2}|^{1/p}}{\left(\int_{B_{r_2}} (r_2 - |x|)^p\,dx\right)^{1/p}} = \frac{1}{\sup\limits_{|x|<r_2} (r_2 - |x|)^+} = \Lambda_2,$$

as claimed. □

By virtue of Lemma 4.2, there exists a sequence $p_i \to \infty$ such that $\lambda_2(p_i)^{1/p_i} \to \hat{\Lambda}$ and $\Lambda_1 \le \hat{\Lambda} \le \Lambda_2$. We show next that this number $\hat{\Lambda}$ is necessarily an ∞-eigenvalue and that it has an ∞-eigenfunction which is not a first eigenfunction.

Lemma 4.3. *Let $u_i \in W_0^{1,p_i}(\Omega)$ be an eigenfunction associated to the second eigenvalue $\lambda_2(p_i)$ such that $\|u_i\|_{p_i,\Omega} = 1$. Then there is a function $u \in W_0^{1,\infty}(\Omega)$, $u \not\equiv 0$, such that for a subsequence we have $u_{i_j} \to u$ uniformly in Ω, and u satisfies $F_{\hat{\Lambda}}(u, \nabla u, D^2 u) = 0$ in Ω. Moreover, u has at least 2 nodal domains.*

Proof. We obtain by Hölder's inequality that

$$\left(\int_\Omega |\nabla u_i|^q \, dx\right)^{1/q} \leq |\Omega|^{1/q-1/p_i} \left(\int_\Omega |\nabla u_i|^{p_i} \, dx\right)^{1/p_i} = \lambda_2(p_i)^{1/p_i} |\Omega|^{1/q-1/p_i}$$

if $q \leq p_i$, that is, the set $\{u_i\}_{q \leq p_i}$ is uniformly bounded in $W_0^{1,q}(\Omega)$ for any $1 < q < \infty$. With the aid of the Sobolev embedding theorem, this implies the existence of a subsequence $p_{i_j} \to \infty$ and a function $u \in W_0^{1,\infty}(\Omega)$ such that $u_{i_j} \to u$ uniformly and in $C^\alpha(\overline{\Omega})$ for any $0 < \alpha < 1$ (see [27] for details).

To show that u is a viscosity solution to $F_{\hat{\Lambda}}(u, \nabla u, D^2 u) = 0$, let us fix $x_0 \in \Omega$. First we consider the case $u(x_0) > 0$. Then there is $\rho > 0$ such that $u_{i_j} > 0$ in $B_\rho(x_0)$ for all j sufficiently large, and we may proceed as in the case of the first eigenvalue, see [27], to conclude that

$$\min\{|\nabla u(x_0)| - \hat{\Lambda} u(x_0), -\Delta_\infty u(x_0)\} = 0$$

in the viscosity sense. The case $u(x_0) < 0$ is similar.

Finally, let us assume that $u(x_0) = 0$. Let $\phi \in C^2(\Omega)$ be such that $u - \phi$ has a strict local maximum at x_0. Since $u_{i_j} \to u$ uniformly, there is a sequence $x_j \to x_0$ such that $u_{i_j} - \phi$ has a local maximum at x_j. Then, as u_{i_j} is a viscosity solution to $-\Delta_{p_j} w = \lambda_2(p_j)|w|^{p_j-2} w$, where $p_j = p_{i_j}$, we obtain

$$-|\nabla\phi(x_j)|^{p_j-2}\Delta\phi(x_j) - (p_j-2)|\nabla\phi(x_j)|^{p_j-4}\Delta_\infty\phi(x_j) \leq \lambda_2(p_j)|u_{i_j}(x_j)|^{p_j-2}u_{i_j}(x_j).$$

Now if $\nabla\phi(x_0) \neq 0$, we divide both sides of the inequality by $(p_j-2)|\nabla\phi(x_j)|^{p_j-4}$, which is different from zero for j large enough. This yields

$$-\Delta_\infty\phi(x_j) \leq \frac{|\nabla\phi(x_j)|^2\Delta\phi(x_j)}{p_j-2} + \left(\frac{\lambda_2(p_j)^{\frac{1}{p_j-4}}|u_{i_j}(x_j)|}{|\nabla\phi(x_j)|}\right)^{p_j-4} \frac{u_{i_j}(x_j)^3}{p_j-2},$$

where the right-hand side tends to zero as $j \to \infty$, because

$$\frac{\lambda_2(p_j)^{\frac{1}{p_j-4}}|u_{i_j}(x_j)|}{|\nabla\phi(x_j)|} \longrightarrow \frac{\hat{\Lambda}|u(x_0)|}{|\nabla\phi(x_0)|} = 0 \qquad \text{as } j \to \infty.$$

Thus $-\Delta_\infty\phi(x_0) \leq 0$, and since this inequality is trivially true if $\nabla\phi(x_0) = 0$, we conclude that u is a viscosity subsolution of (2.2). The fact that it is also a supersolution can be deduced by considering $-u$ and repeating the argument above.

Let us finish the proof by checking that u really does change its sign in Ω. We denote $\Omega_i^+ = \{x \in \Omega : u_i(x) > 0\}$ and $\Omega_i^- = \{x \in \Omega : u_i(x) < 0\}$. Since

$$\left(\int_{\Omega_i^+} |u_i|^{p_i} \, dx\right)^{1/p_i} \leq \left(\frac{|\Omega_i^+|}{\omega_n}\right)^{1/n} \left(\int_{\Omega_i^+} |\nabla u_i|^{p_i} \, dx\right)^{1/p_i}$$

by the Poincaré inequality, we obtain using the fact that u_i is the first p_i-eigenfunction on each of its nodal domains (and there are precisely two of them, see [14]) that

$$|\Omega_i^+| \geq \frac{\omega_n}{\lambda_1(p_i, \Omega_i^+)^{n/p_i}} = \frac{\omega_n}{\lambda_2(p_i, \Omega)^{n/p_i}}.$$

Furthermore,

$$\limsup_{j\to\infty} \Omega_{i_j}^+ := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \Omega_{i_j}^+ \subset \{x \in \Omega : u(x) \geq 0\},$$

and

$$|\limsup_{j\to\infty} \Omega_{i_j}^+| \geq \limsup_{j\to\infty} |\Omega_{i_j}^+| \geq \frac{\omega_n}{\Lambda_2^n}.$$

Thus the set $\{u \geq 0\}$ has positive measure. Similar estimates hold for the set $\{u(x) \leq 0\}$. Now, if u does not change its sign, say u is nonnegative, then by the Harnack inequality u is positive in Ω, which clearly contradicts $|\{u \leq 0\}| > 0$. □

Finally, we show that $\hat{\Lambda} = \Lambda_2$, which means that Λ_2 is an eigenvalue, and that Λ_2 is the second ∞-eigenvalue.

Lemma 4.4. *Let u be a viscosity solution of $F_\Lambda(u, \nabla u, D^2 u) = 0$ in Ω and assume that u has at least two nodal domains. Then $\Lambda_2 \leq \Lambda$.*

Proof. Let N_j, $j = 1, 2$, denote the two nodal domains of u. As u is a first ∞-eigenfunction on both of these domains, Theorem 8.1 in the appendix implies that $\Lambda_1(N_j) = \Lambda$ for $j = 1, 2$. By the geometric characterization of the first ∞-eigenvalue this means that Ω contains 2 disjoint balls B_j of radius $1/\Lambda$, and thus we must have

$$\frac{1}{\Lambda} \leq r_2(\Omega) = \frac{1}{\Lambda_2}$$

by the definition of r_2 and Λ_2. Hence $\Lambda_2 \leq \Lambda$. □

The second ∞-eigenvalue $\Lambda_2(\Omega)$ has a variational characterization similar to that obtained in [13] for the second eigenvalue of the p-Laplacian.

Theorem 4.5. *Let $S = \{v \in W_0^{1,\infty}(\Omega) : \|v\|_{\infty,\Omega} = 1\}$ and let $u_1 \in S$ be any first ∞-eigenfunction in Ω. Then*

$$\Lambda_2 = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} \|\nabla u\|_{\infty,\Omega},$$

where Γ is the family of all continuous paths on S going from u_1 to $-u_1$.

Proof. Choose for a given path $\gamma \in \Gamma$, $\gamma : [0, 1] \to S$, a point $t_0 \in (0, 1)$ such that for $u = \gamma(t_0)$

$$\|u^+\|_{\infty,\Omega} = \|u^-\|_{\infty,\Omega} = 1,$$

where u^+ and u^- denote the positive and negative part of u, respectively. Denote $\Omega_+ = \{x \in \Omega : u(x) > 0\}$, $\Omega_- = \{x \in \Omega : u(x) < 0\}$ and choose two nodal domains $N_\pm \subset \Omega_\pm$ such that $\|u\|_{\infty,N_\pm} = 1$. Then

$$\|\nabla u\|_{\infty,\Omega_\pm} \geq \frac{\|\nabla u\|_{\infty,N_\pm}}{\|u\|_{\infty,N_\pm}} \geq \Lambda_1(N_\pm) = \frac{1}{\sup\{r > 0 : B_r(x) \subset N_\pm\}},$$

and thus necessarily

$$\Lambda_2 \leq \max\{\Lambda_1(N_+), \Lambda_1(N_-)\} \leq \|\nabla u\|_{\infty,\Omega} \leq \sup_{v\in\gamma} \|\nabla v\|_{\infty,\Omega}.$$

This shows that

$$\Lambda_2 \leq \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} \|\nabla u\|_{\infty,\Omega}.$$

For the other direction, we construct a continuous path $\gamma \in \Gamma$ satisfying

$$\|\nabla v\|_{\infty,\Omega} \leq \Lambda_2 \quad \text{for all } v \in \gamma.$$

In order to do this, let us fix two disjoint balls $B_{r_2}(x_1), B_{r_2}(x_2) \subset \Omega$ with radius $r_2 = \Lambda_2^{-1}$. Define the corresponding truncated cones

$$C_1(x) = (1 - \tfrac{1}{r_2}|x - x_1|)^+, \qquad C_2(x) = (1 - \tfrac{1}{r_2}|x - x_2|)^+.$$

We may assume that $u_1 > 0$ in Ω. The path γ will be constructed from 6 pieces. We start from u_1 and go to $C_1 - C_2$, using the paths

$$\begin{aligned}\gamma_1(t) &= \max\{u_1(x), tC_1(x)\},\\ \gamma_2(t) &= \max\{(1-t)u_1(x), C_1(x)\},\\ \gamma_3(t) &= C_1(x) - tC_2(x),\end{aligned}$$

where $0 \leq t \leq 1$. Clearly $\gamma_i(t) \in S$ for all $i = 1, 2, 3$, and

$$\|\nabla\gamma(t)\|_{\infty,\Omega} \leq \max\Big\{\|\nabla u_1\|_{\infty,\Omega}\ ,\ \frac{1}{r_2}\Big\} = \Lambda_2.$$

It is now obvious how to complete the construction with three paths connecting $C_1 - C_2$ to $-u_1$. □

The proof of Theorem 4.5, actually, implies that the second eigenvalue Λ_2 is obtained by a simple variational formula.

Corollary 4.6. *For every bounded domain $\Omega \subset \mathbb{R}^n$,*

$$\Lambda_2 = \inf_{v\in\mathcal{O}} \frac{\|\nabla v\|_{\infty,\Omega}}{\|v\|_{\infty,\Omega}}$$

where

$$\mathcal{O} = \{v \in W_0^{1,\infty}(\Omega) : v \neq 0,\ \|v^+\|_{\infty,\Omega} = \|v^-\|_{\infty,\Omega}\}.$$

5. Higher eigenvalues

In this section, we address the issue of existence of ∞-eigenvalues $\Lambda > \Lambda_2$. In the one dimensional case all eigenvalues and eigenfunctions are explicitly known. Let Ω be the interval $(0, 1)$. Now $\Lambda_1 = 2$ and the first ∞-eigenfunction is

$$u_1(x) = \tfrac{1}{2} - |x - \tfrac{1}{2}|,$$

the distance function. The higher eigenvalues are just $2\Lambda_1$, $3\Lambda_1$, $, 4\Lambda_1, \ldots,$ and the corresponding eigenfunctions u_k are obtained from u_1 by the usual method of first extending u_1 to an odd function on $(-1, 1)$ and then periodically to the whole real line, and finally setting

$$u_k(x) = u_1(kx) \quad \text{for } x \in (0,1),\ \ k = 2, 3, 4, \ldots.$$

Observe that for example the function

$$v(x) = \begin{cases} \frac{1}{4} - |x - \frac{1}{4}|, & x \in (0, \frac{1}{2}], \\ 2(\,|x - \frac{3}{4}| - \frac{1}{4}), & x \in (\frac{1}{2}, 1) \end{cases}$$

is *not* an ∞-eigenfunction because it does not satisfy $-\Delta_\infty v = 0$ at $x = \frac{1}{2}$.

For the case $n \geq 2$, let us define for $k = 3, 4, \ldots$ the numbers

$$r_k = \sup\{r > 0 : \text{there exists } k \text{ disjoint open balls } B_i \subset \Omega \text{ of radius } r\},$$

and

$$\Lambda_k = \Lambda_k(\Omega) = \frac{1}{r_k}. \tag{5.1}$$

By repeating the arguments of the previous section, we can now study the limiting behavior, as $p \to \infty$, of the sequences $\{\lambda_k(p)^{1/p}\}$ for $k = 3, 4, \ldots$ as well as the sequences of corresponding suitably normalized p-eigenfunctions. The important difference from the case $k = 2$ is that we do not have such a good lower bound for the number of nodal domains. As a result, the numbers Λ_k for $k > 2$ only bound the accumulation points of $\{\lambda_k(p)^{1/p}\}$ from above and, in general, they are not known to be ∞-eigenvalues themselves. The proofs of Lemma 5.1 and Lemma 5.2 below are analogous to the proofs of Lemma 4.2 and Lemma 4.3, respectively.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. Then*

$$\limsup_{p\to\infty} \lambda_k(p)^{1/p} \leq \Lambda_k$$

for all $k \in \mathbb{N}$.

Lemma 5.2. *For a fixed $k \in \mathbb{N}$, let $p_i \to \infty$ be any sequence such that the eigenvalues $\lambda_k(p_i)^{1/p_i}$ converge to a number $\Lambda_k^* \leq \Lambda_k$. Let further $u_i \in W_0^{1,p_i}(\Omega)$ be a p-eigenfunction associated to $\lambda_k(p_i)$, normalized so that $\|u_i\|_{p_i,\Omega} = 1$. Then there exists $u \in W_0^{1,\infty}(\Omega)$ such that for a subsequence we have $u_{i_j} \to u$ uniformly in Ω and $F_{\Lambda_k^*}(u, \nabla u, D^2 u) = 0$. In particular, Λ_k^* is an ∞-eigenvalue.*

Lemmas 5.1 and 5.2 do not imply right away that the set of ∞-eigenvalues is unbounded. The sequence $\Lambda_k(\Omega)$ clearly tends to ∞ as $k \to \infty$, but these numbers only bound the ∞-eigenvalues we found from above. If the domain is, for example, a parallelepiped, then one can construct arbitrarily large ∞-eigenvalues with the aid of Schwarz's reflection principle, Theorem 7.1. In a general domain we will use some ideas from [37].

Theorem 5.3. *For every bounded domain Ω the set of ∞-eigenvalues is unbounded.*

Proof. We consider first the case in which the domain is a ball B. Let $\{\varphi_m\} \subset W_0^{1,2}(B)$ be an orthonormal sequence of eigenfunctions of the ordinary Laplacian, and denote

$$E_k^C = E_k^C(p) = \{v \in W_0^{1,p}(B) : \int_B v\varphi_j \, dx = 0 \qquad \text{for all } j = 1, \ldots, k\}.$$

If $2 \le p < \infty$, $\varphi_m \in W_0^{1,p}(B)$ and so Proposition 7.8 in [38] shows that the set E_k^C intersects every closed symmetric subset of $W_0^{1,p}(B)$ whose genus is at least $k+1$. If we set

$$\hat{\lambda}_k(p) = \inf_{A \in \Sigma_k} \sup_{u \in A \cap E_{k-1}^C} \frac{\int_B |\nabla u|^p \, dx}{\int_B |u|^p \, dx},$$

then clearly $\lambda_k(p) \ge \hat{\lambda}_k(p)$. By a diagonal argument and Lemma 5.1 we can find a sequence $p_j \to \infty$ such that for every $k \in \mathbb{N}$

$$\hat{\lambda}_k(p_j)^{1/p_j} \longrightarrow \hat{\Lambda}_k \qquad \text{for some } \hat{\Lambda}_k \le \Lambda_k.$$

By Lemma 5.2, there exists an ∞-eigenvalue $\Lambda_k^* \in [\hat{\Lambda}_k, \Lambda_k]$, and hence it suffices to prove that $\hat{\Lambda}_k \to \infty$ as $k \to \infty$.

To this end, we argue by contradiction and suppose that there is a constant $C > 0$ such that $\hat{\Lambda}_k \le C$ for each $k \in \mathbb{N}$. This implies, after some relabeling, that there is a sequence $k_j \to \infty$ and a number $\hat{\Lambda} \le C$ such that

$$\hat{\lambda}_{k_j}(p_j)^{1/p_j} \longrightarrow \hat{\Lambda} \qquad \text{as } j \to \infty.$$

By the definition of $\hat{\lambda}_k(p)$, this means that there are sets $A_j \in \Sigma_{k_j}(p_j)$ and functions $u_j \in A_j \cap E_{k_j-1}^C(p_j)$ such that for

$$v_j = \frac{u_j}{\left(\int_B |u_j|^{p_j} \, dx\right)^{1/p_j}}$$

we have

$$\int_B |\nabla v_j|^{p_j} \, dx \longrightarrow \hat{\Lambda}$$

as $j \to \infty$. Hence by Hölder $\{v_j\}_{p_j \ge q}$ is uniformly bounded in $W_0^{1,q}(B)$, and with the aid of the Sobolev embedding theorem and a diagonal argument as in [27], we find a function $v \in W^{1,\infty}(B)$ such that $v_j \to v$ uniformly in B. In particular, since $||v_j||_{p_j,B} = 1$ for every j, we obtain $||v||_{\infty,B} = 1$. However, as $v_j \in E_{k_j-1}^C(p_j)$, we must have

$$\int_B v \varphi_m \, dx = \lim_{j \to \infty} \int_B v_j \varphi_m \, dx = 0$$

for every $m \in \mathbb{N}$. Thus $v \equiv 0$, which is clearly a contradiction, and we have proved the claim for a ball.

To deal with the general case, we choose for a given domain Ω a ball B such that $\Omega \subset B$. Then $\lambda_k(p, \Omega) \ge \lambda_k(p, B)$ by the definition of these numbers, and the theorem now follows from the first part of the proof and Lemma 5.2. □

According to [20], [17], for $1 < p < \infty$ there exist constants C_1, C_2 depending on p and n such that

$$C_1 \left(\frac{k}{|\Omega|}\right)^{1/n} \le \lambda_k(p, \Omega)^{1/p} \le C_2 \left(\frac{k}{|\Omega|}\right)^{1/n}$$

for all $k \in \mathbb{N}$. One can probably deduce the unboundedness of the spectrum also from this result, but then the dependence of the constant C_1 on p should be analyzed carefully.

Also Lemma 4.4 in the previous section has its natural counterpart which is proved in analogous manner.

Lemma 5.4. *Let Λ be any ∞-eigenvalue for which there exists an ∞-eigenfunction having at least k nodal domains. Then $\Lambda \geq \Lambda_k$.*

Lemma 5.4 implies, in particular, that if the domain Ω satisfies

$$\Lambda_1(\Omega) < \Lambda_2(\Omega) < \Lambda_3(\Omega),$$

then every ∞-eigenfunction associated to $\Lambda_2(\Omega)$ admits exactly two nodal domains. For the second p-eigenfunctions this is true in any domain, see [14]. It is easy to find an example of a domain for which $\Lambda_2 = \Lambda_3$ that has an ∞-eigenfunction associated to this common value with a given number of nodal domains, see Section 7.

It is natural to ask, whether it could be that the numbers Λ_k in (5.1) exhaust the spectrum. This is clearly true if $n = 1$, but we doubt very much that it holds in higher dimensions. The next example should explain our skepticism.

Example 5.5. The following packing problem was raised by Moser [33], see also [12]: Find the value of ρ_m, the maximal radius of m non-overlapping equal circles in a unit square. Optimal packings and values for ρ_m, and consequently the numbers Λ_m for the domain $Q = (0,1)^2$, are known at least for $m \leq 27$, see [34]. Yet the most striking result obtained is that the natural square lattice packing is not optimal for $m = 49$, see [35]. This means, in particular, that $\Lambda_{49}(Q) < 14$. On the other hand, using the reflection principle, it is easy to see that for $\Lambda = 14$ there exists an ∞-eigenfunction having exactly 49 nodal domains. Observe that this does not yet disprove the conjecture, since it might be that $14 = \Lambda_m$ for some $m > 49$.

6. Properties of the spectrum and eigenfunctions

We record some properties of the spectrum and eigenfunctions. First we bound the number of nodal domains in a rather explicit way.

Theorem 6.1. *Let $u \in C(\Omega)$, $u|_{\partial\Omega} = 0$, be a solution of $F_\Lambda(u, \nabla u, D^2 u) = 0$ in the bounded domain $\Omega \subset \mathbb{R}^n$. Then the number of nodal domains of u is at most $\Lambda^n \omega_n^{-1} |\Omega|$, where ω_n is the measure of the unit ball of $\mathbb{R}^n$. Conversely, if u has k nodal domains, then $\Lambda^n \geq k\omega_n |\Omega|^{-1}$.*

Proof. By Theorem 8.1 in the appendix, $\Lambda_1(N) = \Lambda$ for any nodal domain $N \subset \Omega$. Thus

$$|\Omega| \geq \sum |N| \geq \omega_n \left(\frac{1}{\Lambda}\right)^n \#\{\text{nodal domains}\},$$

whence the result. □

It is not difficult to see that the bound found above is optimal. Just consider as the domain Ω a chain of k disjoint balls of radius 1, connected by corridors of width $\varepsilon > 0$, and let $\varepsilon \to 0$. The ∞-eigenfunction with the right number of nodal domains can be constructed with the aid of the reflection principle, Theorem 7.1.

Next we link together the eigenvalue and the value of the Rayleigh quotient for the eigenfunction. For a finite p, the corresponding result is obtained easily by using the p-eigenfunction itself as a test function in the weak formulation of the problem.

Lemma 6.2. *Let $u \not\equiv 0$ be a solution to $F_\Lambda(u, \nabla u, D^2 u) = 0$ in Ω. Then*

$$\frac{||\nabla u||_{\infty,\Omega}}{||u||_{\infty,\Omega}} = \Lambda.$$

Proof. Let us choose a nodal domain N so that $\|u\|_{\infty,N} = \|u\|_{\infty,\Omega}$. Then necessarily $\|\nabla u\|_{\infty,N} = \|\nabla u\|_{\infty,\Omega}$. Indeed, if this is not the case, there is another nodal domain N' so that

$$\frac{\|\nabla u\|_{\infty,N'}}{\|u\|_{\infty,N'}} > \frac{\|\nabla u\|_{\infty,N}}{\|u\|_{\infty,N}}.$$

But this is impossible due to Theorem 8.1 in the appendix, and consequently

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}} = \frac{\|\nabla u\|_{\infty,N}}{\|u\|_{\infty,N}} = \Lambda.$$

□

In the previous section the spectrum was seen to be unbounded. Next we show that it is also a closed set.

Theorem 6.3. *For every bounded domain Ω the set of ∞-eigenvalues is closed.*

Proof. Let $\Lambda_i \in \mathbb{R}$ be ∞-eigenvalues such that $\Lambda_i \to \Lambda$ as $i \to \infty$, and choose eigenfunctions u_i satisfying $F_{\Lambda_i}(u_i, \nabla u_i, D^2 u_i) = 0$ and $\|u_i\|_{\infty,\Omega} = 1$. Since

$$\|\nabla u_i\|_{\infty,\Omega} \leq \Lambda_i < 2\Lambda$$

for all i large enough, we see that the sequence $\{u_i\}$ is uniformly bounded and equicontinuous. Thus, by Ascoli's theorem, we may assume that $u_i \to u$ uniformly in Ω where $u \in W_0^{1,\infty}(\Omega)$.

If the function $F_\Lambda : \mathbb{R} \times \mathbb{R}^n \times S_{n\times n} \to \mathbb{R}$ in (2.2) were continuous, the theorem would now follow immediately from the standard stability results in the theory of viscosity solutions. However, since the equation is discontinuous at $\{u = 0\}$, we need to check that the limit function really satisfies $-\Delta_\infty u = 0$ at those points. So let $x \in \Omega$ and $\phi \in C^2(\Omega)$ be such that $u(x) = 0$ and $\phi - u$ has a strict local minimum at x. In order to show that u is a viscosity subsolution, we have to prove that $-\Delta_\infty \phi(x) \leq 0$. We will derive a contradiction from the antithesis $-\Delta_\infty \phi(x) > 0$. By the uniform convergence $u_i \to u$ we find a sequence $x_i \to x$ such that $\phi - u_i$ has a local minimum at x_i. Because $-\Delta_\infty \phi(x_i) > 0$ for i large, we obtain using the equation that $u_i(x_i) > 0$. Indeed, if $u_i(x_i) \leq 0$ for i large, then $-\Delta_\infty \phi(x_i) \leq 0$ contradicting the antithesis. Thus $u_i(x_i) > 0$ and

$$0 \geq \min\{|\nabla \phi(x_i)| - \Lambda_i u_i(x_i), -\Delta_\infty \phi(x_i)\} = |\nabla \phi(x_i)| - \Lambda_i u_i(x_i)$$

for i large. But this implies that $\nabla \phi(x) = 0$ and consequently $-\Delta_\infty \phi(x) = 0$, establishing that u is a subsolution. An analogous argument shows that u is also a supersolution, and hence u is a solution in Ω. □

Next we will prove counterparts for two famous results of the linear theory. The first one shows that the Payne-Pólya-Weinberger conjecture, regarding the ratio of the first two eigenvalues, is true for the ∞-eigenvalue problem. In the linear case, this was proved by Ashbaugh and Benguria in [4] for $n = 2$ and in [5] for any n. For the eigenvalues of the p-Laplacian, $p \neq 2$, the validity of the conjecture is an open question.

Theorem 6.4. *For a bounded domain Ω, the ratio of the first two ∞-eigenvalues satisfies*

$$\frac{\Lambda_2(\Omega)}{\Lambda_1(\Omega)} \leq 2,$$

and equality holds if and only if Ω is a ball.

Proof. First, it is easy to see that the supremum of the ratios is attained by a ball B. Indeed, for any bounded domain $\Omega \subset \mathbb{R}^n$, a maximal inscribed ball of radius $\Lambda_1(\Omega)^{-1}$ contains two disjoint balls of radius $(2\Lambda_1(\Omega))^{-1}$, and thus

$$\frac{\Lambda_2(\Omega)}{\Lambda_1(\Omega)} \leq \frac{2\Lambda_1(\Omega)}{\Lambda_1(\Omega)} = 2 = \frac{\Lambda_2(B)}{\Lambda_1(B)}$$

for the ball B.

Hence it remains to show that if Ω is not a ball, then $\frac{\Lambda_2(\Omega)}{\Lambda_1(\Omega)} < 2$. To this end, let us denote $R = 1/\Lambda_1(\Omega)$ and suppose that $B_R(0) \subset \Omega$. Since $\Omega \setminus B_R(0) \neq \emptyset$, there exists $\hat{x} \in \partial B_R(0)$ and $\tau > 0$ small such that $B_\tau(\hat{x}) \subset \Omega$. Now the proof reduces to the elementary problem of finding two disjoint balls of the same size contained in $B_R(0) \cup B_\tau(\hat{x})$ such that their common radius exceeds $R/2$.

Without loss of generality, we may assume that $R = 2$ and $\hat{x} = 2e_1 = (2, 0, \ldots, 0)$. Since

$$\min\{\, \mathrm{dist}(x, \partial B_2(0)) \,:\, x \in \partial B_1(e_1) \setminus B_{\tau/2}(2e_1) \,\} > 0$$

and $B_\tau(2e_1) \subset \Omega$, there exists $\varepsilon > 0$ such that $B_{1+\varepsilon}(e_1) \subset\subset \Omega$. We claim that the balls

$$B_{1+\frac{\varepsilon}{4}}((1 + \tfrac{3\varepsilon}{4})e_1) \quad \text{and} \quad B_{1+\frac{\varepsilon}{4}}((\tfrac{\varepsilon}{4} - 1)e_1)$$

are disjoint and contained in Ω. Indeed, clearly $B_{1+\frac{\varepsilon}{4}}((1 + \frac{3\varepsilon}{4})e_1) \subset B_{1+\varepsilon}(e_1) \subset \Omega$ and $B_{1+\frac{\varepsilon}{4}}((\frac{\varepsilon}{4} - 1)e_1) \subset B_2(0) \subset \Omega$, and the balls are disjoint because

$$|(1 + \tfrac{3\varepsilon}{4})e_1 - (\tfrac{\varepsilon}{4} - 1)e_1| = 2 + \tfrac{\varepsilon}{2} = 2(1 + \tfrac{\varepsilon}{4}),$$

that is, the distance between the centers is the same as the sum of the radii of the balls. This concludes the proof of the theorem. □

The second result deals with the zero set of a second eigenfunction. In 1967 Payne [36] conjectured that in any bounded domain $\Omega \subset \mathbb{R}^2$ a second eigenfunction of the Laplacian cannot have a closed nodal line. This was proved to be correct if Ω is convex by Melas [32] and Alessandrini [1]. On the other hand, an example constructed by Hoffmann-Ostenhof et al. [23] shows that the conclusion need not be true if the domain is not convex and not simply connected. To the best of our knowledge, nothing similar is known about the nodal line of a second eigenfunction of the p-Laplacian. In the case of the ∞-eigenvalue problem, we show that a weaker version of Payne's conjecture holds: every nodal domain of a second ∞-eigenfunction in a convex domain necessarily meets the boundary. The proof works in any dimension and is again very simple due to the explicit characterization of $\Lambda_2(\Omega)$.

Theorem 6.5. *Let Ω be a bounded convex domain and let u be a second ∞-eigenfunction in Ω. Then the closure of each nodal domain of u intersects the boundary $\partial\Omega$.*

Proof. Suppose that the claim is not true. Then there exists a nodal domain N, associated to u, such that $d := \operatorname{dist}(N, \partial\Omega) > 0$. Since u is a second ∞-eigenfunction, this means that for some small $\varepsilon > 0$ there exist disjoint balls $B_r(x_1) \subset \Omega$ and $B_{r+\varepsilon}(x_2) \subset \Omega$, where $r^{-1} = \Lambda_2(\Omega)$ and $x_2 \in N$. Indeed, since N contains a ball B of radius r and $(1 + d/r)B$, the concentric ball with radius $r + d$, is contained in Ω, the ball $B_{r+\varepsilon}(x_2)$ can be found inside $(1 + d/r)B \setminus B_r(x_1)$. Without loss of generality, we may assume that $x_1 = 0$ and $x_2 = e_1 = (1, 0, \dots, 0)$, in which case $0 < r < \frac{1}{2}$.

Denote

$$K = \text{convex hull}\big(B_r(0) \cup B_{r+\varepsilon}(e_1)\big) \subset \Omega.$$

Then for $\mu > 0$ sufficiently small, $\operatorname{dist}(\mu e_1, \partial K) > r$, that is, $B_{r(\mu)}(\mu e_1) \subset\subset K$ for some $r < r(\mu) < r + \mu$.

Let us choose $r < \rho < r + \frac{\varepsilon}{8}$ and $\mu < \frac{\varepsilon}{8}$. Then the balls $B_\rho((1 + r + \varepsilon - \rho)e_1)$ and $B_{r(\mu)}(\mu e_1)$ are both contained in $K \subset \Omega$ and have radius larger than r. They are also disjoint, because

$$\begin{aligned} |(1 + r + \varepsilon - \rho)e_1 - \mu e_1| &= 1 + r + \varepsilon - \rho - \mu > 1 + \tfrac{3}{4}\varepsilon \\ &> 2r + \tfrac{\varepsilon}{4} > (r + \mu) + (r + \tfrac{\varepsilon}{8}) \\ &> r(\mu) + \rho. \end{aligned}$$

Thus

$$\Lambda_2(\Omega) \le \frac{1}{\min\{r(\mu), \rho\}} < \frac{1}{r},$$

which contradicts the definition of r. The claim now follows. □

In Theorem 6.5 it is important that u is a second ∞-eigenfunction. The conclusion is not true for higher eigenvalues even in a ball. Indeed, one can check that the function

$$u(x) = \begin{cases} 1 - |x|, & \text{if } |x| \le 2 \\ |x| - 3, & \text{if } 2 < |x| < 3 \end{cases}$$

satisfies (2.2) in $B_3(0)$ with $\Lambda = 1 > \Lambda_2(B_3(0))$ (and u even has two nodal domains), but the zero set of u does not meet the boundary of the ball $B_3(0)$.

7. Examples

In this section, we present some examples displaying various features of the ∞-eigenvalue problem. Most notably, we consider domains for which the first and the second ∞-eigenvalue coincide.

We base most of our examples on the following *reflection principle*. It enables us to build higher eigenfunctions from first eigenfunctions of suitable subdomains.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain that is symmetric in the x_n direction, i.e., $(x', x_n) \in \Omega$ if and only if $(x', -x_n) \in \Omega$. Let u be a first ∞-eigenfunction* *in* $\Omega \cap \{x \in \mathbb{R}^n : x_n > 0\}$*, and define* $\hat{u} : \Omega \to \mathbb{R}$ *by*

$$\hat{u}(x) = \hat{u}(x', x_n) = \begin{cases} u(x), & \text{if } x_n > 0, \\ 0, & \text{if } x_n = 0, \\ -u(x', -x_n), & \text{if } x_n < 0. \end{cases}$$

Then $\hat{u}$ *is an* ∞*-eigenfunction in* Ω*, associated to the eigenvalue* $\Lambda_1(\Omega \cap \{x_n > 0\})$*.*

Proof. It is enough to show that $-\Delta_\infty \hat{u} = 0$ on $\Gamma = \Omega \cap \{x_n = 0\}$. To check this, let $x_0 \in \Gamma$ and $\phi \in C^2(\Omega)$ be such that $\phi - \hat{u}$ has a local minimum, relative to Ω, at x_0. Without loss of generality, we may assume that $x_0 = 0$ and $\phi(0) = 0$. Since the function $\phi|_\Gamma$ of $(n-1)$ -variables has a minimum at 0 we have

$$\nabla\phi(0) = (0, \dots, 0, \phi_{x_n}(0)).$$

This implies, in particular, that

$$-\Delta_\infty \phi(0) = -\phi_{x_n x_n}(0) \left(\phi_{x_n}(0)\right)^2.$$

In order to estimate $\phi_{x_n x_n}(0)$, we note that

$$\hat{u}(te_n) \le \phi(te_n) = \phi_{x_n}(0)t + \frac{1}{2}\phi_{x_n x_n}(0)t^2 + o(t^2)$$

and

$$\hat{u}(-te_n) \le \phi(-te_n) = -\phi_{x_n}(0)t + \frac{1}{2}\phi_{x_n x_n}(0)t^2 + o(t^2)$$

for $t \ge 0$ sufficiently small. Here $e_n = (0, \dots, 0, 1)$. Since $\hat{u}(-te_n) = -\hat{u}(te_n)$ we thus have

$$\phi_{x_n}(0)t + \frac{1}{2}\phi_{x_n x_n}(0)t^2 \ge \phi_{x_n}(0)t - \frac{1}{2}\phi_{x_n x_n}(0)t^2 + o(t^2)$$

which implies

$$\phi_{x_n x_n}(0) \ge 0.$$

Consequently

$$-\Delta_\infty \phi(0) \le 0,$$

and we conclude that $\hat{u}$ is a subsolution. To see that $\hat{u}$ is also a supersolution, repeat the argument above for $-\hat{u}$. □

Examples 7.2. (i) Let $\Omega = (0,1) \times (-1,1) \subset \mathbb{R}^2$. Then clearly $\Lambda_1(\Omega) = \Lambda_2(\Omega) = 2$, and we may construct a sign changing ∞-eigenfunction by taking a positive first ∞-eigenfunction of the unit square $(0,1) \times (0,1)$ and then extending the function to the whole of Ω by using the odd reflection above. Furthermore, each integer $k \ge 2$ is an ∞-eigenvalue of Ω, and so is also for example $2\sqrt{2}$. These facts follow from the reflection principle as well.

It is obvious how one can modify the example to find, for a given $k \in \mathbb{N}$, a domain such that $\Lambda_1 = \Lambda_2 = \cdots = \Lambda_k$.

(ii) Let $\Omega = (0,1) \times (0,T) \subset \mathbb{R}^2$ for $T > 2$. Then again $\Lambda_1(\Omega) = \Lambda_2(\Omega) = 2$, and for each $s \in [1, T-1]$ we can construct an ∞-eigenfunction u_s with precisely two nodal domains in such a way that

$$\{u_s = 0\} = \{(x,y) \in \Omega : y = s\}.$$

The construction is based on "gluing" together pieces of a (fixed) first ∞-eigenfunction of the unit square and the distance function of Ω, cf. [28], and applying the reflection principle at the nodal line. The set $\{u_s\}_{1\le s\le T-1}$ is linearly independent.

8. Appendix

In [27], it was proved that if the domain Ω satisfies $\partial\Omega = \partial\overline{\Omega}$ then a positive ∞-eigenfunction is necessarily a first ∞-eigenfunction. We will now remove the assumption about the boundary. This improvement is crucial in the study of nodal domains whose regularity properties are in general unknown. Our argument here is different from that used in [27], where a logarithmic comparison principle was the main ingredient of the proof.

Theorem 8.1. *Let Ω be any bounded domain of $\mathbb{R}^n$ and let Λ_1 be the first ∞-eigenvalue of Ω. If $u \in C(\overline{\Omega})$ is a positive viscosity solution to the equation*

$$\min\{|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0$$

in Ω with $u = 0$ on the boundary $\partial\Omega$, then $\Lambda = \Lambda_1$. Furthermore, we have

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}} = \Lambda_1.$$

To prove this, we need the following Harnack inequality.

Theorem 8.2. *Let u be a nonnegative viscosity supersolution of $-\Delta_\infty u = 0$ in Ω, and let $\delta(x) = dist(x, \partial\Omega)$ for $x \in \Omega$. Then*

$$|\nabla \log u(x)| \le |\nabla \log \delta(x)| \qquad \text{for a.e. } x \in \Omega.$$

Proof. For the ∞-harmonic functions this was proved in [30], and there it was also noticed that the estimate holds for any supersolution that is a limit, as $p \to \infty$, of a sequence of p-superharmonic functions. However, it turns out that this approximation property is true for all supersolutions, see [31], [26], and thus the Harnack inequality follows. Another proof, based on the "comparison with cones" property of supersolutions of $-\Delta_\infty u = 0$, is due to Crandall, see [3]. □

Now we turn to the proof of Theorem 8.1.

Proof. Notice first that if $\Lambda \le 0$, then the eigenvalue equation above reduces to the ∞-Laplace equation $-\Delta_\infty u = 0$, whose only solution with zero boundary values is $u \equiv 0$, according to Jensen's uniqueness theorem, cf. [24], [3]. Thus necessarily $\Lambda > 0$.

Let us normalize u so that $\sup u = \frac{1}{\Lambda}$. Then $\Lambda u \le 1$ in Ω, which implies that

$$\min\{|\nabla u| - 1, -\Delta_\infty u\} \le 0 \qquad \text{in } \Omega$$

in the viscosity sense. Since the distance function $\delta(x) = \text{dist}(x, \partial\Omega)$ satisfies

$$\min\{|\nabla \delta| - 1, -\Delta_\infty \delta\} = 0, \tag{8.1}$$

see [26], and $\delta = u$ on $\partial\Omega$, we obtain by Jensen's comparison principle for the equation (8.1), see [24], that $u(x) \leq \delta(x)$ for all $x \in \Omega$. Hence, as $|\nabla\delta| = 1$ a.e.,

$$|\nabla u(x)| \leq \frac{u(x)}{\delta(x)} \leq 1 \qquad \text{a.e. in } \Omega$$

by the Harnack inequality (Theorem 8.2) and, consequently,

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}} \leq \frac{1}{\|u\|_{\infty,\Omega}} = \Lambda. \tag{8.2}$$

Because Λ_1 is the minimum of the ∞-Rayleigh quotient, we must have $\Lambda_1 \leq \Lambda$.

To prove the reverse inequality, we approximate $v = \log u$ by its semiconcave inf-convolutions (see e.g. [25])

$$v^{\epsilon}(x) = \inf_{y\in\overline{\Omega}_\sigma} \left\{ v(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\}, \ \varepsilon > 0,$$

in the set $\Omega_\sigma = \{x \in \Omega : \delta(x) > \sigma\}$. Since $|\nabla v| \geq \Lambda$ in the viscosity sense by the assumptions and v^ε is twice differentiable a.e., it follows from the properties of the inf-convolution (see [25]) that $|\nabla v^\varepsilon(x)| \geq \Lambda$ for a.e. x in a smaller set $\Omega_{\sigma,\varepsilon} = \{x \in \Omega_\sigma : \mathrm{dist}(x, \partial\Omega_\sigma) > C\sqrt{\varepsilon}\}$. Moreover, the function e^{v^ε} is a positive supersolution of $-\Delta_\infty w = 0$ in $\Omega_{\sigma,\varepsilon}$. Thus we obtain, using Theorem 8.2 and then letting $\varepsilon \to 0$, $\sigma \to 0$ that

$$\Lambda \leq |\nabla \log\delta(x)| = \frac{1}{\delta(x)} \qquad \text{a.e. in } \Omega,$$

and so

$$\Lambda \leq \frac{1}{\sup_{x\in\Omega}\delta(x)} = \Lambda_1.$$

This completes the proof of the first assertion, and the second one now follows immediately from (8.2) and the definition of Λ_1. □

References

[1] G. Alessandrini, *Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains*, Comment. Math. Helv. 69 (1994), no. 1, pp. 142– 154.

[2] A. Anane, and N. Tsouli, *On the second eigenvalue of the p-Laplacian*, Nonlinear partial differential equations (Fés, 1994) Pitman Res. Notes Math. Ser., 343, Longman, Harlow, (1996) pp. 1–9.

[3] G. Aronsson, M. G. Crandall, and P. Juutinen, *A Tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (to appear).

[4] M. S. Ashbaugh, and R. Benguria, *Proof of the Payne-Pólya-Weinberger conjecture*, Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 1, pp. 19–29.

[5] M. S. Ashbaugh, and R. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math. (2) 135 (1992), no. 3, pp. 601–628.

[6] M. Belloni, and B. Kawohl, *A direct uniqueness proof for equations involving the p-Laplace operator*, Manuscripta Math. 109 (2002), no. 2, pp. 229–231.

[7] M. Belloni, and B. Kawohl, *The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$*, ESAIM Control Optim. Calc. Var. 10 (2004), pp. 28–52.

[8] L. A. Caffarelli, and X. Cabré, *Fully Nonlinear Elliptic Equations*, American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995.

[9] R. Courant, and D. Hilbert, *Methods of Mathematical Physics, Vol. I*, Interscience Publishers, Inc., New York, N.Y., 1953.

[10] M. G. Crandall, *Viscosity solutions: a primer* Viscosity solutions and applications (Montecatini Terme, 1995), Lecture Notes in Math., 1660, Springer, Berlin (1997) pp. 1–43.

[11] M. G. Crandall, H. Ishii, and P-.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[12] H. Croft, K. Falconer, and R. Guy, *Unsolved problems in geometry*, Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, New York, (1994).

[13] M. Cuesta, D. de Figueiredo, J.-P. Gossez *The beginning of the Fučik spectrum for the p-Laplacian* J. Differential Equations 159 (1999) pp. 212–238.

[14] M. Cuesta, D. de Figueiredo, J.-P. Gossez, *A nodal domain property for the p-Laplacian*, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) pp. 669–673.

[15] E. DiBenedetto, *$C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., 7 (1983), pp. 827–850.

[16] V. Fridman, and B. Kawohl, *Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant*, Comm. Math. Univ. Carol. (to appear).

[17] L. Friedlander, *Asymptotic behavior of the eigenvalues of the p-Laplacian*, Comm. Partial Differential Equations 14 (1989), no. 8-9, pp. 1059–1069.

[18] N. Fukagai, M. Ito, and K. Narukawa, *Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of the Poincaré type*, Differential Integral Equations 12 (1999), no. 2, pp. 183–206.

[19] J. P. García, and I. Peral, *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues*, Comm. Partial Differential Equations 12 (1987) pp. 1389–1430.

[20] J. P. García, and I. Peral, *Comportement asymptotique des valeurs propres du p-laplacien. (French) [Asymptotic behavior of the eigenvalues of the p-Laplacian]*, C. R. Acad. Sci. Paris Sér. I Math. 307 (1988) pp. 75–78.

[21] R. Hall, W. Hayman, and A. Weitsman, *On asymmetry and capacity*, J. Analyse Math. 56 (1991), 87–123.

[22] W. Hansen, and N. Nadirashvili, *Isoperimetric inequalities in potential theory*, Potential Anal. 3 (1994), no. 1, 1–14.

[23] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, *The nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$ can be closed*, Duke Math. J. 90 (1997), no. 3, pp. 631–640.

[24] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Rational Mech. Anal. 123 (1993) pp. 51–74.

[25] R. Jensen, P.-L. Lions, and P. E. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*, Proc. Amer. Math. Soc. 102 (1988) pp. 975–978.

[26] P. Juutinen, *Minimization problems for Lipschitz functions via viscosity solutions*, Ann. Acad. Sci. Fenn. Math. Diss. 115 (1998) pp. 1–53.

[27] P. Juutinen, P. Lindqvist, and J. J. Manfredi, *The ∞-eigenvalue problem*, Arch. Rational Mech. Anal. 148 (1999) pp. 89–105.

[28] P. Juutinen, P. Lindqvist, and J. J. Manfredi, *The infinity Laplacian: examples and observations*, Papers on analysis, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, (2001) pp. 207–217.

[29] P. Lindqvist, *On the equation div* $\left(|\nabla u|^{p-2}\nabla u\right) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 No. 1 (1990) pp. 157–164.

[30] P. Lindqvist, and J. J. Manfredi, *The Harnack inequality for ∞-harmonic functions*, Electronic Journal of Differential Equations 5 (1995) pp. 1–5.

[31] P. Lindqvist, and J. J. Manfredi, *Note on ∞-harmonic functions*, Revista Matemática de la Universidad Complutense 10 (1997) pp. 471–480.

[32] A. Melas, *On the nodal line of the second eigenfunction of the Laplacian in* $\mathbb{R}^2$, J. Differential Geom. 35 (1992), no. 1, pp. 255–263.

[33] L. Moser, *Problem 24 (corrected)* Canad. Math. Bull. 3 (1960) pp. 78.

[34] K. Nurmela, and P. Östergård, *More optimal packings of equal circles in a square*, Discrete Comput. Geom. 22 no. 3 (1999) pp. 439–457.

[35] K. Nurmela, and P. Östergård, *Packing up to 50 Equal Circles in a Square*, Disc. Comput. Geom. 18 (1997) pp. 111–120.

[36] L. E. Payne, *Isoperimetric inequalities and their applications*, SIAM Rev. 9 1967 pp. 453–488.

[37] I. Peral, *Some results on quasilinear elliptic equations: growth versus shape*, Nonlinear functional analysis and applications to differential equations (Trieste, 1997), 153–202, World Sci. Publishing, River Edge, NJ (1998).

[38] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1986).

[39] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian systems.* Second edition. Springer-Verlag, Berlin, (1996).

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