

The ∞ -Eigenvalue Problem

PETRI JUUTINEN, PETER LINDQVIST & JUAN J. MANFREDI

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Abstract

The Euler-Lagrange equation of the nonlinear Rayleigh quotient

$$\left(\int_{\Omega} |\nabla u|^p dx \right) / \left(\int_{\Omega} |u|^p dx \right)$$

is

$$-\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) = \Lambda_p^p |u|^{p-2} u,$$

where Λ_p^p is the minimum value of the quotient. The limit as $p \rightarrow \infty$ of these equations is found to be

$$\max \left\{ \Lambda_{\infty} - \frac{|\nabla u(x)|}{u(x)}, \Delta_{\infty} u(x) \right\} = 0,$$

where the constant $\Lambda_{\infty} = \lim_{p \rightarrow \infty} \Lambda_p$ is the reciprocal of the maximum of the distance to the boundary of the domain Ω .

§0. Introduction

Let Ω be a bounded domain in \mathbb{R}^n . The minimum of the Rayleigh quotient

$$\frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}$$

among all functions with zero boundary values is the first eigenvalue of the Laplacian in the domain Ω . This minimum value λ is achieved by the unique positive solution, up to multiplication by constants, of the Euler-Lagrange equation $\Delta u + \lambda u = 0$

with zero boundary values. Given a number p , $1 < p < \infty$, consider minimizing the nonquadratic Rayleigh quotient

$$\frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}.$$

This problem leads to a nonlinear Euler-Lagrange equation, except in the case $p = 2$. As expected, the cases $p = 1$ and $p = \infty$ present additional difficulties. The objective of this paper is to study the limiting case $p = \infty$.

Formally, one has to minimize the ratio

$$\frac{\|\nabla u\|_{\infty, \Omega}}{\|u\|_{\infty, \Omega}} = \lim_{p \rightarrow \infty} \frac{\|\nabla u\|_{p, \Omega}}{\|u\|_{p, \Omega}}.$$

The minimum is the reciprocal of the radius of the largest possible ball inscribed in the domain Ω . Unfortunately, this min-max problem has too many solutions. In fact, outside the largest possible ball inscribed in the domain, one can modify a solution rather freely without changing the ratio. A more careful limiting procedure as $p \rightarrow \infty$ is called for to identify the genuine ∞ -eigenfunctions.

The correct Euler-Lagrange equation turns out to be

$$(0.1) \quad \max \left\{ \Lambda_{\infty} - \frac{|\nabla u(x)|}{u(x)}, \Delta_{\infty} u(x) \right\} = 0.$$

That is, at each point $x \in \Omega$, the larger of the two expressions is zero. Here

$$\Lambda_{\infty} = \frac{1}{\max\{\text{dist}(x, \partial\Omega) : x \in \Omega\}},$$

and

$$\Delta_{\infty} u(x) = \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

is the so called ∞ -Laplacian. A most peculiar feature is that the ‘‘principal frequency’’ Λ_{∞} has such a simple geometric characterization. The presence of the operator Δ_{∞} is natural but, at first sight, the dichotomy of the equation is astonishing.

The equation has to be properly interpreted in the viscosity sense. For example, when Ω is a ball, the only solution is the distance function

$$\delta(x) = \text{distance}(x, \partial\Omega)$$

up to constant multiples. In this case, the distance function satisfies $\Delta_{\infty} \delta(x) = 0$ and $\Lambda_{\infty} < |\nabla \log \delta(x)|$ except at the center x_0 of the ball. Indeed, at the center $\Lambda_{\infty} = |\nabla \log \delta(x_0)|$ and $\Delta_{\infty} \delta(x_0) < 0$ in the viscosity sense. Notice that the second derivatives needed to evaluate $\Delta_{\infty} \delta(x_0)$ do not exist in the ordinary sense. This illustrates the usefulness of viscosity solutions as weak solutions of nonlinear partial differential equations.

It is easy to see (Section 2 below) that the distance function always satisfies the minimization problem. That is,

$$\Lambda_\infty = \frac{\|\nabla\delta\|_{\infty,\Omega}}{\|\delta\|_{\infty,\Omega}}.$$

However, the distance is often not a genuine ∞ -eigenfunction, since it is not a solution of the Euler-Lagrange equation (0.1). This happens already when Ω is a square or a parallelepiped. Moreover, this example shows that the solution is not a concave function when Ω is convex, although its logarithm is, indeed, concave. This follows from SAKAGUCHI's generalization of the Brascamp-Lieb theorem; see [S].

What about the existence of positive solutions to the equation

$$(0.2) \quad \max \left\{ \Lambda - \frac{|\nabla u(x)|}{u(x)}, \Delta_\infty u(x) \right\} = 0$$

for values of Λ other than Λ_∞ ? It follows from an appropriate Harnack inequality that $\Lambda \leq \Lambda_\infty$. If, in addition, the solution has zero boundary values, then $\Lambda = \Lambda_\infty$ indeed. This later result lies deeper, its proof being based on a uniqueness result for the equation

$$\max \left\{ \Lambda - |\nabla v|, \Delta_\infty v + |\nabla v|^4 \right\} = 0$$

satisfied by $v = \log u$, where u satisfies (0.2).

In Section 1 we present the relevant definitions and first results, and prove the basic fact that limits of p -eigenfunctions are indeed viscosity solutions of (0.2). Note that in order to use the terminology in [CIL] we consider equation (0.2) with a minus sign in front. See equation (1.22) below. In Section 2 we present a proof of a comparison principle for the logarithms of genuine ∞ -eigenfunctions. This is our main result; its proof is based on the construction of a new sensitive test function. An application of this comparison principle is presented in Section 3, where we prove that Λ_∞ is the only "right" Λ . We finish by presenting some explicit computations in the case of a square, which are discussed in Section 4.

§1. Definitions and First Results

For a bounded domain Ω in \mathbb{R}^n , the distance function $\delta(x) = \text{distance}(x, \partial\Omega)$ is Lipschitz continuous, satisfies $|\nabla\delta(x)| = 1$ for a.e. $x \in \Omega$, and vanishes on the boundary of Ω . Let ϕ be any other Lipschitz continuous function vanishing on $\partial\Omega$. Fix $x \in \Omega$ and choose $y \in \partial\Omega$ such that $\delta(x) = |x - y|$. We have

$$|\phi(x)| = |\phi(x) - \phi(y)| \leq \|\nabla\phi\|_\infty \delta(x).$$

Therefore,

$$(1.1) \quad \frac{\|\nabla\phi\|_\infty}{|\phi(x)|} \geq \frac{\|\nabla\delta\|_\infty}{|\delta(x)|}$$

and we see that the distance function satisfies

$$(1.2) \quad \Lambda_\infty = \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty} \leq \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty},$$

for all $\phi \in W^{1,\infty}(\Omega)$ vanishing on $\partial\Omega$. The constant $\Lambda_\infty = 1/\|\delta\|_\infty$ depends only on the domain Ω , and for reasons that will be clear later on we think of Λ_∞ as the smallest ∞ -eigenvalue of the domain Ω .

Consider the problem corresponding to (1.2) for finite $p > 1$:

$$(1.3) \quad \Lambda_p = \inf \left\{ \frac{\left(\frac{1}{|\Omega|} \int_\Omega |\nabla\phi(x)|^p dx \right)^{1/p}}{\left(\frac{1}{|\Omega|} \int_\Omega |\phi(x)|^p dx \right)^{1/p}} : \phi \in W_0^{1,p}(\Omega) \right\}.$$

There is a minimizer $u_p \in W_0^{1,p}(\Omega)$, unique up to a multiplicative constant, that satisfies the Euler equation

$$(1.4) \quad -\operatorname{div} \left(|\nabla u_p|^{p-2} \nabla u_p \right) = \Lambda_p^p |u_p|^{p-2} u_p.$$

It is well known that $u_p > 0$ in Ω so that we can replace the right-hand side of (1.4) by u_p^{p-1} . References to these facts can be found in [L]. We normalize u_p by requiring that $\|u_p\|_p = 1$, where $\|f\|_p = \left(\frac{1}{|\Omega|} \int_\Omega |f|^p dx \right)^{1/p}$. The name given to Λ_∞ is justified by the following lemma.

1.5. Lemma.

$$\lim_{p \rightarrow \infty} \Lambda_p = \Lambda_\infty.$$

Proof. Using $\delta(x)$ as a test function in (1.3) we get

$$\Lambda_p \leq \frac{1}{\left(\frac{1}{|\Omega|} \int_\Omega |\delta(x)|^p dx \right)^{1/p}},$$

which implies that

$$\limsup_{p \rightarrow \infty} \Lambda_p \leq \Lambda_\infty.$$

Note that

$$\left(\frac{1}{|\Omega|} \int_\Omega |\nabla u_p(x)|^p dx \right)^{1/p} \leq \Lambda_p$$

is uniformly bounded in p . Fix an exponent $m > n$. For $p > m$ by Hölder's inequality we have

$$\left(\frac{1}{|\Omega|} \int_\Omega |\nabla u_p(x)|^m dx \right)^{1/m} \leq \Lambda_p.$$

We conclude that $\{u_p\}_{p \geq m}$ is uniformly bounded in $W_0^{1,m}(\Omega)$. We can select a subsequence u_{p_i} that converges to a function denoted by u_∞ weakly in $W^{1,m}(\Omega)$ and

uniformly in $C^\alpha(\Omega)$ for $\alpha = 1 - n/m$. The limit function u_∞ is an ∞ -superharmonic function as defined in [LM2], where it is also proved that nonnegative ∞ -superharmonic functions satisfy an inequality of Harnack type that implies that $u_\infty(x) > 0$ for all $x \in \Omega$. For q large enough, using the weak lower semicontinuity of the L^q -norm and the fact that u_{p_i} converges to u_∞ weakly also in $W^{1,q}(\Omega)$, we have the inequality

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leq \liminf_{p_i \rightarrow \infty} \frac{\left(\frac{1}{|\Omega|} \int_\Omega |\nabla u_{p_i}(x)|^q dx\right)^{1/q}}{\left(\frac{1}{|\Omega|} \int_\Omega |u_{p_i}(x)|^q dx\right)^{1/q}}.$$

Multiplying and dividing this inequality by $\left(\frac{1}{|\Omega|} \int_\Omega |u_{p_i}(x)|^{p_i} dx\right)^{1/p_i}$ and using Hölder's inequality we obtain

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leq \liminf_{p_i \rightarrow \infty} \left(\Lambda_{p_i} \frac{\|u_{p_i}\|_\infty}{\|u_{p_i}\|_q} \right).$$

We can take limits as $p_i \rightarrow \infty$ in the right-hand side to get

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leq \left(\liminf_{p_i \rightarrow \infty} \Lambda_{p_i} \right) \frac{\|u_\infty\|_\infty}{\|u_\infty\|_q}$$

for a fixed q . Letting $q \rightarrow \infty$ and using the minimizing property (1.2) we have

$$\Lambda_\infty \leq \liminf_{p_i \rightarrow \infty} \Lambda_{p_i}.$$

This is enough to conclude the lemma, since we can apply this process to any subsequence of $\{u_p\}$. \square

1.6. Remark. As a matter of fact, the above proof shows that any such u_∞ is extremal for the problem (1.2), that is,

$$\Lambda_\infty = \frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty}.$$

As we noted in the introduction, it is quite easy to find examples in which this minimum is attained by more than one function.

Suppose for a moment that the u_p are smooth so that we can differentiate (1.4) to get

$$(1.7) \quad -[|\nabla u_p|^{p-2} \Delta u_p + (p-2)|\nabla u_p|^{p-4} \Delta_\infty u_p] = \Lambda_p^p |u_p|^{p-2} u_p.$$

This equation is fully nonlinear and it makes sense to talk about viscosity subsolutions and supersolutions of it. The following lemma tells us that u_p is always a viscosity solution of (1.7). This is a somewhat delicate lemma since it is not clear that the comparison principle holds for equation (1.9) below.

1.8. Lemma. A continuous weak (sub-)supersolution $u \in W_{\text{loc}}^{1,p}(\Omega)$ of the equation

$$(1.9) \quad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Lambda_p^p |u|^{p-2}u$$

is always a viscosity (sub-)supersolution of (1.7).

Before proving Lemma 1.8, let us recall the definition of viscosity (sub-)supersolution in our case. Let $z \in \mathbb{R}^n$, $X \in \mathbb{R}^n$ and S be a real symmetric matrix. Consider the continuous function

$$F_p(z, X, S) = -[|X|^{p-2}\operatorname{trace}(S) + (p-2)|X|^{p-4}\langle S \cdot X, X \rangle] - \Lambda_p^p |z|^{p-2}z.$$

Since we are interested in solutions of the partial differential equation

$$(1.10) \quad F_p(u, \nabla u, D^2u) = 0$$

when $p \rightarrow \infty$, we always assume that p is large enough.

1.11. Definition. An upper semicontinuous function u defined in Ω is a *viscosity subsolution* of (1.10) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

(a) $u(x_0) = \phi(x_0)$ and (b) $u(x) < \phi(x)$ if $x \neq x_0$, then

$$F_p(\phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \leq 0.$$

1.12. Definition. A lower semicontinuous function u defined in Ω is a *viscosity supersolution* of (1.10) if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

(a) $u(x_0) = \phi(x_0)$ and (b) $u(x) > \phi(x)$ if $x \neq x_0$, then

$$F_p(\phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

Condition (b) in both definitions can be relaxed quite a bit. The strict inequality is not really required and the condition only needs to hold in a neighborhood of x_0 . We refer to [CIL] for the theory of viscosity solutions in general and to [Ju] for viscosity solutions of operators related to the ∞ -Laplacian.

Proof of Lemma 1.8. We present the details for the case of supersolutions. Fix $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$ for $x \neq x_0$. We want to show that

$$\begin{aligned} & -[|\nabla\phi(x_0)|^{p-2}\Delta\phi(x_0) + (p-2)|\nabla\phi(x_0)|^{p-4}\Delta_\infty\phi(x_0)] \\ & \quad - \Lambda_p^p |\phi(x_0)|^{p-2}\phi(x_0) \geq 0. \end{aligned}$$

Suppose that this is not the case. Then, by continuity there exists a small $r > 0$ such that, if $|x - x_0| < r$, we have

$$- [|\nabla\phi(x)|^{p-2}\Delta\phi(x) + (p-2)|\nabla\phi(x)|^{p-4}\Delta_\infty\phi(x)] < \Lambda_p^p |\phi(x)|^{p-2}\phi(x).$$

Set $m = \inf\{u(x) - \phi(x) : |x - x_0| = r\} > 0$ and write $\Phi = \phi + \frac{1}{2}m$. The function Φ satisfies $\Phi < u$ on $\partial B(x_0, r)$, $\Phi(x_0) > u(x_0)$ and

$$(1.13) \quad -\operatorname{div}(|\nabla\Phi(x)|^{p-2}\nabla\Phi(x)) < \Lambda_p^p |\phi(x)|^{p-2}\phi(x).$$

The function $(\Phi - u)^+$ extended as the zero function outside of the ball $B(x_0, r)$ is a good test function for equation (1.9). Since we are assuming that u is a weak supersolution, we get

$$(1.14) \quad \int_{\{\Phi > u\}} |\nabla u|^{p-2} \langle \nabla u, \nabla(\Phi - u) \rangle dx \geq \Lambda_p^p \int_{\{\Phi > u\}} |u|^{p-2} u (\Phi - u) dx.$$

Multiply both sides of (1.13) by $(\Phi - u)^+$ and integrate the product by parts to obtain

$$(1.15) \quad \int_{\{\Phi > u\}} |\nabla \Phi|^{p-2} \langle \nabla \Phi, \nabla(\Phi - u) \rangle dx < \Lambda_p^p \int_{\{\Phi > u\}} |\phi|^{p-2} \phi (\Phi - u) dx.$$

Subtracting (1.14) from (1.15) we arrive at

$$\begin{aligned} \int_{\{\Phi > u\}} \langle |\nabla \Phi|^{p-2} \nabla \Phi - |\nabla u|^{p-2} \nabla u, \nabla(\Phi - u) \rangle dx \\ < \Lambda_p^p \int_{\{\Phi > u\}} \left(|\phi|^{p-2} \phi - |u|^{p-2} u \right) (\Phi - u) dx. \end{aligned}$$

Since the left-hand side is bounded below by a positive constant, depending on p and n , times

$$\int_{\{\Phi > u\}} |\nabla \Phi - \nabla u|^p dx,$$

and the right-hand side is negative, we conclude that $\Phi \leq u$ in $B(x_0, r)$, contradicting the fact that $\Phi(x_0) > u(x_0)$. \square

Next, we compute the limit of the $F_p(z, X, S)$ as $p \rightarrow \infty$ in the viscosity sense. That is, we consider the sequence of viscosity solutions $\{u_p\}$ and we would like to find out what equation is satisfied by any cluster point of this sequence, which we denote by u_∞ . Explicitly, we assume that for a subsequence $p_i \rightarrow \infty$ we have $\lim_{p_i \rightarrow \infty} u_{p_i} = u_\infty$ uniformly in Ω .

Fix a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and the inequality $u_\infty(x) > \phi(x)$ holds for $x \neq x_0$. Also fix $R > 0$ so that $B(x_0, 2R) \subset \Omega$. For $0 < r < R$ we certainly have

$$\inf\{u_\infty(x) - \phi(x) : x \in B(x_0, R) \setminus B(x_0, r)\} > 0.$$

Since $u_{p_i} \rightarrow u_\infty$ uniformly in the closure of $B(x_0, R)$, we conclude that for $i \geq i_r$,

$$\inf\{u_{p_i}(x) - \phi(x) : x \in B(x_0, R) \setminus B(x_0, r)\} > u_{p_i}(x_0) - \phi(x_0).$$

Therefore, for such indices i , $u_{p_i} - \phi$ attains its minimum at a point $x_i \in B(x_0, r)$, and we see by letting $r \rightarrow 0$ that $x_i \rightarrow x_0$ as $i \rightarrow \infty$. For notational simplicity we drop the subindices and write p_i for p_{i_r} and x_i for $x_{p_{i_r}}$. Since u_{p_i} is a viscosity supersolution of (1.7) we get

$$(1.16) \quad \begin{aligned} - \left[|\nabla \phi(x_i)|^{p_i-2} \Delta \phi(x_i) + (p_i - 2) |\nabla \phi(x_i)|^{p_i-4} \Delta_\infty \phi(x_i) \right] \\ \geq \Lambda_{p_i}^{p_i} |u_{p_i}(x_i)|^{p_i-2} u_{p_i}(x_i). \end{aligned}$$

Recall that $u_\infty(x) > 0$, and so $u_{p_i}(x_i) > 0$ for large i , which itself implies that $|\nabla\phi(x_i)| \neq 0$ as follows from (1.16). Dividing by $|\nabla\phi(x_i)|^{p_i-4}$ and by $p_i - 2$ we arrive at

$$(1.17) \quad -\frac{|\nabla\phi(x_i)|^2 \Delta\phi(x_i)}{p_i - 2} - \Delta_\infty\phi(x_i) \geq \left(\frac{\Lambda_{p_i} u_{p_i}(x_i)}{|\nabla\phi(x_i)|} \right)^{p_i-4} \frac{\Lambda_{p_i}^4 u_{p_i}(x_i)^3}{p_i - 2}.$$

Suppose that $\frac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} > 1$. Letting $p_i \rightarrow \infty$ we get a contradiction. Therefore we must have

$$(1.18) \quad \frac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} \leq 1.$$

Since the right-hand side of (1.17) is nonnegative, letting $p_i \rightarrow \infty$ we see that

$$(1.19) \quad -\Delta_\infty\phi(x_0) \geq 0.$$

These two equations (1.18) and (1.19) can be combined into one by writing

$$(1.20) \quad \min \{ |\nabla\phi(x_0)| - \Lambda_\infty\phi(x_0), -\Delta_\infty\phi(x_0) \} \geq 0.$$

We have established that u_∞ is a viscosity supersolution of the equation

$$\min \{ |\nabla u| - \Lambda_\infty u, -\Delta_\infty u \} = 0.$$

It is therefore natural to define

$$F_\infty(z, X, S) = \min \{ |X| - \Lambda_\infty z, -\langle S \cdot X, X \rangle \}.$$

We can now state the main theorem of this section:

1.21. Theorem. *A function u_∞ obtained as a limit of a subsequence of $\{u_p\}$ is a viscosity solution of the equation*

$$(1.22) \quad F_\infty(u, \nabla u, D^2 u) = \min \{ |\nabla u| - \Lambda_\infty u, -\Delta_\infty u \} = 0.$$

Before finishing the proof of the theorem, note that

i) u_∞ is ∞ -superharmonic, since

$$-\Delta_\infty u_\infty \geq 0$$

in the viscosity sense, and

ii) $|\nabla u_\infty| \geq \Lambda_\infty u_\infty$ in the viscosity sense. Moreover, at least heuristically, if one of these inequalities is strict, the other must be an equality.

Proof. It remains to be proved that u_∞ is a viscosity subsolution. The proof is similar to the supersolution case but not symmetric. Fix a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and the inequality $u_\infty(x) < \phi(x)$ holds for $x \neq x_0$. We want to check that

$$\min\{|\nabla\phi(x_0)| - \Lambda_\infty\phi(x_0), -\Delta_\infty\phi(x_0)\} \leq 0.$$

Observe that if $\nabla\phi(x_0) = 0$, there is nothing to prove. As a matter of fact, we may assume that $|\nabla\phi(x_0)| - \Lambda_\infty\phi(x_0) > 0$. We now repeat the procedure that we followed in the supersolution case. The analogue of (1.16) is

$$\begin{aligned} -\left[|\nabla\phi(x_i)|^{p_i-2}\Delta\phi(x_i) + (p_i - 2)|\nabla\phi(x_i)|^{p_i-4}\Delta_\infty\phi(x_i)\right] \\ \leq \Lambda_p^p|u_{p_i}(x_i)|^{p-2}u_{p_i}(x_i), \end{aligned}$$

and the analogue of (1.17) is

$$-\frac{|\nabla\phi(x_i)|^2\Delta\phi(x_i)}{p_i - 2} - \Delta_\infty\phi(x_i) \leq \left(\frac{\Lambda_{p_i}u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\right)^{p_i-4} \frac{\Lambda_{p_i}^4u_{p_i}(x_i)^3}{p_i - 2}.$$

Letting $p_i \rightarrow \infty$ we get $-\Delta_\infty\phi(x_0) \leq 0$. \square

§2. Comparison Principles

Consider again the equation (1.22):

$$F_\infty(u, \nabla u, D^2u) = \min\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0.$$

Note that $F_\infty(z, X, S)$ is decreasing in S and decreasing in z . In the language of [CIL], the function F_∞ is degenerate elliptic but it is not proper. Therefore, the usual tools to prove uniqueness to solutions to a Dirichlet problem associated with equation (1.22) do not apply. However, we know that every u_∞ is strictly positive. This suggests considering the equation that $v_\infty = \log(u_\infty)$ satisfies.

2.1. Lemma. *Let u be a nonnegative viscosity solution of (1.22) in Ω . Then $v = \log(u)$ is a viscosity solution of the equation*

$$(2.2) \quad \min\{|\nabla v| - \Lambda_\infty, -\Delta_\infty v - |\nabla v|^4\} = 0$$

in Ω .

Proof. The lemma follows from a simple calculation. We provide the details in the supersolution case. Let $\phi \in C^2(\Omega)$ such that $v(x_0) = \phi(x_0)$ and $v(x) > \phi(x)$ for $x \neq x_0$. Write $\Phi(x) = e^{\phi(x)}$. Then Φ is a good test function for u at the point x_0 . Therefore, we have

$$\min\{|\nabla\Phi(x_0)| - \Lambda_\infty\Phi(x_0), -\Delta_\infty\Phi(x_0)\} \geq 0.$$

Writing this inequality in terms of ϕ we get

$$\min\{e^{\phi(x_0)}(|\nabla\phi(x_0)| - \Lambda_\infty), -e^{3\phi(x_0)}(\Delta_\infty\phi(x_0) + |\nabla\phi(x_0)|^4)\} \geq 0,$$

from which the lemma follows easily. \square

Since equation (2.2) is now proper, we can try to prove the comparison principle for solutions of (2.2). Because the equation is degenerate elliptic, the usual techniques of [CIL] need to be augmented. In the case of the ∞ -harmonic equation $\Delta_\infty u = 0$ the comparison principle is given in JENSEN [J]. A nice proof of this comparison principle for the ∞ -harmonic functions based on the “comparison principle for semicontinuous functions” is due to JUUTINEN [Ju]. Equation (2.2) is different on two counts. First, in the viscosity sense we have $|\nabla v| \geq \Lambda_\infty$, which will make possible the uniqueness proof presented below and second, it contains the term $|\nabla v|^4$. The main result of this section is:

2.3. Theorem. *Let Ω be a bounded domain, let u be a viscosity subsolution of (2.2) in Ω and let v be viscosity supersolution of (2.2) in Ω . Suppose that both functions are continuous in $\overline{\Omega}$. Then, the following comparison principle holds:*

$$(2.4) \quad \sup_{x \in \overline{\Omega}} (u(x) - v(x)) = \sup_{x \in \partial\Omega} (u(x) - v(x)).$$

Proof. Without loss of generality we may assume that u and v are positive by adding a large constant to both of them. We proceed by contradiction. Suppose that (2.4) does not hold. Then, we must have

$$(2.5) \quad \sup_{x \in \overline{\Omega}} (u(x) - v(x)) > \sup_{x \in \partial\Omega} (u(x) - v(x)).$$

This inequality still holds if we replace v by a function w for which $\|v - w\|_{L^\infty(\Omega)}$ is small enough. We construct a function w that is a strict supersolution of (2.2), and then we apply the comparison for semicontinuous functions from [CIL].

2.6. Lemma. *Let $A > 1$ and $\alpha > 1$ be given. The function*

$$f(t) = \frac{1}{\alpha} \log(1 + A(e^{\alpha t} - 1))$$

has the following properties:

- (i) $f(0) = 0$, $f'(t) > 1$ and $f''(t) < 0$ for all $t \geq 0$,
- (ii) f is invertible,
- (iii) f satisfies the differential inequality

$$1 - \frac{1}{f'(t)} + \frac{f''(t)}{(f'(t))^2} < 0,$$

- (iv) f is an approximation of the identity as $A \rightarrow 1^+$ in the sense that

$$0 < f(t) - t < \frac{A - 1}{\alpha}$$

for all $t \geq 0$.

The proof of this lemma is elementary. Notice that f satisfies the differential equation

$$1 - \frac{1}{f'(t)} + \frac{1}{\alpha} \frac{f''(t)}{(f'(t))^2} = 0$$

so that (iii) follows from the fact that $1 - 1/\alpha > 0$. Observe that if we write $f_A(t) = \frac{1}{\alpha} \log(1 + A(e^{\alpha t} - 1))$, then for any positive A and B we have

$$f_A \circ f_B = f_{AB}.$$

In particular, $f_A^{-1} = f_{A^{-1}}$ since $AA^{-1} = 1$ and f_1 is the identity.

By taking A close enough to 1, we can guarantee that $w = f(v)$ satisfies (2.5). The equation for which w is a supersolution is obtained as follows. Let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ be such that $w(x_0) = \phi(x_0)$ and $w(x) \geq \phi(x)$ for $x \neq x_0$. Set

$$\Phi = f^{-1}(\phi), \text{ so that } f(\Phi) = \phi.$$

Since f^{-1} is monotone, Φ is a good test function for v at the point x_0 . Since v is a supersolution of (2.2), we have

$$(2.7) \quad \min \{ |\nabla \Phi(x_0)| - \Lambda_\infty, -\Delta_\infty \Phi(x_0) - |\nabla \Phi(x_0)|^4 \} \geq 0.$$

Differentiating we obtain

$$\nabla \Phi = \frac{1}{f'(\Phi)} \nabla \phi,$$

$$D^2 \Phi = \frac{1}{f'(\Phi)} D^2 \phi - \frac{f''(\Phi)}{(f'(\Phi))^3} (\nabla \phi \otimes \nabla \phi).$$

From (2.7) we deduce that

$$(2.8) \quad |\nabla \Phi(x_0)| - \Lambda_\infty \geq 0$$

$$(2.9) \quad -\Delta_\infty \Phi(x_0) - |\nabla \Phi(x_0)|^4 \geq 0.$$

From (2.8) it follows that

$$(2.10) \quad |\nabla \phi(x_0)| \geq f'(\Phi(x_0)) \Lambda_\infty,$$

$$(2.11) \quad |\nabla \phi(x_0)| - \Lambda_\infty \geq [f'(\Phi(x_0)) - 1] \Lambda_\infty.$$

We compute starting from (2.9). Omitting the point x_0 for notational simplicity, we obtain

$$\begin{aligned} - \left\langle \left(\frac{1}{f'(\Phi)} D^2 \phi - \frac{f''(\Phi)}{(f'(\Phi))^3} (\nabla \phi \otimes \nabla \phi) \right) \frac{1}{f'(\Phi)} \nabla \phi, \frac{1}{f'(\Phi)} \nabla \phi \right\rangle \\ - \frac{1}{f'(\Phi)^4} |\nabla \phi|^4 \geq 0. \end{aligned}$$

After elementary manipulations this inequality becomes

$$-\Delta_\infty \phi - \left[\frac{1}{f'} - \frac{f''}{(f')^2} \right] |\nabla \phi|^4 \geq 0.$$

Thus, we have obtained the inequality

$$-\Delta_\infty \phi - |\nabla \phi|^4 \geq - \left[1 - \frac{1}{f'} - \frac{f''}{(f')^2} \right] |\nabla \phi|^4.$$

Now using (iii) of Lemma 2.6, (2.10) and the fact that $\Phi(x_0) = v(x_0)$ we get

$$(2.12) \quad -\Delta_\infty \phi(x_0) - |\nabla \phi(x_0)|^4 \geq - \left[1 - \frac{1}{f'(v(x_0))} - \frac{f''(v(x_0))}{(f'(v(x_0)))^2} \right] (f'(v(x_0)))^4 \Lambda_\infty^4.$$

From (2.11) and (2.12) we deduce that

$$(2.13) \quad \min\{|\nabla \phi(x_0)| - \Lambda_\infty, -\Delta_\infty \phi(x_0) - |\nabla \phi(x_0)|^4\} \geq \mu(x_0) > 0,$$

where

$$\mu(x) = \min \left\{ [f'(v(x)) - 1] \Lambda_\infty, - \left[1 - \frac{1}{f'(v(x))} - \frac{f''(v(x))}{(f'(v(x)))^2} \right] (f'(v(x)))^4 \Lambda_\infty^4 \right\}.$$

Since $\mu(x) > 0$, inequality (2.13) expresses that w is a strict supersolution of (2.2).

We are now ready to complete the proof. Let (x_τ, y_τ) be a maximum point of

$$u(x) - w(y) - \frac{\tau}{2}|x - y|^2$$

in $\overline{\Omega} \times \overline{\Omega}$. From the results of [CIL] it follows that through a subsequence

$$x_{\tau_i} \rightarrow x_0 \in \overline{\Omega},$$

where x_0 is a maximum point of $u - w$ in $\overline{\Omega}$. By (2.5) x_0 is in fact an interior point of Ω . We note also that $y_{\tau_i} \rightarrow x_0$. From now on we just write τ for τ_i for notational simplicity. Applying the maximum principle for semicontinuous functions we get symmetric matrices $\mathbb{X}_\tau, \mathbb{Y}_\tau$ such that

$$(2.14) \quad (\tau(x_\tau - y_\tau), \mathbb{X}_\tau) \in \overline{D}^{2,+} u(x_\tau),$$

$$(2.15) \quad (\tau(x_\tau - y_\tau), \mathbb{Y}_\tau) \in \overline{D}^{2,-} w(y_\tau),$$

$$(2.16) \quad \langle \mathbb{X}_\tau \xi, \xi \rangle - \langle \mathbb{Y}_\tau \eta, \eta \rangle \leq 3\tau |\xi - \eta|^2.$$

The maximum principle for semicontinuous functions as well as the definition of the semijets $\overline{D}^{2,+}$ and $\overline{D}^{2,-}$ can be found in [CIL].

Since u is a subsolution of (2.2), we have

$$(2.17) \quad \min \{ |\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2 \langle \mathbb{X}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau) \rangle - \tau^4 |x_\tau - y_\tau|^4 \} \leq 0.$$

Since w is a strict supersolution of (2.2), we get from (2.13) that

$$(2.18) \quad \min \{ |\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2 \langle \mathbb{Y}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau) \rangle - \tau^4 |x_\tau - y_\tau|^4 \} \geq \mu(y_\tau) > 0.$$

We now subtract (2.17) from (2.18) to get

$$(2.19) \quad \begin{aligned} \mu(y_\tau) &\leq \min \{ |\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2 \langle \mathbb{Y}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau) \rangle - \tau^4 |x_\tau - y_\tau|^4 \} \\ &\quad - \min \{ |\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2 \langle \mathbb{X}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau) \rangle - \tau^4 |x_\tau - y_\tau|^4 \} \\ &\leq \tau^2 \max \{ 0, \langle (\mathbb{X}_\tau - \mathbb{Y}_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau) \rangle \} \\ &= 0. \end{aligned}$$

Since $\mu(y_\tau) > 0$, we have arrived at a contradiction, and the theorem is thereby proved. \square

2.20. Remark. It can be read off from the proof that the comparison principle also holds when *one* of the functions takes the value $-\infty$ on the whole boundary, as $\log u_\infty$ does for instance.

§3. The Principal Frequency of Δ_∞ in a Domain

As an application of the comparison principle (2.3) we are able to prove that Λ_∞ has a property typical of more conventional eigenvalue problems.

3.1. Theorem. *Let Ω be bounded domain in \mathbb{R}^n satisfying $\partial\Omega = \partial\bar{\Omega}$. If u is a continuous positive solution in Ω of the equation*

$$(3.2) \quad \min \{ |\nabla u| - \Lambda u, -\Delta_\infty u \} = 0,$$

with zero boundary values, then $\Lambda = \Lambda_\infty$.

Proof. Fix a point $x_0 \in \Omega$ so that

$$\delta(x_0) = \frac{1}{\Lambda_\infty}.$$

Without loss of generality we may assume that $x_0 = 0$. Suppose that $\Lambda > \Lambda_\infty$. Then the ball $B(0, 1/\Lambda)$ is strictly contained in Ω . Indeed it is away from $\partial\Omega$. Let $\rho(x)$ be the distance function to the boundary of the ball $B(0, 1/\Lambda)$. Both $C\rho(x)$ and $u(x)$ are solutions of (3.2) in $B(0, 1/\Lambda)$ for any positive constant C . By the comparison principle we have

$$\log C\rho(x) \leq \log u(x)$$

in the ball $B(0, 1/\Lambda)$, leading to a contradiction as $C \rightarrow \infty$. Therefore we must have $\Lambda \leq \Lambda_\infty$.

If $\Lambda < 0$, then $|\nabla u| - \Lambda u > 0$ because u is positive. Thus, equation (3.2) becomes $\Delta_\infty u = 0$ whose only solution with zero boundary values is the zero function. Therefore $\Lambda \geq 0$.

We claim that $\Lambda \neq 0$. If not, equation (3.2) becomes

$$(3.3) \quad \min \{ |\nabla u|, -\Delta_\infty u \} = 0.$$

Using the definition of a viscosity solution, we easily check that, in fact, (3.3) is equivalent to $-\Delta_\infty u = 0$, again forcing u to vanish.

So far we have proved that $0 < \Lambda \leq \Lambda_\infty$. Suppose that $\Lambda < \Lambda_\infty$ and denote $\Omega_\varepsilon = \{x \in \mathbb{R}^n \mid \text{dist}(x, \overline{\Omega}) < \varepsilon\}$. Since $\partial\Omega = \partial\overline{\Omega}$ and $\overline{\Omega}$ is compact, we have for small $\varepsilon > 0$ that $\Lambda_\infty(\Omega_\varepsilon) > \Lambda$. Now let Ω_Λ be the domain obtained by connecting Ω_ε to a ball of radius $1/\Lambda$ with a sufficiently narrow tube. For this new domain the reciprocal of the maximum of the distance from the boundary is now Λ and also $\overline{\Omega} \subset \Omega_\Lambda$. Consider a genuine ∞ -eigenfunction of Ω_Λ , say u_Λ . Both Cu_Λ and u are solutions to the same equation in Ω . The comparison principle (2.3) can be used in this situation, since u_Λ is positive on $\partial\Omega$. It gives

$$\log u(x) \leq \log Cu_\Lambda(x)$$

for $x \in \Omega$. We arrive at a contradiction by letting $C \rightarrow 0^+$. \square

3.4. Remark. It is quite easy to give an example of a domain Ω and a number $0 < \Lambda < \Lambda_\infty$ for which the above argument cannot be applied. Nevertheless we think that the result itself is true even without the assumption $\partial\Omega = \partial\overline{\Omega}$.

§4. Examples

We now use the limit equation (1.22) to conclude that the distance function

$$\delta(x, y) = \frac{1 - (|x| + |y|)}{\sqrt{2}}$$

is not a genuine ∞ -eigenfunction of the square

$$\Omega = \{(x, y) : |x| + |y| < 1\}$$

centered at the origin. In other words, $\delta(x, y)$ is not the limit of eigenfunctions $u_p(x, y)$ as $p \rightarrow \infty$. Note that $\Lambda_\infty = \sqrt{2}$. The ridge set of Ω (the set of points at which δ is not in C^1) consists of the intersection of Ω with the coordinate axes.

4.1. Proposition. *Along the ridge of Ω the distance function $\delta(x, y)$ is not a viscosity subsolution of*

$$(4.2) \quad \min \{ |\nabla u| - \sqrt{2}u, -\Delta_\infty u \} = 0.$$

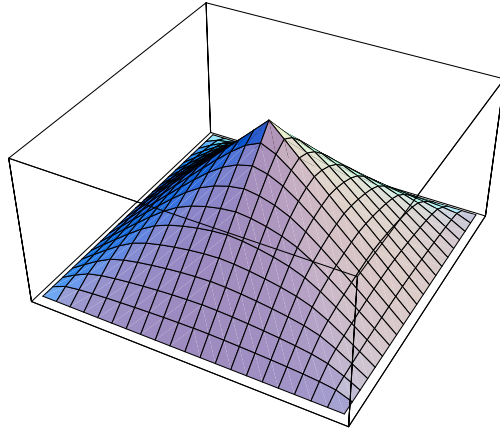


Fig. 4.1.

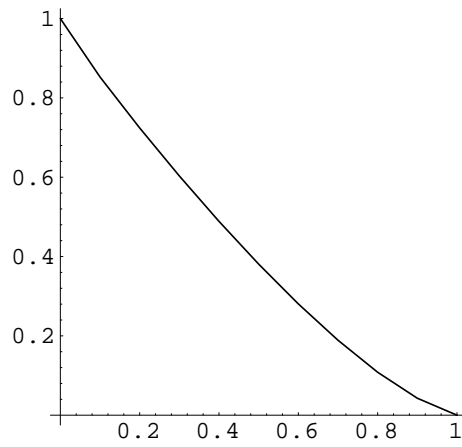


Fig. 4.2.

Proof. Select a point in the ridge, for example, the point $(0, \frac{1}{2})$. We will exhibit a C^2 function $\phi(x, y)$ satisfying

$$(4.3) \quad \delta(0, \frac{1}{2}) = \phi(0, \frac{1}{2}) = \frac{1}{2\sqrt{2}},$$

$$(4.4) \quad \delta(x, y) < \phi(x, y) \text{ in a neighborhood of } (0, \frac{1}{2}),$$

$$(4.5) \quad \min \{ |\nabla\phi(0, \frac{1}{2})| - \sqrt{2}\phi(0, \frac{1}{2}), -\Delta_\infty\phi(0, \frac{1}{2}) \} > 0.$$

This shows that $\delta(x, y)$ cannot be a subsolution of $\min \{ |\nabla u| - \sqrt{2}u, -\Delta_\infty u \} = 0$. To find this ϕ start out with

$$\phi_0(x, y) = \frac{1}{2\sqrt{2}} + ax - \frac{1}{\sqrt{2}}\left(y - \frac{1}{2}\right) + bx^2 + cx\left(y - \frac{1}{2}\right) + d\left(y - \frac{1}{2}\right)^2$$

and require that

$$\frac{1 - (|x| + |y|)}{\sqrt{2}} < \phi_0(x, y)$$

in a neighborhood of $(0, \frac{1}{2})$. Elementary considerations show that the choice $a = 1/2\sqrt{2}$, $b = -1$, $c = 0$ and $d = 0$ gives us a function ϕ_0 satisfying (4.3), (4.4) with “ \leq ” instead of “ $<$ ”, and (4.5). To get the strict inequality just consider $\phi(x, y) = \phi_0(x, y) + x^4 + (y - \frac{1}{2})^4$. \square

In the case of the square one can prove uniqueness of smooth (C^1) genuine ∞ -eigenfunctions. Although we believe that solutions are indeed of class C^1 off the center of the square, we have not yet been able to prove it.

Normalizing u_∞ so that $u_\infty(0, 0) = \delta(0, 0) = 1/\sqrt{2}$ we have

$$\frac{1}{\sqrt{2}} - \sqrt{x^2 + y^2} \leq u_\infty(x, y) \leq \delta(x, y)$$

by comparison. The lower bound is the distance to the largest inscribed circle. On the lines $x = \pm y$ we have equality $u_\infty = \delta$. This shows that u_∞ cannot be a concave function. However, $\log u_\infty$ is concave; cf. [S]. The graph of the solution on a square is shown in Figure 4.1 and the graph of the diagonal cross-section, showing the lack of concavity, is shown in Figure 4.2.

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Department of Mathematics
University of Jyväskylä
40351 Jyväskylä, Finland

Department of Mathematics
Norwegian Institute of Technology
Trondheim N-7034, Norway

and

Department of Mathematics
University of Pittsburgh
Pittsburgh, Pennsylvania 15260

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