LARGE SOLUTIONS FOR THE INFINITY LAPLACIAN

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Abstract. In this paper, we study existence, uniqueness and asymptotic behavior near the boundary of solutions to

$$\Delta_\infty u = \left(D^2u(x)\frac{Du(x)}{|Du(x)|}\right)\cdot\frac{Du(x)}{|Du(x)|} = u^q$$

in Ω with an explosive boundary condition $u(x)\to+\infty$ as $x\to\partial\Omega$. We find that there exists a solution if and only if $q>1$. Moreover, when the domain Ω is sufficiently regular, such a solution is unique and verifies

$$u(x)\sim\left(\frac{2(q+1)}{(q-1)^2}\right)^{\frac{1}{q-1}}\operatorname{dist}(x,\partial\Omega)^{-\frac{2}{q-1}}$$

as $\operatorname{dist}(x,\partial\Omega)\to 0$.

1. Introduction

In this paper we study existence, uniqueness and asymptotic behavior near the boundary for solutions to the following problem:

$$\begin{cases} \Delta_\infty u = u^q & \text{in } \Omega, \\ \lim\limits_{x\to z} u(x) = +\infty & \text{for all } z\in\partial\Omega. \end{cases} \tag{1.1}$$

Here

$$\Delta_\infty u(x) := \left(D^2u(x)\frac{Du(x)}{|Du(x)|}\right)\cdot\frac{Du(x)}{|Du(x)|} \tag{1.2}$$

is the (1-homogeneous) infinity Laplace operator, $q>0$ and Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 1$.

By a solution of (1.1) we understand a non-negative function $u\in C(\Omega)$ verifying the equation in the viscosity sense (see Section 2 for the precise definition) and the boundary condition in the classical sense, that is, $u(x)\to\infty$ as $x\to z\in\partial\Omega$.

The solutions to problem (1.1) are known as "large" solutions due to the explosive boundary condition. In fact, a motivation for the name

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is as follows: if u is a large solution, then, since there is a comparison principle for the equation (see Section 3), any solution to $\Delta_\infty v = v^q$ in Ω verifies $v(x) \le u(x)$. Hence, the large solutions provide uniform bounds for all other solutions in Ω, regardless of the boundary data.

We refer to the pioneering papers [7], [21] and [22], and to the survey [24] for an extensive list of references on the subject of large solutions.

The infinity Laplacian (1.2) in turn is a very topical differential operator that appears in many contexts. For example, the infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) appear naturally as limits of p-harmonic functions (solutions to $-\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = 0$) and have applications to optimal transport problems, image processing, etc. See [2], [6], [14] and references therein. Moreover, the infinity Laplacian plays a fundamental role in the calculus of variations of L^∞ functionals, see e.g. [1], [4], [5], [8], [10], [11], [18], [23], [25] and the survey [2]. Note that this elliptic operator is not in divergence form and is non-degenerate only in the direction of the gradient.

As a motivation, besides its own interest, to our study of large solutions, we recall that the study of the infinity Poisson equation $-\Delta_\infty u(x) = f(x)$ has been recently initiated by Peres, Schramm, Sheffield and Wilson in [23] (see also [4]) via a game-theoretic interpretation of the equation. Since, as we have mentioned, the large solutions provide uniform bounds for any solution in Ω, regardless of the boundary data, properties of large solutions (like its asymptotic behavior near the boundary) could be exploited in the study of the more general problems of the form $-\Delta_\infty u(x) = f(x, u(x))$, which seems like the next logical thing to do.

Large solutions to the p-Laplacian, $\Delta_p u = u^q$, were obtained in [13] for $q > p$ and their limit as p (and hence also q) goes to infinity was analyzed in [15]. It was shown in [15] that the limit equation for large solutions to $\Delta_p u = u^q$ is $\max\{-\Delta_\infty u,\ -|Du| + u^Q\} = 0$, where $Q = \lim \frac{q}{p}$. Note that our problem (1.1) is *not* obtained as a limit of p-Laplacian type equations.

Now we state our main result:

Theorem 1.1. *There exists a finite viscosity solution to* (1.1) *if and only if* $q > 1$. *Any such solution is positive.*

Moreover, if there exists a neighborhood N *of* $\partial\Omega$ *such that* $\mathrm{dist}(x, \partial\Omega) \in C^1(N \cap \Omega)$, *then the solution is unique and it verifies the precise boundary behavior*

$$u(x) \sim \left(\frac{2(q+1)}{(q-1)^2} \right)^{\frac{1}{q-1}} \mathrm{dist}(x, \partial\Omega)^{-\frac{2}{q-1}} \qquad \textit{as} \ \ \mathrm{dist}(x, \partial\Omega) \to 0.$$

Let us next comment briefly on the ideas and methods used in the proofs. The main idea behind the proof of existence is to take advantage of the comparison principle that holds for viscosity solutions of $\Delta_\infty u = u^q$ and perform the usual approximation technique that consists of solving the problem with $u = M$ as boundary condition and then taking the limit of these solutions as $M \to \infty$. To conclude that the limit is finite, we use again comparison with a radial large solution obtained by analyzing the corresponding ODE, a task that turns out be somewhat subtle. The positivity of solutions to (1.1) for $q > 1$ follows from the strong minimum principle, see [3]; here the important fact is that q is larger than the degree of homogeneity of (1.2).

We have two different proofs for the nonexistence of solution when $q \le 1$. The first one is simpler and uses only ODE arguments for radial solutions in a ball. The second one is more involved but contains an application of a new tool in the theory of the infinity Laplacian, the comparison with quadratic functions introduced in [23]. In passing, we also show that (1.1) has no positive solutions for any $q \le 0$.

The estimates for the asymptotic behavior near the boundary are obtained by using again comparison arguments with suitable super and subsolutions. The lower bound,

$$u(x) \ge \left(\frac{2(q+1)}{(q-1)^2} \right)^{\frac{1}{q-1}} \operatorname{dist}(x, \partial\Omega)^{-\frac{2}{q-1}}, \qquad x \in \Omega,$$

is valid in any domain, whereas in the proof of the corresponding upper bound we exploit the fact that $x \mapsto \operatorname{dist}(x, \partial\Omega)$ is a solution of $\Delta_\infty v = 0$ near $\partial\Omega$ if the regularity assumption of Theorem 1.1 holds. Using these estimates and the comparison principle we can obtain uniqueness of solutions. We would like to emphasize again that to obtain existence in Theorem 1.1 we do not assume anything about the regularity of the domain Ω, but in order to get the precise asymptotic behavior (and to obtain uniqueness) we need to assume some smoothness.

If one looks carefully at our proofs it can be checked that our existence/nonexistence results for large solutions can be generalized for the equation $\Delta_\infty u = f(u)$ where f is increasing for large values of u. In this case we obtain that the condition that is necessary and sufficient for existence of solutions is the well known Keller-Osserman condition,

$$\int_K^\infty \frac{ds}{\sqrt{F(s)}} < +\infty$$

where $F(u) = \int_0^u f(s)\, ds$ is a primitive of f. Indeed, it is known that this condition is necessary and sufficient for the existence of large solutions to $u''(r) = f(u(r))$ in an interval (see [21] and [22]). Since our arguments rely on comparison with radial solutions in a ball, they can be carried over without significant changes. We leave the details to the reader.

The rest of the paper is organized as follows: in Section 2 we gather some definitions concerning viscosity solutions to $\Delta_\infty u = u^q$; in Section 3 we prove a comparison principle for this equation; in Section 4 we obtain existence of solutions to (1.1) when $q > 1$; in Section 5 we show nonexistence for $q \le 1$ and finally in Section 6 we analyze the behavior near the boundary and use it to obtain uniqueness of solutions to (1.1).

2. Definition of viscosity solutions

We need to recall here the precise definition of a viscosity solution to our problem.

Due to the fact that (1.2) is singular at the points where the gradient vanishes, we have to use the semicontinuous extensions of the function $(\xi, X) \mapsto (X\frac{\xi}{|\xi|})\cdot\frac{\xi}{|\xi|}$ when defining the viscosity solutions of (1.1). This is a standard procedure, see e.g. [9], that coheres, for example, with the useful stability properties of viscosity solutions. To this end, for a symmetric $n\times n$-matrix A, we denote its largest and smallest eigenvalue by $M(A)$ and $m(A)$, respectively. That is,

$$M(A) = \max_{|\eta|=1} (A\eta)\cdot\eta = \limsup_{\substack{B\to A \\ \xi\to 0,\,\xi\neq 0}} \left(B\frac{\xi}{|\xi|}\right)\cdot\frac{\xi}{|\xi|}$$

and

$$m(A) = \min_{|\eta|=1} (A\eta)\cdot\eta = \liminf_{\substack{B\to A \\ \xi\to 0,\,\xi\neq 0}} \left(B\frac{\xi}{|\xi|}\right)\cdot\frac{\xi}{|\xi|}.$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A non-negative upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution* of (1.1) in Ω if, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $0 = u(\hat{x}) - \varphi(\hat{x}) > u(x) - \varphi(x)$ for all $x \neq \hat{x}$ then

$$\begin{cases} \Delta_\infty\varphi(\hat{x}) \ge \varphi(\hat{x})^q & \text{if } D\varphi(\hat{x}) \neq 0, \\ M(D^2\varphi(\hat{x})) \ge \varphi(\hat{x})^q & \text{if } D\varphi(\hat{x}) = 0. \end{cases} \tag{2.1}$$

A non-negative lower semicontinuous function $v : \Omega \to \mathbb{R}$ is a *viscosity supersolution* of (1.1) in Ω if, whenever $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $0 = v(\hat{x}) - \varphi(\hat{x}) < v(x) - \varphi(x)$ for all $x \neq \hat{x}$ then

$$\begin{cases} \Delta_\infty\varphi(\hat{x}) \le \varphi(\hat{x})^q & \text{if } D\varphi(\hat{x}) \neq 0, \\ m(D^2\varphi(\hat{x})) \le \varphi(\hat{x})^q & \text{if } D\varphi(\hat{x}) = 0. \end{cases} \tag{2.2}$$

Finally, a continuous function $h : \Omega \to \mathbb{R}$ is a *viscosity solution* of (1.1) in Ω if it is both a viscosity subsolution and a viscosity supersolution.

Observe that if $u \in C^2(\Omega)$, then it is a viscosity solution to (1.1) if and only if $\Delta_\infty u = u^q$ in $\Omega \cap \{Du \neq 0\}$ and $m(D^2u) \leq u^q \leq M(D^2u)$ in $\Omega \cap \{Du = 0\}$.

3. Comparison principles

In this section, we prove the comparison results needed later on. Although the operator $-\Delta_\infty u + u^q$ is degenerate elliptic and strictly increasing in the "u variable" for $q > 0$, the general comparison result stated in [12, Theorem 3.3] does not apply as such because of the singularity at the points where the gradient vanishes and the explosive boundary condition. For the reader's convenience, we provide full details of the needed extension.

Theorem 3.1. *Let $w \geq 0$ and $v \geq 0$ be a subsolution and a supersolution to $\Delta_\infty u = u^q$, $q \geq 1$, in Ω, respectively, and suppose that*

$$\limsup \frac{w(x)}{v(x)} \leq 1 \qquad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0. \tag{3.1}$$

Then $w \leq v$ in Ω.

Remark 3.2. In precise terms the assumption (3.1) means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{w(x)}{v(x)} \leq 1 + \varepsilon \qquad \text{for all } x \in \Omega \text{ for which } \operatorname{dist}(x, \partial\Omega) < \delta.$$

Moreover, (3.1) implicitly contains the assumption that $v > 0$ near $\partial\Omega$ so as to make the ratio $\frac{w}{v}$ well-defined.

Proof. Suppose that the claim does not hold and $\sup_\Omega (w - v) > 0$. Then, owing to (3.1), there exists $\varepsilon > 0$ and $x_0 \in \Omega$ such that

$$\sup_{x \in \Omega} \Big(w(x) - (1+\varepsilon)v(x) \Big) = w(x_0) - (1+\varepsilon)v(x_0) > 0. \tag{3.2}$$

Moreover, there is an open set V such that $x_0 \in V$, $\overline{V} \subset \Omega$ and

$$\sup_V \Big(w - (1+\varepsilon)v \Big) > \sup_{\partial V} \Big(w - (1+\varepsilon)v \Big). \tag{3.3}$$

Observe that since $q \geq 1$, we have for $v_\varepsilon = (1+\varepsilon)v$ that

$$\Delta_\infty v_\varepsilon = (1+\varepsilon)\Delta_\infty v \leq (1+\varepsilon)v^q = (1+\varepsilon)^{1-q} v_\varepsilon^q \leq v_\varepsilon^q,$$

that is, v_ε is a viscosity supersolution to $\Delta_\infty u = u^q$ as well.

Consider the functions

$$f_j(x,y) = w(x) - v_\varepsilon(y) - \Psi_j(x,y), \quad j = 1, 2, \ldots,$$

where

$$\Psi_j(x,y) = \tfrac{j}{4}|x-y|^4,$$

and let (x_j, y_j) be a maximum point of f_j relative to $\overline{V} \times \overline{V}$. By (3.3) and [12, Proposition 3.7], we see that for j sufficiently large, (x_j, y_j) is an interior point of $V \times V$ and that we may assume $(x_j, y_j) \to (x_0, x_0)$ as $j \to \infty$.

Let us first show that $x_j \neq y_j$ for j large enough. To this end, since

$$w(x) - v_\varepsilon(y) - \Psi_j(x, y) \leq w(x_j) - v_\varepsilon(y_j) - \Psi_j(x_j, y_j)$$

for all $x, y \in V$, we obtain by choosing $y = y_j$ and

$$\phi_j(x) = w(x_j) - \Psi_j(x_j, y_j) + \Psi_j(x, y_j)$$

that $w(x) \leq \phi_j(x)$ for all $x \in V$. Since $w(x_j) = \phi_j(x_j)$ and w is a subsolution, this implies

$$\Delta_\infty \phi_j(x_j) \geq \phi_j^q(x_j),$$

if $D\phi_j(x_j) \neq 0$, and

$$M(D^2\phi_j(x_j)) \geq \phi_j^q(x_j)$$

if $D\phi_j(x_j) = 0$. If $x_j = y_j$, then $D\phi_j(x_j) = 0$, $D^2\phi_j(x_j) = 0$, and the second alternative gives

$$0 = M(D^2\phi_j(x_j)) \geq \phi_j^q(x_j) = w^q(x_j) > 0,$$

which is a contradiction. Here the last inequality follows from

$$\begin{aligned} w(x_j) &\geq w(x_j) - v_\varepsilon(y_j) \geq w(x_j) - v_\varepsilon(y_j) - \Psi_j(x_j, y_j) \\ &\geq w(x_0) - v_\varepsilon(x_0) > 0, \end{aligned} \tag{3.4}$$

where the definition of (x_j, y_j) and (3.2) were used.

The rest of the proof is now a fairly standard application of the maximum principle for semicontinuous functions, see e.g. [12]. Since (x_j, y_j) is a local maximum point of $f_j(x, y)$, we conclude that there exist symmetric $n \times n$ matrices X_j, Y_j, $X_j \leq Y_j$, such that

$$(\eta_j, X_j) \in \overline{J}^{2,+} w(x_j), \qquad (\eta_j, Y_j) \in \overline{J}^{2,-} v_\varepsilon(y_j).$$

Here

$$\eta_j := D_x\Psi_j(x_j, y_j) = -D_y\Psi_j(x_j, y_j) = j|x_j - y_j|^2(x_j - y_j) \neq 0$$

since $x_j \neq y_j$, and $\overline{J}^{2,+} w(x_j)$ and $\overline{J}^{2,-} v_\varepsilon(y_j)$ denote the closures of the second order superjet of w at x_j and the second order subjet of v_ε at y_j, respectively. By the definition and the properties of these jets and their closures, see e.g. [12], there exist $z_{j,k} \to x_j$, $\eta_{j,k} \to \eta_j$ and $X_{j,k} \to X_j$ such that $w(z_{j,k}) \to w(x_j)$ and functions $\varphi_{j,k} \in C^2(V)$ satisfying

$$\begin{aligned} \varphi_{j,k}(x) = &\, w(z_{j,k}) + \eta_{j,k} \cdot (x - z_{j,k}) + \tfrac{1}{2} X_{j,k}(x - z_{j,k}) \cdot (x - z_{j,k}) \\ &+ o(|x - z_{j,k}|^2) \end{aligned}$$

and

$$0 = w(z_{j,k}) - \varphi_{j,k}(z_{j,k}) \geq w(x) - \varphi_{j,k}(x) \qquad \text{for } x \in V.$$

Since w is a subsolution and $\eta_{j,k} \neq 0$ for k large, this implies

$$X_{j,k}\frac{\eta_{j,k}}{|\eta_{j,k}|} \cdot \frac{\eta_{j,k}}{|\eta_{j,k}|} \geq w(z_{j,k})^q,$$

and hence, taking limit as $k \to \infty$ and using $w(z_{j,k}) \to w(x_j)$,

$$X_j\frac{\eta_j}{|\eta_j|} \cdot \frac{\eta_j}{|\eta_j|} \geq w(x_j)^q.$$

Similarly, since v_ε is a supersolution and $(\eta_j, Y_j) \in \overline{J}^{2,-}v_\varepsilon(y_j)$, we have

$$Y_j\frac{\eta_j}{|\eta_j|} \cdot \frac{\eta_j}{|\eta_j|} \leq v_\varepsilon(y_j)^q.$$

Combining these inequalities with $X_j \leq Y_j$ yields

$$w^q(x_j) - v_\varepsilon^q(y_j) \leq 0.$$

However, from (3.4) we infer that

$$w(x_j) - v_\varepsilon(y_j) \geq w(x_0) - v_\varepsilon(x_0) > 0,$$

a contradiction. □

In the preceding proof, the assumption $q \geq 1$ was used to guarantee that $(1+\varepsilon)v$ is a supersolution if v is, and for the rest of the argument to work it is enough that $q > 0$ (or $q = 0$ and both functions w and v are positive). Hence we obtain the following comparison result that will be needed to prove nonexistence of solutions to (1.1) for $0 < q \leq 1$.

Theorem 3.3. *Let $w\colon \overline{\Omega} \to [0,\infty)$ be upper semicontinuous in $\overline{\Omega}$ and $v\colon \overline{\Omega} \to [0,\infty)$ lower semicontinuous in $\overline{\Omega}$. If w is a subsolution and v a supersolution to $\Delta_\infty u = u^q$, with $q > 0$, in Ω and $w \leq v$ on $\partial\Omega$, then $w \leq v$ in Ω.*

4. Existence for $q > 1$.

In this section, we prove the existence of solutions to (1.1) for $q > 1$. First, we analyze the one-dimensional case in which we are looking just to an ODE problem. Next, we deal with radial solution on a ball (which gives again an ODE problem) and finally we tackle the problem in a general domain.

4.1. **One dimensional case.**

Let us look for a solution to (1.1) on an open interval $(-R, R)$, $R > 0$. To this end, we need to solve the ordinary differential equation

$$u''(r) = u^q(r), \tag{4.1}$$

and choose as the initial condition

$$u(0) = K, \qquad u'(0) = 0. \tag{4.2}$$

Here $K > 0$, and our aim is to show that it can be chosen so that

$$\lim_{r\nearrow R} u(r) = +\infty, \qquad \lim_{r\searrow -R} u(r) = +\infty. \tag{4.3}$$

The necessarily even solution to the initial value problem above is given implicitly by

$$\int_K^{u(r)} \frac{ds}{\sqrt{s^{q+1} - K^{q+1}}} = \sqrt{\frac{2}{q+1}}|r| \tag{4.4}$$

Notice that for $K = 0$ the integral on the left diverges unless $u \equiv 0$; this is in accordance with the fact that the only solution to (4.1), (4.2) with $K = 0$ is $u \equiv 0$. For $K > 0$, one can apply the change of variables $z^{q+1} = s^{q+1} - K^{q+1}$ to obtain

$$\int_K^{u} \frac{ds}{\sqrt{s^{q+1} - K^{q+1}}} = \int_0^v \frac{z^{\frac{q-1}{2}}\,dz}{(z^{q+1} + K^{q+1})^{\frac{q}{q+1}}},$$

where $v = (u^{q+1} - K^{q+1})^{\frac{1}{q+1}}$. Since $\frac{q-1}{2} > 0$, this integral is finite for any $v > 0$, and thus for any $u > K$. On the other hand,

$$\begin{aligned}\lim_{v\to\infty}\left(\int_0^v \frac{z^{\frac{q-1}{2}}\,dz}{(z^{q+1} + K^{q+1})^{\frac{q}{q+1}}}\right) &\le \int_0^K K^{-q} z^{\frac{q-1}{2}}\,dz + \lim_{v\to\infty}\int_K^v \frac{dz}{z^{\frac{q+1}{2}}} \\ &= \frac{4q}{q^2-1} K^{\frac{1-q}{2}}\end{aligned}$$

and

$$\lim_{v\to\infty}\left(\int_0^v \frac{z^{\frac{q-1}{2}}\,dz}{(z^{q+1} + K^{q+1})^{\frac{q}{q+1}}}\right) \ge \int_0^K \frac{z^{\frac{q-1}{2}}}{2^{\frac{q}{q+1}} K^q}\,dz = \frac{2^{\frac{1}{q+1}}}{q+1} K^{\frac{1-q}{2}}.$$

By continuity, this means that for any fixed $K > 0$ the function

$$u \mapsto \int_K^{u(r)} \frac{ds}{\sqrt{s^{q+1} - K^{q+1}}}$$

is a bijection from $[K, \infty)$ to $[0, l_K)$ for some constant $l_K > 0$ that tends to 0 as $K \to \infty$, and tends to ∞ as $K \to 0$. Thus by choosing K so that $l_K = \sqrt{\frac{2}{q+1}}R$, we can see that the function u defined by the integral (4.4) satisfies the asserted boundary conditions (4.3).

Remark 4.1. The reasoning above implies that there are constants $a, b > 0$ depending only on q such that

$$aK^{\frac{1-q}{2}} \le \int_K^{\infty} \frac{ds}{\sqrt{s^{q+1} - K^{q+1}}} \le bK^{\frac{1-q}{2}},$$

that is, $aK^{\frac{1-q}{2}} \le l_K \le bK^{\frac{1-q}{2}}$. In particular, since $u(0) = K$ and $l_K = \sqrt{\frac{2}{q+1}}R$, there are $\alpha, \beta > 0$ depending only on q such that

$$\alpha R^{-\frac{2}{q-1}} \le u(0) \le \beta R^{-\frac{2}{q-1}}. \tag{4.5}$$

This implies that the solution u we have obtained satisfies

$$\alpha(R-|r|)^{-\frac{2}{q-1}} \le u(r) \le \beta(R-|r|)^{-\frac{2}{q-1}} \tag{4.6}$$

for all $r \in (-R,R)$. Indeed, if $0<r<R$ and v_r is the solution to (4.1) on $(r-\rho_\varepsilon, r+\rho_\varepsilon) \subset (-R,R)$ with $\rho_\varepsilon = R-r-\varepsilon$ for $\varepsilon>0$ small enough, that verifies

$$\lim_{s\to r\pm\rho_\varepsilon} v(s) = \infty,$$

then by the comparison principle $v_r \ge u$ in the interval $(r-\rho_\varepsilon, r+\rho_\varepsilon)$. In particular, owing to (4.5),

$$u(r) \le v_r(r) \le \beta\rho_\varepsilon^{-\frac{2}{q-1}},$$

which yields the upper bound upon letting $\varepsilon \to 0$. The estimate for the case $-R<r<0$ also follows from this because u is even. As regards the lower bound, see Remark 4.2 below.

4.2. **Radial case.** Let $\Omega = B_R = B_R(0)$ and let

$$u : (-R,R) \to [K_R, \infty)$$

be the solution to

$$\begin{cases} u''(r) = u^q(r) & \text{in } (-R,R), \\ u(0) = K_R, \\ u'(0) = 0, \end{cases}$$

with $\lim_{r\to\pm R} u(r) = +\infty$ that we have constructed above.

We claim that then the function $U : B_R \to [K_R,\infty)$, $U(x) = u(|x|)$ is a (positive) radial viscosity solution to (1.1) in B_R. Indeed, since $Du(x) = u'(|x|)\frac{x}{|x|}$ and

$$D^2U(x) = u''(|x|)\frac{x}{|x|}\otimes\frac{x}{|x|} + \frac{u'(|x|)}{|x|}\Big(I - \frac{x}{|x|}\otimes\frac{x}{|x|}\Big)$$

for $x \neq 0$, it is easy to verify that $\Delta_\infty U(x) = u''(|x|) = U^q(x)$ outside the origin. Moreover, since $u'(s) = K_R^q s + o(s)$, we see that $U \in C^2(B_R)$ and $D^2U(0) = K_R^q \cdot I$. Hence

$$M(D^2U(0)) = m(D^2U(0)) = K_R^q = U^q(0),$$

which implies that U is a viscosity solution to (1.1) also at the origin.

Remark 4.2. If we seek a solution to (1.1) in the form $u(x) = g(R-|x|)$, then again the equation $\Delta_\infty u = u^q$ formally reduces to $g'' = g^q$. Solving this ordinary differential equation yields a function

$$u_{R,q}(x) := \left(\frac{2(q+1)}{(q-1)^2}\right)^{\frac{1}{q-1}} (R-|x|)^{-\frac{2}{q-1}},$$

and since $-\frac{2}{q-1} < 0$, it satisfies $\lim_{|x|\to R} u_{R,q}(x) = \infty$. However, although this function is smooth outside the origin, and thus a classical solution to $\Delta_\infty u = u^q$ in $B_R \setminus \{0\}$, it is not differentiable at $x = 0$. In fact, there are no test-functions touching $u_{R,q}$ from above at the origin, which readily implies that $u_{R,q}$ is a viscosity subsolution of (1.1) in B_R. But it is *not* a viscosity supersolution at the origin; this follows from the fact that we can test from below by test functions with small but non-zero gradient and any arbitrary Hessian.

Observe that if u is any solution to (1.1) in B_R, then $u_{R,q} \le u$. Indeed, $u_{R+\varepsilon,q} \le u$ in B_R for any $\varepsilon > 0$ by the comparison principle, Theorem 3.1, and clearly $u_{R+\varepsilon,q} \to u_{R,q}$ locally uniformly as $\varepsilon \to 0$.

4.3. **Existence in a general bounded domain of $\mathbb{R}^n$.** First, we prove existence and uniqueness of solutions with $u = M$ as boundary datum.

Lemma 4.3. *For each $M > 0$ there exists a unique, non-negative viscosity solution $u_M \in C(\overline{\Omega})$ to*

$$\begin{cases} \Delta_\infty u = u^q & \text{in } \Omega, \\ u(x) = M & \text{for all } x \in \partial\Omega. \end{cases} \tag{4.7}$$

Proof. The uniqueness is a direct consequence of the comparison principle, Theorem 3.3. The existence in turn can be obtained by using the standard Perron's method. To this end, it suffices to find a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (4.7) such that $\underline{u} = \overline{u} = M$ on $\partial\Omega$.

The supersolution is easy to find since we may take $\overline{u} \equiv M$. On the other hand, if for $C \ge 1$ and $z \in \partial\Omega$ we set $v_z(x) = M - C|x - z|^{1/2}$, then there is $\delta > 0$, independent of $C \ge 1$ and z, such that

$$\Delta_\infty v_z(x) = \frac{C}{4}|x - z|^{-3/2} \ge M^q \ge v_z^q(x)$$

for all $x \in B_{2\delta}(z)$. By choosing C so large that $v_z \le 0$ outside $B_\delta(z)$, it follows that

$$\underline{u}(x) = \max\left\{0, \sup_{z \in \partial\Omega}\left(M - C|x - z|^{1/2}\right)\right\}$$

is a viscosity subsolution of $\Delta_\infty u = u^q$ in Ω and $\underline{u} = M$ on $\partial\Omega$. □

Remark 4.4. Observe that Lemma 4.3 holds for any $q > 0$, not just for $q > 1$. This fact will be used in the proof of non-existence of solutions to (1.1) for $0 < q \le 1$.

In view of the comparison principle, the sequence (u_M) obtained in Lemma 4.7 is increasing, and hence the limit function

$$u_\infty \colon \overline{\Omega} \to [0, \infty], \qquad u_\infty(x) := \lim_{M \to \infty} u_M(x)$$

exists. Next we show that it is finite in Ω and gives the desired solution to (1.1).

Lemma 4.5. *The function u_∞ defined above is a solution to* (1.1)*.*

Proof. We show first that u_∞ is locally bounded in Ω. To this end, let us fix $x_0 \in \Omega$ and $r > 0$ so that $B_r(x_0) \subset \Omega$. Let U_r be the radial function satisfying

$$\begin{cases} \Delta_\infty U_r = U_r^q & \text{in } B_r(x_0), \\ \lim_{x\to z} U_r(x) = +\infty & \text{for all } z \in \partial B_r(x_0), \end{cases}$$

constructed in Section 4.2 above. By the comparison principle we obtain $u_M \le U_r$ in $B_r(x_0)$ for every $M > 0$, and so $u_\infty \le U_r$ in $B_r(x_0)$.

Since (u_M) is a locally uniformly bounded sequence of non-negative viscosity subsolutions of the infinity Laplace equation $-\Delta_\infty u = 0$, it follows from e.g. Lemma 2.9 of [2] that (u_M) is also equicontinuous. Hence, due to monotonicity, $u_M \to u_\infty$ locally uniformly as $M \to \infty$, which readily implies that $\Delta_\infty u_\infty = u_\infty^q$ in Ω, see e.g. [9].

In order to show that $\lim_{x\to z} u_\infty(x) = \infty$ for all $z \in \Omega$, we introduce the barrier functions

$$w(x) = A_q\Big(|x-z| + \varepsilon\Big)^{-\frac{2}{q-1}}.$$

Here $z \in \partial\Omega$, $\varepsilon > 0$, and $A_q = \left(\frac{2(q+1)}{(q-1)^2}\right)^{\frac{1}{q-1}}$. It is easy to verify that $\Delta_\infty w = w^q$ in $\mathbb{R}^n \setminus \{z\}$, and therefore $u_M \ge w$ in Ω for all $M \ge A_q \varepsilon^{-\frac{2}{q-1}}$ by comparison principle. Letting first $M \to \infty$ and then $\varepsilon \to 0$ we conclude that

$$u_\infty(x) \ge A_q |x-z|^{-\frac{2}{q-1}} \tag{4.8}$$

for all $x \in \Omega$ and for any fixed $z \in \partial\Omega$. □

5. Non-existence for $q \le 1$.

Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. The goal in this section is to show that if $0 < q \le 1$, then (1.1) has no solution.

We present two different proofs of this fact. The first one is elementary and uses only simple ODE methods. The second one is more involved and uses the "quadratic comparison with cones", introduced in [23].

5.1. First proof.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $0 < q \le 1$. Then problem* (1.1) *has no positive solution.*

Proof. We argue by contradiction. Assume that there is a solution in a bounded domain Ω. We may assume that $0 \in \Omega$. Fix R large such that $\Omega \subset B_R(0)$.

Let $u(x)$ be the large solution in Ω. Then, by comparison, we have

$$u(x) \geq v_M(x) \qquad \text{for all } x \in \Omega,$$

where v_M is the unique solution of the equation $\Delta_\infty \phi = \phi^q$ in the ball $B_R(0)$ with $v_M(R) = M$, provided by Lemma 4.3. Therefore, the functions v_M are uniformly bounded in Ω. In particular,

$$v_M(0) \leq u(0) < +\infty.$$

These functions v_M are radial (by uniqueness) and hence solutions to

$$v_M''(r) = v_M^q(r)$$

with

$$(v_M)'(0) = 0, \qquad v_M(0) < \max\{1, u(0)\} + 1 := K.$$

Let $z(r)$ be the solution to

$$z''(r) = z(r)$$

with

$$z'(0) = 0, \qquad z(0) = K > 1.$$

By integration we obtain the explicit solution

$$z(r) = \frac{K}{2}(e^r + e^{-r}).$$

Note that $z(r) > 1$ for every $r \geq 0$, and hence z verifies

$$z''(r) = z(r) \geq z^q(r).$$

That is, z is a subsolution to our equation. Also, we have $v_M(0) < z(0)$.

We claim that

$$v_M(r) < z(r) \qquad \text{for all } r \in [0, R].$$

In order to prove this claim, we argue by contradiction and assume that it is false. Then there exists $R_0 \in (0, R]$ with $v_M(R_0) = z(R_0)$, and we have in $B_{R_0}(0)$ a solution v_M and a subsolution z that agree on the boundary. By the comparison principle this yields $v_M \geq z$, a contradiction with the fact that $v_M(0) < z(0)$. This proves the claim.

Now, we just have to observe that, setting $r = R$ we get,

$$M = v_M(R) < z(R) = \frac{K}{2}(e^R + e^{-R}),$$

a contradiction if M is large enough. $\square$

5.2. **Second proof.** The main tool in our second proof is the "quadratic comparison with cones", introduced in [23]. Let us briefly recall some relevant definitions.

Definition 5.2. Let $Q(r) = ar^2 + br + c$, $a, b, c \in \mathbb{R}$, and for $z \in \mathbb{R}^n$ let $\varphi(x) = Q(|x - z|)$. Then φ is said to be a $\star$-*increasing* quadratic distance function on an open set $V \subset \mathbb{R}^n$ if one of the following two conditions holds:

(1) $z \notin V$ and $Q'(|x - z|) > 0$ for every $x \in V$,
(2) $z \in V$, $b = 0$, and $a > 0$.

The function φ is called $\star$-*decreasing* if $-\varphi$ is $\star$-increasing.

Definition 5.3. Let $g\colon \Omega \to \mathbb{R}$ be a given continuous function. A continuous function $u\colon \Omega \to \mathbb{R}$ satisfies *g-quadratic comparison from above* in Ω if, whenever $V \subset\subset \Omega$ and φ is a $\star$-increasing quadratic distance function in V with $a \le \frac{1}{2} \inf_V g$, the inequality $\varphi \ge u$ on ∂V implies $\varphi \ge u$ in V.

Similarly, a continuous function $u\colon \Omega \to \mathbb{R}$ satisfies *g-quadratic comparison from below* in Ω if, whenever $V \subset\subset \Omega$ and φ is a $\star$-decreasing quadratic distance function in V with $a \ge \frac{1}{2} \sup_V g$, the inequality $\varphi \le u$ on ∂V implies $\varphi \le u$ in V.

The notion of g-quadratic comparison characterizes the viscosity solutions of the inhomogeneous equation $\Delta_\infty u = g$:

Theorem 5.4. [23, Theorem 1.7] *Let $\Omega \subset \mathbb{R}$ be any bounded domain and $g\colon \Omega \to \mathbb{R}$ be a given continuous function. Then $\Delta_\infty u = g$ in Ω in the viscosity sense if and only if u satisfies g-quadratic comparison from above and from below.*

Now we are ready to present our second non-existence proof for $0 < q \le 1$:

Proposition 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $0 < q \le 1$. Then the problem* (1.1) *has no solution.*

Proof. We argue by contradiction and suppose that there is a non-negative function $u \in C(\Omega)$ satisfying (1.1). For $\lambda \ge 1$, denote $\Omega_\lambda := \{x \in \Omega\colon u(x) < \lambda\}$. We fix $z \in \partial\Omega$ and let $\varphi_z(x) = Q(|x - z|)$ be the quadratic distance function where $Q(r) = ar^2 + br + M$,

$$a = \frac{1}{2}\lambda^q, \qquad b = -\lambda^q \mathrm{diam}(\Omega),$$

and M is determined below. Notice that since

$$Q'(|x - z|) = 2a|x - z| + b = \lambda^q \Big(|x - z| - \mathrm{diam}(\Omega) \Big) < 0$$

for all $x \in \Omega_\lambda$, φ_z is $\star$-decreasing in Ω_λ. Next let $\eta := \mathrm{dist}(\partial\Omega, \partial\Omega_\lambda) > 0$ and define M by the relation

$$a\eta^2 + b\eta + M = \lambda.$$

The idea here is to choose M as the largest value for which $\varphi_z(x) \le u(x)$ at the point $x \in \partial\Omega_\lambda$ that is closest to z. Since φ_z is $\star$-decreasing and $u \equiv \lambda$ on $\partial\Omega_\lambda$, this guarantees that $\varphi_z \le u$ on $\partial\Omega_\lambda$.

Now, since $a = \frac{1}{2}\lambda^q = \sup_{\Omega_\lambda} \frac{u^q}{2}$ and $\Delta_\infty u = u^q$, we may use the fact that u satisfies u^q-quadratic comparison from below on Ω and conclude that $\varphi_z \le u$ in Ω_λ. This implies that for any $x \in \Omega_\lambda$, $\lambda \ge 1$, we have

$$\begin{aligned} u(x) &\ge \sup_{z\in\partial\Omega} \varphi_z(x) = a\,\delta(x)^2 + b\,\delta(x) + M \\ &= \lambda + \lambda^q\Big[\frac{1}{2}(\delta(x)^2 - \eta^2) - \mathrm{diam}(\Omega)(\delta(x) - \eta)\Big] \\ &\ge \lambda - (\delta(x) - \eta)\mathrm{diam}(\Omega)\lambda^q \\ &\ge \lambda\Big[1 - \delta(x)\mathrm{diam}(\Omega)\Big], \end{aligned} \tag{5.1}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. If we fix any $x \in \Omega$ so that $\delta(x)\mathrm{diam}(\Omega) < \frac{1}{2}$, then $x \in \Omega_\lambda$ for all λ large enough, and we infer from (5.1) that $u(x) \ge \frac{1}{2}\lambda$. Letting $\lambda \to \infty$ leads to a desired contradiction. □

5.3. **The case $q \le 0$.** So far in this paper we have all the time assumed that $q > 0$. In this subsection, we briefly deal with the case $q \le 0$ and show that there is no positive solution to (1.1).

Proposition 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $q \le 0$. Then the problem* (1.1) *has no (positive) solution.*

Proof. Suppose that there is a positive solution u to (1.1) with $q \le 0$. Then there exists $C_0 > 0$ such that

$$0 \le u^q(x) \le C_0 \qquad \text{for all } x \in \Omega.$$

In particular, we have that $-\Delta_\infty u \ge -C_0$ in Ω in the viscosity sense. Now let $w(x) = L + \frac{C_0}{2}|x|^2$ for $L \in \mathbb{R}$. Then $-\Delta_\infty w = -C_0$ in $\mathbb{R}^n$ and since $\lim_{x\to z} u(x) = +\infty$ for all $z \in \partial\Omega$, we have by the comparison principle (see e.g. [19, Theorem 3.6]) that $w \le u$ in Ω for any $L \in \mathbb{R}$. Letting $L \to \infty$ we obtain that $u \equiv \infty$. □

Remark 5.7. Notice that the argument above does not apply if u is only non-negative, that is, it is allowed to have zeroes in Ω. However, in that case, the equation $\Delta_\infty u = u^q$ becomes "doubly singular", and it is not even completely clear what is the correct definition of a solution.

6. Growth estimates near the boundary and uniqueness

It follows from the proof of Lemma 4.5 that the lower growth estimate (4.8) holds in fact for *all* possible solutions of (1.1), that is, if u is a viscosity solution to (1.1) in a bounded domain $\Omega \subset \mathbb{R}^n$, then

$$u(x) \geq A_q |x-z|^{-\frac{2}{q-1}} \qquad \text{for all } x \in \Omega \text{ and } z \in \partial\Omega. \tag{6.1}$$

Here the constant $A_q = \left(\frac{2(q+1)}{(q-1)^2}\right)^{\frac{1}{q-1}}$ depends only on $q > 1$. As an immediate consequence we have that

$$u(x) \geq A_q \operatorname{dist}(x, \partial\Omega)^{-\frac{2}{q-1}} \qquad \text{for all } x \in \Omega.$$

We now look for the corresponding upper growth estimates and start with the case where the domain Ω is fairly nice.

Lemma 6.1. *Let u be a viscosity solution to* (1.1) *in Ω and assume that there exists a neighborhood N of $\partial\Omega$ such that* $\operatorname{dist}(x, \partial\Omega) \in C^1(N \cap \Omega)$. *Then there are constants $\gamma > 0$ and $\mu > 0$ depending only on the domain Ω such that*

$$u(x) \leq A_q \operatorname{dist}(x, \partial\Omega)^{-\frac{2}{q-1}} + \gamma \tag{6.2}$$

for all $x \in \Omega$ for which $\operatorname{dist}(x, \partial\Omega) < \mu$.

Remark 6.2. In view of Remark 4.1 and the proof below, we actually have that $\gamma \approx \mu^{-\frac{2}{q-1}}$. In particular, there is a constant $\beta > 0$ depending only on q such that $\gamma \leq \beta \mu^{-\frac{2}{q-1}}$.

Proof. Let us denote $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ and let $\mu > 0$ be such that $\Omega_\mu := \{x \in \Omega \colon \delta(x) < \mu\} \subset N \cap \Omega$. Then $|D\delta(x)| = 1$ for all $x \in \Omega_\mu$. Moreover, $-\Delta_\infty \delta(x) = 0$ in the viscosity sense in Ω_μ, see, for example, [2, Example 4.3].

As a first step, we estimate the number $m_\mu := \sup_{\delta(x)=\mu} u(x)$. To this end, fix x_0 such that $\delta(x_0) = \mu$, and for $0 < r < \mu$ let U_r be the radial function satisfying

$$\begin{cases} \Delta_\infty U_r = U_r^q & \text{in } B_r(x_0), \\ \lim\limits_{x \to z} U_r(x) = \infty & \text{for all } z \in \partial B_r(x_0). \end{cases}$$

By the comparison principle, $u(x_0) \leq U_r(x_0)$ for all $r > 0$, and thus

$$m_\mu \leq U_\mu(x_0) =: \gamma.$$

Now for $0 < \varepsilon < \mu$ and $x \in \Omega_\mu \setminus \overline{\Omega}_\varepsilon$, let

$$w_\varepsilon(x) := A_q (\delta(x) - \varepsilon)^\alpha,$$

where $\alpha = -\frac{2}{q-1} < 0$, and $A_q = \left(\frac{2(q+1)}{(q-1)^2}\right)^{1/(q-1)}$. In view of these choices, a formal calculation yields

$$\begin{aligned}\Delta_\infty w_\varepsilon(x) = & \ A_q\alpha(\delta(x)-\varepsilon)^{\alpha-1}\Delta_\infty\delta(x) \\ & +A_q\alpha(\alpha-1)(\delta(x)-\varepsilon)^{\alpha-2}|D\delta(x)|^2 \\ = & \ A_q\alpha(\alpha-1)(\delta(x)-\varepsilon)^{\alpha-2} \\ = & \ w_\varepsilon^q(x).\end{aligned}$$

This can be easily made rigorous (in the viscosity sense) after observing that if $w_\varepsilon - \phi$ has a local minimum (resp. maximum) at $\hat{x} \in \Omega_\mu \setminus \overline{\Omega}_\varepsilon$, then $\delta - \varphi$, where $\varphi(x) = (\frac{1}{A_q}\phi(x))^{1/\alpha} + \varepsilon$, has a local maximum (resp. minimum) at $\hat{x}$, and then recalling that $-\Delta_\infty\delta(x) = 0$ in the viscosity sense in Ω_μ. We leave the details to the reader.

As $\lim_{x\to z} w_\varepsilon(x) = \infty$ for all z for which $\delta(z) = \varepsilon$, we infer from the comparison principle that $w_\varepsilon(x)+\gamma \geq u(x)$ in $\Omega_\mu \setminus \overline{\Omega}_\varepsilon$. Here we use the fact that $\Delta_\infty(w_\varepsilon+\gamma) \leq (w_\varepsilon+\gamma)^q$, which is a consequence of $\Delta_\infty w_\varepsilon = w_\varepsilon$ and $\gamma, q > 0$. The estimate (6.2) follows upon letting $\varepsilon \to 0$. □

Remark 6.3. It is well-known that the assumption on the domain in Lemma 6.1 above holds for all C^2-domains (see e.g. [17]), but it holds also for Lipschitz domains satisfying a uniform interior ball condition, see [20, Remark 3.1]. In fact, the condition $\operatorname{dist}(x, \partial\Omega) \in C^1(N \cap \Omega)$ is equivalent to the geometric condition that for any $x \in N \cap \Omega$ there exists a *unique* $z \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - z|$.

If $\Omega = B_R(x_0)$, then, owing to Remark 4.1, we already have an upper estimate that holds for all points in the domain. This observation yields the following corollary:

Lemma 6.4. *Let Ω be any bounded domain and u be a viscosity solution to* (1.1) *in Ω. Then*

$$u(x) \leq \beta \operatorname{dist}(x, \partial\Omega)^{-\frac{2}{q-1}}$$

for all $x \in \Omega$, where the constant β is from Remark 4.1.

Proof. Fix $x_0 \in \Omega$ and let $r = \operatorname{dist}(x_0, \partial\Omega)$. By comparing u with the radial solution in $B_{r-\varepsilon}(x_0)$ and then letting $\varepsilon \to 0$ we obtain

$$u(x_0) \leq \beta r^{-\frac{2}{q-1}} = \beta \operatorname{dist}(x_0, \partial\Omega)^{-\frac{2}{q-1}}.$$

□

Finally, we just observe that for sufficiently nice domains (see Remark 6.3) these results on the asymptotic behavior of solutions near the boundary imply uniqueness of large solutions to (1.1).

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded domain and assume that there exists a neighborhood N of $\partial\Omega$ such that* $\operatorname{dist}(x, \partial\Omega) \in C^1(N \cap \Omega)$*. Then for $q > 1$ there exists a unique viscosity solution to* (1.1)*.*

Proof. Let u and v be solutions to (1.1), and let $\mu > 0$ be as in Lemma 6.1. Then, by (6.1) and Lemma 6.1,

$$\frac{u(x)}{v(x)} \leq \frac{A_q \operatorname{dist}(x, \partial\Omega)^{-\frac{2}{q-1}} + \gamma}{A_q \operatorname{dist}(x, \partial\Omega)^{-\frac{2}{q-1}}}$$

for all $x \in \Omega$ for which $\operatorname{dist}(x, \partial\Omega) < \mu$. Hence

$$\limsup \frac{u(x)}{v(x)} \leq 1 \qquad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

and then, by Theorem 3.1, we obtain $u(x) \leq v(x)$ in the whole Ω. A symmetric argument shows that $u = v$. □

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