PRINCIPAL EIGENVALUE OF A VERY BADLY DEGENERATE OPERATOR AND APPLICATIONS

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Abstract. In this paper, we define and investigate the properties of the principal eigenvalue of the singular infinity Laplace operator

$$\Delta_\infty u = (D^2 u \frac{Du}{|Du|}) \cdot \frac{Du}{|Du|}.$$

This operator arises from the optimal Lipschitz extension problem and it plays the same fundamental role in the calculus of variations of L^∞ functionals as the usual Laplacian does in the calculus of variations of L^2 functionals. Our approach to the eigenvalue problem is based on the maximum principle and follows the outline of the celebrated work of Berestycki, Nirenberg and Varadhan [5] in the case of uniformly elliptic linear operators. As an application, we obtain existence and uniqueness results for certain related non-homogeneous problems and decay estimates for the solutions of the evolution problem associated to the infinity Laplacian.

1. Introduction

Eigenvalue problems are an integral part of the theory of second order elliptic partial differential equations and appear frequently in various applications. In the most classical case of a linear self-adjoint operator L in divergence form

$$Lu = -\operatorname{div}[A(x)Du + B(x)u] + B(x)\cdot Du + c(x)u,$$

the principal eigenvalue of L, i.e., the least number $\lambda \in \mathbb{R}$ for which the Dirichlet problem

$$\begin{cases} Lu + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a non-trivial solution, can be characterized as the infimum of the associated Rayleigh quotient

$$\frac{\langle Lu, u\rangle}{\langle u, u\rangle} = \frac{\int_\Omega A(x)Du\cdot Du + 2u\,B(x)\cdot Du + c(x)u^2\,dx}{\int_\Omega u^2\,dx}$$

in $W_0^{1,2}(\Omega)\setminus\{0\}$. Moreover, the minimizers of this quotient are precisely the principal eigenfunctions. See e.g. [17]. Here, and throughout the paper, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain.

The method involving the Rayleigh quotient uses heavily the variational structure and cannot be applied to operators in non-divergence form. Hence

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another approach is needed. In their famous paper [5], Berestycki, Nirenberg and Varadhan showed that it is possible to define the principal eigenvalue of a linear operator with the aid of the maximum principle. More precisely, they proved that for uniformly elliptic linear operators the number

$$\lambda_1 = \sup\{\lambda \in \mathbb{R} : L + \lambda I \text{ satisfies the maximum principle}\}$$

is the least eigenvalue of L. Recall that $L+\lambda I$ satisfies the maximum principle in Ω if any subsolution of the equation $Lu + \lambda u = 0$ that is non-positive on $\partial\Omega$ is non-positive in Ω. Several other properties such as simplicity and stability of the principal eigenvalue were also studied thoroughly in [5].

The task of developing an eigenvalue theory for nonlinear operators in non-divergence form has been taken up recently by several authors. The Pucci extremal operators were treated by Busca, Esteban and Quaas in [10] (see also [15] and [29]), and their results have been improved and extended to fully nonlinear, uniformly elliptic operators in [30] by Quaas and Sirakov. Similar results have been obtained independently by Ishii and Yoshimura [19]. However, closest to the framework of this paper is the work by Birindelli and Demengel [8], who allow singular operators and, in particular, do not assume uniform ellipticity. Instead they require, among other assumptions, that the operator $F(Du, D^2u)$ satisfies

$$a|p|^\alpha \operatorname{trace}(N) \le F(p, M+N) - F(p,M) \le A|p|^\alpha \operatorname{trace}(N) \tag{1.1}$$

for some $\alpha > -1$, $0 < a \le A$ and for all $N \ge 0$. A typical example is $F(Du, D^2u) = |Du|^\alpha \mathcal{M}_{a,A}(D^2u)$, where $\mathcal{M}_{a,A}$ is one of Pucci's operators, but their theory also applies to the p-Laplacian $-\Delta_p u = -\operatorname{div}(|Du|^{p-2}Du)$, $1 < p < \infty$.

In this paper, we are interested in the eigenvalue problem

$$\begin{cases} -\Delta_\infty u(x) = \lambda u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where

$$\Delta_\infty u := \left(D^2u \frac{Du}{|Du|} \right) \cdot \frac{Du}{|Du|} \tag{1.3}$$

is known as the infinity Laplace operator. Note that (1.1) does not hold for the infinity Laplacian, and therefore the problem (1.2) is not covered by the work of Birindelli and Demengel. In fact, the infinity Laplacian is non-degenerate only in the direction of the gradient.

The motivation to study (1.2) stems partially from the usefulness of the infinity Laplace operator in certain applications. The by-now well known geometric interpretation of the viscosity solutions of the infinity Laplace equation $-\Delta_\infty u = 0$ as absolutely minimizing Lipschitz extensions, see [1], [2], has attracted considerable interest for example in image processing, the main usage being in the reconstruction of damaged digital images, see e.g. [11]. On the other hand, while the equation $-\Delta_\infty u = 0$ has been studied extensively after the fundamental paper by Jensen [21], a systematic investigation of the infinity Poisson equation $-\Delta_\infty u = f(x)$ has barely begun. Most of the known results are due to Peres, Schramm, Sheffield and Wilson [28] (see also [3]) and are obtained via a game-theoretic interpretation of the equation. In order to broaden the study to include right-hand sides of the form $f(x,u)$, it seems well motivated to consider the eigenvalue problem associated to (1.3).

We define the principal eigenvalue λ_1 as in [5], [8], by setting

$$\lambda_1 = \sup\{\lambda : \exists\, v > 0 \text{ in } \overline{\Omega} \text{ such that } -\Delta_\infty v \geq \lambda v\}. \tag{1.4}$$

It turns out that this number is positive and it can be explicitly computed in the case of a ball; this yields reasonably good upper and lower bounds for λ_1 in the general case. We are able to show that λ_1 is an eigenvalue and that it is the least eigenvalue of the infinity Laplacian. Moreover, it admits a positive eigenfunction and can be characterized as the supremum of the values λ for which $\Delta_\infty + \lambda I$ satisfies the maximum principle. These results are then applied to obtain existence and uniqueness results for the equation

$$-\Delta_\infty u(x) = \lambda u(x) + f(x)$$

and decay estimates for the solutions of the corresponding evolution equation

$$h_t(x,t) = \Delta_\infty h(x,t)$$

with zero data on the lateral boundary. A key tool in the proofs is a logarithmic change of dependent variable.

We want to emphasize that all our results hold for an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$. Moreover, it will be evident that with minor modifications in our main arguments one can prove most of the results of this paper for a class of quasilinear operators of the form

$$F(x, Du, D^2u) = -\operatorname{trace}(A(x, Du)D^2u).$$

See Remark 3.3 below for more details.

In the literature, there are several papers, most notably [24], [23], [16], and [12], whose topic includes both of the terms "eigenvalue" and "infinity Laplacian". Let us state very clearly that these deal with a problem that is different from the one considered in this work. Indeed, the above mentioned papers are concerned with the asymptotic behavior, as $p \to \infty$, of the p-Laplace eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u.$$

The limit equation in case of the principal eigenvalue is found to be

$$\min\{|Du| - \Lambda u, -\Delta_\infty u\} = 0, \tag{1.5}$$

where

$$\Lambda = \frac{1}{\sup_\Omega \operatorname{dist}(x, \partial\Omega)}$$

and the solutions of (1.5) minimize

$$\frac{\sup_\Omega |Du|}{\sup_\Omega |u|}$$

over $W_0^{1,\infty}(\Omega) \setminus \{0\}$. We want to point out that although the equation $-\Delta_\infty u = 0$ is the limit of equations $-\Delta_p u = 0$ as $p \to \infty$, see e.g. [2], [21], the infinity Laplace operator is *not* a limit of the p-Laplacians. For example, if $u(x) = |x|$, then $\Delta_p u = \frac{n-1}{|x|}$ in $\mathbb{R}^n \setminus \{0\}$ for all $1 < p < \infty$, but

$\Delta_\infty u = 0$[1]. Hence there is no reason to expect that (1.2) and (1.5) would be equivalent or even strongly related. We will provide explicit examples that corroborate this. On the other hand, since both (1.2) and (1.5) involve the infinity Laplacian, it is natural to compare the results we obtain to those known in the case of (1.5).

2. Definitions

Due to the fact that (1.3) is singular at the points where the gradient vanishes, we have to use the semicontinuous extensions of the function $(\xi, X) \mapsto (X\frac{\xi}{|\xi|})\cdot\frac{\xi}{|\xi|}$ when defining the viscosity solutions of (1.2). To this end, for a symmetric $n\times n$-matrix A, we denote its largest and smallest eigenvalue by $M(A)$ and $m(A)$, respectively. That is,

$$M(A) = \max_{|\eta|=1} (A\eta)\cdot\eta$$

and

$$m(A) = \min_{|\eta|=1} (A\eta)\cdot\eta.$$

Definition 2.1. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $\lambda\in\mathbb{R}$. An upper semicontinuous function $u:\Omega\to\mathbb{R}$ is a *viscosity subsolution* of (1.2) in Ω if, whenever $\hat{x}\in\Omega$ and $\varphi\in C^2(\Omega)$ are such that $0 = u(\hat{x})-\varphi(\hat{x}) > u(x)-\varphi(x)$ for all $x\neq\hat{x}$ then

$$\begin{cases} -\Delta_\infty\varphi(\hat{x}) \le \lambda\varphi(\hat{x}) & \text{if } D\varphi(\hat{x})\neq 0, \\ -M(D^2\varphi(\hat{x})) \le \lambda\varphi(\hat{x}) & \text{if } D\varphi(\hat{x}) = 0. \end{cases} \tag{2.1}$$

A lower semicontinuous function $v:\Omega\to\mathbb{R}$ is a *viscosity supersolution* of (1.2) in Ω if $-v$ is a viscosity subsolution, that is, whenever $\hat{x}\in\Omega$ and $\varphi\in C^2(\Omega)$ are such that $0 = v(\hat{x})-\varphi(\hat{x}) < v(x)-\varphi(x)$ for all $x\neq\hat{x}$ then

$$\begin{cases} -\Delta_\infty\varphi(\hat{x}) \ge \lambda\varphi(\hat{x}) & \text{if } D\varphi(\hat{x})\neq 0, \\ -m(D^2\varphi(\hat{x})) \ge \lambda\varphi(\hat{x}) & \text{if } D\varphi(\hat{x}) = 0. \end{cases} \tag{2.2}$$

Finally, a continuous function $h:\Omega\to\mathbb{R}$ is a *viscosity solution* of (1.2) in Ω if it is both a viscosity subsolution and a viscosity supersolution.

Now the number λ_1 is defined as in [5]:

Definition 2.2. Given a bounded domain $\Omega\subset\mathbb{R}^n$, let $E\subset\mathbb{R}$ be the set of those $\lambda\in\mathbb{R}$ for which there exists $v\in C(\overline{\Omega})$ such that $v(x)>0$ for all $x\in\overline{\Omega}$ and $-\Delta_\infty v\ge\lambda v$ in Ω in the viscosity sense. Then we define

$$\lambda_1 = \sup E.$$

Since constant functions satisfy the equation $-\Delta_\infty u = 0$, the number λ_1 is well-defined and non-negative. Moreover, it follows immediately from the definition that if $\Omega_1\subset\Omega_2$ then $\lambda_1(\Omega_2)\le\lambda_1(\Omega_1)$. This allows us to estimate λ_1 for a general domain once we obtain a formula for the principal eigenvalue of a ball.

[1]The limiting behaviour, as $p\to\infty$, of the p-Poisson equations $-\Delta_p u = f(x)$ (and their connection with the mass transfer problems) has been investigated in detail in e.g. [6], [14], [18], [20] and [25].

3. Comparison results

We begin by establishing a series of comparison results that are needed in the verification of the fact that λ_1 is the least eigenvalue. Similar results were obtained by Birindelli and Demengel in [8] by utilizing their earlier results in [7]. To our taste, the self-contained argument presented below is simpler than that in [8] and it also makes the proof somewhat shorter. Moreover, we have the opportunity to correct a minor error that appears in [8]. [2]

Theorem 3.1. *Suppose that $\mu < \lambda_1$ and let $u \in C(\overline{\Omega})$ satisfy $-\Delta_\infty u \le \mu u$ in Ω and $u \le 0$ on $\partial\Omega$. Then $u \le 0$ in Ω.*

Theorem 3.1 is a special case of the following slightly more general result:

Proposition 3.2. *Let $\mu < \lambda$ and suppose that $v \in C(\overline{\Omega})$ is such that $v(x) > 0$ for all $x \in \overline{\Omega}$ and $-\Delta_\infty v \ge \lambda v$. If $u \in C(\overline{\Omega})$ satisfies $-\Delta_\infty u \le \mu u$ in Ω and $u \le 0$ on $\partial\Omega$, then $u \le 0$ in Ω.*

Proof. Our proof is by contradiction, and we suppose that u is not non-positive. Since $v > 0$ in $\overline{\Omega}$ and $u \le 0$ on $\partial\Omega$, this means that the function $\frac{u(x)}{v(x)}$ attains a positive maximum at an interior point $\hat{x} \in \Omega$.

Let us denote $w(x) = \log u(x)$ and $g(x) = \log v(x)$, where w is defined only in a neighborhood $\hat{\Omega}$ of $\hat{x}$ where u is positive. Then it is easy to check that

$$-\Delta_\infty g - |Dg|^2 \ge \lambda \qquad \text{in } \Omega \tag{3.1}$$

and

$$-\Delta_\infty w - |Dw|^2 \le \mu \qquad \text{in } \hat{\Omega} \tag{3.2}$$

in the viscosity sense. Here we interpret the infinity Laplacian at the points where the gradient vanishes as in Definition 2.1. Note also that $\hat{x}$ is a local maximum point of $w(x) - g(x) = \log \frac{u(x)}{v(x)}$.

Consider next the functions

$$\Psi_j(x, y) = w(x) - g(y) - \theta_j(x, y), \quad j \in \mathbb{N},$$

where $\theta_j(x, y) = \frac{j}{4}|x-y|^4$, and let $(x_j, y_j) \in \hat{\Omega}\times\hat{\Omega}$ be such that $\Psi_j(x_j, y_j) = \sup_{\hat{\Omega}\times\hat{\Omega}} \Psi_j(x, y)$. Without loss of generality, we may assume that $x_j \to \hat{x}$ and $y_j \to \hat{x}$ as $j \to \infty$, cf. [13, Lemma 3.1]. Moreover, $j|x_j - y_j|^4 \to 0$ as $j \to \infty$.

Next we apply the maximum principle for semicontinuous functions from [13]. There exist symmetric $n \times n$ matrices X_j, Y_j such that

$$\begin{aligned} (\eta_j, X_j) &\in \overline{J}^{2,+} w(x_j), \\ (\eta_j, Y_j) &\in \overline{J}^{2,-} g(y_j), \end{aligned} \tag{3.3}$$

where $\eta_j = j|x_j - y_j|^2(x_j - y_j)$, and

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le D^2\theta_j(x_j, y_j) + \frac{1}{j}\left[D^2\theta_j(x_j, y_j)\right]^2. \tag{3.4}$$

[2] Birindelli and Demengel have themselves also detected this error and have addressed the issue in their recent paper [9].

See [13] for the notation used above. Recalling the definition of θ_j and denoting $z_j = x_j - y_j$, (3.4) can be rewritten as

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le j(|z_j|^2 + 2|z_j|^4) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 16j|z_j|^2 \begin{pmatrix} z_j \otimes z_j & -z_j \otimes z_j \\ -z_j \otimes z_j & z_j \otimes z_j \end{pmatrix}.$$

In particular, by evaluating the corresponding quadratic forms at $\begin{pmatrix} \xi \\ \xi \end{pmatrix} \in \mathbb{R}^{2n}$, we see that $X_j\xi \cdot \xi \le Y_j\xi \cdot \xi$ for all $\xi \in \mathbb{R}^n$, i.e., $Y_j - X_j$ is positive semidefinite. Hence if $x_j \ne y_j$, we have by using the fact that g and w satisfy (3.1) and (3.2), respectively, that

$$\lambda \le -\left(Y_j \frac{\eta_j}{|\eta_j|}\right) \cdot \frac{\eta_j}{|\eta_j|} - |\eta_j|^2 \le -\left(X_j \frac{\eta_j}{|\eta_j|}\right) \cdot \frac{\eta_j}{|\eta_j|} - |\eta_j|^2 \le \mu,$$

contradicting the assumption $\lambda > \mu$. On the other hand, if $x_j = y_j$, then $\eta_j = z_j = 0$ and it follows from (3.4) that

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $X_j \le 0 \le Y_j$, and we obtain from (3.1) and (3.2) that

$$\lambda \le -m(Y_j) \le 0 \le -M(X_j) \le \mu;$$

again a contradiction. □

Remark 3.3. It is clear that the argument used in the proof of Proposition 3.2 works in a more general setting than just in the case of the infinity Laplacian. For example, it applies to quasilinear operators of the form

$$F(x, Du, D^2u) = -\operatorname{trace}(A(x, Du)D^2u)$$

under the assumptions that the matrix valued function $A = A(x,p)$ is positive semidefinite, homogeneous of degree 0 in the second variable and has a Lipschitz continuous (in x) square root. In fact, most of the results obtained in this paper can be quite easily generalized to this class of operators.

Corollary 3.4. *Suppose that $\lambda < \lambda_1$ and let $u \in C(\overline{\Omega})$ satisfy*

$$\begin{cases} -\Delta_\infty u(x) = \lambda u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \equiv 0$ in Ω. In particular, λ is not an eigenvalue.

Combining this with the fact that λ_1 is an eigenvalue (which will be proved in Theorem 5.3 below) we have, analogously to [5] and [8], that

Corollary 3.5. *The number λ_1 can be characterized as the supremum of those values $\lambda \in \mathbb{R}$ for which the operator $\Delta_\infty + \lambda I$ satisfies the maximum principle.*

In order to show that λ_1 actually is an eigenvalue, we need another comparison result that also yields uniqueness for certain related problems.

Theorem 3.6. *Let $\lambda < \lambda_1$, and let u and v be a viscosity subsolution and a supersolution, respectively, of the equation*

$$-\Delta_\infty \phi(x) = \lambda\phi(x) + f(x), \tag{3.5}$$

where $f \in C(\Omega)$. Suppose that either

$$f(x) > 0 \text{ for all } x \in \Omega \tag{3.6}$$

or

$$f(x) \geq 0 \text{ for all } x \in \Omega \text{ and } \lambda > 0. \tag{3.7}$$

Then, if $v \geq u$ and $v > 0$ on $\partial\Omega$, we have $v \geq u$ in Ω.

Proof. The basic strategy of the proof is the same as that of Proposition 3.2. We argue by contradiction and suppose that the set $\{x \in \Omega : u(x) > v(x)\}$ is not empty. Since $\lambda < \lambda_1$ and $-\Delta_\infty v \geq \lambda v$ in the viscosity sense, it follows from Theorem 3.1 (applied to $-v$) that v is nonnegative. For $\lambda \geq 0$ this together with the Harnack inequality for the supersolutions of $-\Delta_\infty \varphi = 0$ (see Lemma 5.1 below) implies that in fact $v > 0$ in $\overline{\Omega}$. In the case $\lambda < 0$ the same conclusion can be easily reached by noticing that $\varphi \equiv 0$ is a test-function (from below) at the points where v vanishes and then using the assumption $f(x) > 0$ for all $x \in \Omega$.

Let now $\hat{x} \in \Omega$ be such that

$$1 < \frac{u(\hat{x})}{v(\hat{x})} = \sup_{x\in\overline{\Omega}} \frac{u(x)}{v(x)}. \tag{3.8}$$

Without loss of generality, by scaling f if necessary, we may assume that $u > v > 1$ in some neighborhood $\hat{\Omega}$ of $\hat{x}$.

If we denote $w(x) = \log u(x)$ and $g(x) = \log v(x)$, it is easy to check that they are a subsolution and a supersolution, respectively, to

$$-\Delta_\infty \phi(x) - |D\phi(x)|^2 - \lambda - f(x)e^{-\phi(x)} = 0 \tag{3.9}$$

in the subdomain $\hat{\Omega}$. Notice that this equation can be written in the form $F(x, w, Dw, D^2w) = 0$, where the function

$$F(x, r, \xi, X) = -\left(X\tfrac{\xi}{|\xi|}\right)\cdot\tfrac{\xi}{|\xi|} - |\xi|^2 - \lambda - f(x)e^{-r}$$

is increasing in the variable r if f is positive in Ω.

By applying the maximum principle for semicontinuous functions to the functions

$$\Psi_j(x, y) = w(x) - g(y) - \theta_j(x, y), \quad j \in \mathbb{N},$$

where $\theta_j(x, y) = \frac{j}{4}|x - y|^4$, we conclude, as in the proof of Proposition 3.2, that there exist symmetric $n \times n$ matrices X_j, Y_j, $X_j \leq Y_j$ such that

$$\begin{aligned}(\eta_j, X_j) &\in \overline{J}^{2,+} w(x_j),\\ (\eta_j, Y_j) &\in \overline{J}^{2,-} g(y_j),\end{aligned} \tag{3.10}$$

where $\Psi_j(x_j, y_j) = \sup_{\hat{\Omega}\times\hat{\Omega}} \Psi_j(x, y)$, and $\eta_j = j|x_j - y_j|^2(x_j - y_j)$. Moreover, if $x_j = y_j$, then $X_j \leq 0 \leq Y_j$. We may also assume without loss of generality that $(x_j, y_j) \to (\hat{x}, \hat{x})$ as $j \to \infty$.

Now if $x_j \neq y_j$, it follows from $X_j \le Y_j$ and the fact that w and g are a subsolution and a supersolution to (3.9) that

$$\begin{aligned}\lambda + f(y_j)e^{-g(y_j)} &\le -\left(Y_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - |\eta_j|^2 \\ &\le -\left(X_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - |\eta_j|^2 \le \lambda + f(x_j)e^{-w(x_j)}.\end{aligned}$$

On the other hand, if $x_j = y_j$, then $\eta_j = 0$ and we obtain

$$\lambda + f(y_j)e^{-g(y_j)} \le -m(Y_j) \le 0 \le -M(X_j) \le \lambda + f(x_j)e^{-w(x_j)}.$$

Thus in any case $f(y_j)e^{-g(y_j)} \le f(x_j)e^{-w(x_j)}$ for each j, and if $f(\hat{x}) > 0$, we obtain by letting $j \to \infty$ that $g(\hat{x}) \ge w(\hat{x})$, contradicting (3.8).

If f is merely a non-negative function, we perturb g slightly so that it becomes a strict supersolution. More precisely, for $\alpha > 1$ and $A > 1$ let

$$h(t) = \frac{1}{\alpha}\log\left(1 + A(e^{\alpha t} - 1)\right).$$

Then $h'(t) > 1$ and $h'(t) - h'(t)^2 - h''(t) > 0$ for all $t \ge 0$. Moreover, $0 < h(t) - t < \frac{A-1}{\alpha}$ for $t \ge 0$, and thus $h(t) \to t$ uniformly if $A \to 1^+$. See [24] for details. Now a formal computation yields that the function

$$G(x) := h(g(x))$$

satisfies

$$\begin{aligned}-\Delta_\infty G - |DG|^2 &= h'(g)(-\Delta_\infty g) - h''(g)|Dg|^2 - h'(g)^2|Dg|^2 \\ &\ge h'(g)\left[\lambda + fe^{-g}\right] + |Dg|^2\left[h'(g) - h'(g)^2 - h''(g)\right] \\ &\ge h'(g)\left[\lambda + fe^{-g}\right] \\ &> \lambda + f(x)e^{-G(x)},\end{aligned}$$

where the last inequality follows from the facts $\lambda > 0$, $h'(t) > 1$ and $h(t) > t$ for all $t \ge 0$. Since h is smooth and increasing, it is straightforward to verify that indeed

$$-\Delta_\infty G(x) - |DG(x)|^2 > \lambda + f(x)e^{-G(x)} \tag{3.11}$$

in the viscosity sense. By choosing $A > 1$ sufficiently small, we see that also $w - G$ achieves its positive maximum in $\hat{\Omega}$ at an interior point.

Now the rest of the argument runs as in the case $f > 0$. We apply the maximum principle for semicontinuous functions to

$$\Psi_j(x,y) = w(x) - G(y) - \theta_j(x,y), \quad j \in \mathbb{N},$$

and conclude, as above, that there exist symmetric $n \times n$ matrices $X_j \le Y_j$ such that

$$\begin{aligned}&(\eta_j, X_j) \in \overline{J}^{2,+}w(x_j), \\ &(\eta_j, Y_j) \in \overline{J}^{2,-}G(y_j),\end{aligned} \tag{3.12}$$

and $X_j \le 0 \le Y_j$ if $x_j = y_j$; here, as before, x_j, y_j are points such that $\Psi_j(x_j,y_j) = \sup_{\hat{\Omega}\times\hat{\Omega}} \Psi_j(x,y)$ and $\eta_j = j|x_j - y_j|^2(x_j - y_j)$. Using $X_j \le Y_j$, (3.11) and the fact that w is a subsolution of (3.9), this yields

$$\begin{aligned}\lambda + f(y_j)e^{-G(y_j)} &< -\left(Y_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - |\eta_j|^2 \\ &\le -\left(X_j\frac{\eta_j}{|\eta_j|}\right)\cdot\frac{\eta_j}{|\eta_j|} - |\eta_j|^2 \le \lambda + f(x_j)e^{-w(x_j)}\end{aligned}$$

if $x_j \neq y_j$, and

$$\lambda + f(y_j)e^{-G(y_j)} < -m(Y_j) \le 0 \le -M(X_j) \le \lambda + f(x_j)e^{-w(x_j)}$$

if $x_j = y_j$. Both alternatives lead to a contradiction upon letting $j \to \infty$ and the proof is complete. □

Corollary 3.7. *Let $\lambda < \lambda_1$ and suppose that $f : \overline{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are continuous functions such that g is positive and either*

$$f \text{ is positive in } \Omega$$

or

$$f \text{ is non-negative in } \Omega \text{ and } \lambda > 0.$$

Then the Dirichlet problem

$$\begin{cases} -\Delta_\infty \phi(x) = \lambda\phi(x) + f(x), & \text{in } \Omega, \\ \phi(x) = g(x) & \text{on } \Omega, \end{cases} \tag{3.13}$$

has at most one solution.

Remark 3.8. We do not know if the assumptions of the corollary above are optimal. An example constructed in [28] shows that in the case $\lambda = 0$ there exists a Lipschitz continuous function f, defined in the closed unit disc B_1 of $\mathbb{R}^2$, such that the problem

$$\begin{cases} -\Delta_\infty v = f(x) & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1, \end{cases}$$

has more than one solution. This function f takes values of both signs. On the other hand, if f is uniformly continuous and either $f \equiv 0$ or $\inf|f| > 0$, then the Dirichlet problem

$$\begin{cases} -\Delta_\infty v(x) = f(x) & \text{in } \Omega, \\ v(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

has a unique viscosity solution for any uniformly continuous boundary data g and for any bounded domain $\Omega \subset \mathbb{R}^n$. See [28]. Note that in the case $\lambda = 0$ the positivity of g is not a restriction since a constant can be added to a solution, and hence Corollary 3.7 in fact slightly improves the above-mentioned uniqueness result in [28].

In the case $\lambda = \lambda_1$ and $f \equiv 0$, $g \equiv 0$, the uniqueness for (3.13) fails because any constant multiple of an eigenfunction is also an eigenfunction. See Theorem 5.3 below. On the other hand, for $\lambda < 0$ the equation (3.5) is increasing in the ϕ-variable, and thus the Dirichlet problem (3.13) has at most one solution for *any* continuous f and g by the general uniqueness result [13, Theorem 3.3]. Finally, since the infinity Laplacian is odd, the uniqueness for (3.13) holds also when $\lambda < \lambda_1$ and both f and g are negative.

We do not know whether the principal eigenvalue λ_1 is simple. However, arguing as in the proof of Theorem 3.6, we can obtain local uniqueness for the positive principal eigenfunction. The result is analogous to what is known about the first eigenfunctions of the infinity eigenvalue problem (1.5), see [24].

Theorem 3.9. *If u and v are positive eigenfunctions, associated to the same eigenvalue $\lambda \geq \lambda_1 > 0$, then*

$$\sup_{\Omega'} \frac{u}{v} = \sup_{\partial\Omega'} \frac{u}{v}$$

for any $\Omega' \subset\subset \Omega$.

4. The principal eigenvalue in a ball

If the domain Ω is a ball, it is natural to expect that the principal eigenvalue is simple and that the associated eigenfunction is radial. Moreover, there is also hope to find explicit formulas for the eigenvalue and the eigenfunction.

With these goals in mind, let $\Omega = B_R = B_R(0)$ and let us look for positive radial solutions to (1.2). Setting $h(x) = g(|x|)$, we have $Dh(x) = g'(|x|)\frac{x}{|x|}$ and

$$D^2h(x) = g''(|x|)\frac{x}{|x|} \otimes \frac{x}{|x|} + g'(|x|)\frac{1}{|x|}\left(I - \frac{x}{|x|} \otimes \frac{x}{|x|}\right)$$

for $x \neq 0$. Hence

$$\Delta_\infty h(x) = g''(|x|),$$

and the equation $-\Delta_\infty h = \lambda h$ reduces to $-g'' = \lambda g$. This formal calculation becomes rigorous once we observe that if φ is a smooth test-function for g at $r_0 \in]0, R[$, then $\psi(x) := \varphi(|x|)$ is a smooth test-function for h at all points x for which $|x| = r_0$. Taking into account the boundary condition $g(R) = 0$, it is not hard to see that g must be of the form

$$g(r) = C_1 \cos(\sqrt{\lambda}r) + C_2 \sin(\sqrt{\lambda}r).$$

The function

$$h(x) = C_1 \cos(\sqrt{\lambda}|x|), \quad \lambda > 0$$

is twice differentiable everywhere and satisfies the equation (1.2) in B_R (in the viscosity sense). On the contrary, the function $x \mapsto C_2 \sin(\sqrt{\lambda}|x|)$ is only a viscosity sub- or supersolution, depending on the sign of the constant C_2. In fact, near $x = 0$, this function looks like a cone having vertex at the origin, and the conical shape prevents testing from one side (hence automatically a sub/supersolution), but allows test-functions with non-zero gradient and arbitrary Hessian from the other side.

In conclusion, we have proved that the only radial viscosity solutions to (1.2) in a ball B_R are the functions

$$h_k(x) = C_1 \cos(\sqrt{\lambda_k}|x|), \qquad \text{with} \quad \lambda_k = \left(\frac{(2k-1)\pi}{2R}\right)^2.$$

In particular, the only positive radial eigenfunction is $h_1(x) = \cos(\frac{\pi}{2R}|x|)$. We show next that the number $\left(\frac{\pi}{2R}\right)^2$ really is the principal eigenvalue of B_R.

Lemma 4.1. *For a ball B_R we have*

$$\lambda_1(B_R) = \left(\frac{\pi}{2R}\right)^2.$$

Proof. By the above calculations and the definition of λ_1, we easily see that $\lambda_1(B_R) \geq \left(\frac{\pi}{2R}\right)^2$. Suppose that we had

$$\lambda_1(B_R) > \mu > \left(\frac{\pi}{2R}\right)^2$$

and let $0 < \rho < R$ be such that $\mu = \left(\frac{\pi}{2\rho}\right)^2$. Define a function w by

$$w(x) = \begin{cases} \cos(\sqrt{\mu}|x|), & \text{if } |x| \leq \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Then $-\Delta_\infty w \leq \mu w$ in B_R and $w \leq 0$ on ∂B_R, which by Theorem 3.1 should imply $w \leq 0$ in B_R. Clearly this is not the case and therefore $\lambda_1(B_R) = \left(\frac{\pi}{2R}\right)^2$. □

Since $\lambda_1(\Omega_1) \leq \lambda_1(\Omega_2)$ if $\Omega_2 \subset \Omega_1$, we can deduce from the above lemma the estimate

$$\left(\frac{\pi}{2R_E}\right)^2 \leq \lambda_1(\Omega) \leq \left(\frac{\pi}{2R_I}\right)^2, \tag{4.1}$$

where

$$R_E = \inf\{r > 0 : \Omega \subset B_r(x) \text{ for some } x\}$$

and

$$R_I = \sup\{r > 0 : B_r(x) \subset \Omega \text{ for some } x\}.$$

In particular, $\lambda_1(\Omega) > 0$ for all $\Omega \subset \mathbb{R}^n$.

We do not know whether λ_1 is simple even in the case of a ball. The function $h_1(x) = \cos(\frac{\pi}{2R}|x|)$ is only radial principal eigenfunction, but there could exist non-radial principal eigenfunctions as well.

Remark 4.2. The above reasoning shows that the eigenvalue problem considered in this paper is quite different from the so-called ∞-eigenvalue problem (1.5) studied in e.g. [24], [23], [16]. Namely, in case of a ball B_R the first eigenvalue of the ∞-eigenvalue problem is $\frac{1}{R}$ and the corresponding eigenfunction, unique up multiplication by a constant, is $x \mapsto R - |x|$.

5. Existence results

Our main goal in this section is to show that the number λ_1, defined by (1.4), really is an eigenvalue of the infinity Laplacian. This amounts to showing that the problem

$$\begin{cases} -\Delta_\infty u(x) = \lambda_1 u(x) & \text{in } \Omega, \\ \quad u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial solution. The general strategy for the proof is more or less the same as in [5], but the details are quite different.

Before getting started with the actual proof, we need to recall a local Lipschitz continuity estimate for the supersolutions of the infinity Laplace equation. The proof can be found for example in [2] or [26].

Lemma 5.1. *Let $u \in C(\Omega)$ be a non-negative function such that $-\Delta_\infty u \geq 0$ in the viscosity sense in a domain Ω. If $x_0 \in \Omega$ and $0 < r < R \leq \operatorname{dist}(x_0, \partial\Omega)$, then*

$$u(y) \leq u(z) e^{\frac{|y-z|}{R-r}} \qquad \text{for all } y, z \in B_r(x_0). \tag{5.1}$$

Moreover,

$$|Du(x)| \leq \frac{u(x)}{\operatorname{dist}(x, \partial\Omega)} \qquad \text{for a.e. } x \in \Omega. \tag{5.2}$$

The following lemma gives us a useful characterization of the number λ_1.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $0 < \lambda < \lambda_1(\Omega)$. Then there exists a function $w \in C(\overline{\Omega})$ such that $w > 0$ in Ω, $w = 0$ on $\partial\Omega$, and $-\Delta_\infty w = 1 + \lambda w$.*

Proof. Since $0 < \lambda < \lambda_1(\Omega)$, there exists a positive function $u \in C(\overline{\Omega})$ such that $-\Delta_\infty u \geq \lambda u$ in the viscosity sense. Let $\eta_0 := \min_{x \in \partial\Omega} u(x) > 0$ and notice that for $0 < \eta < \eta_0$ the function $u_\eta := u - \eta$ is positive by the maximum principle and satisfies

$$-\Delta_\infty u_\eta \geq \lambda u_\eta + \lambda \eta.$$

Hence $\hat{u}(x) := \frac{1}{\lambda\eta} u_\eta(x)$ is a positive supersolution of $-\Delta_\infty v = 1 + \lambda v$. In order to find a supersolution that vanishes on the boundary, we notice that, given $z \in \partial\Omega$, the function $u_z(x) = |x - z|^{1/2}$ satisfies

$$-\Delta_\infty u_z(x) = \frac{1}{4}|x-z|^{-3/2} = \frac{1}{8|x-z|^2} u_z(x) + \frac{1}{8}|x-z|^{-3/2}$$

in Ω. Thus there exists $\rho > 0$, depending only on λ, such that $-\Delta_\infty u_z \geq \lambda u_z + 1$ in $B_\rho(z) \cap \Omega$. By choosing $C \geq 1$ so that, say, $C\sqrt{\frac{\rho}{2}} \geq \sup_\Omega \hat{u}$, we have that $\min\{C u_z(x), \hat{u}(x)\} = \hat{u}(x)$ outside the set $B_{\rho/2}(z) \cap \Omega$. Hence it follows that the function

$$U(x) = \inf_{z \in \partial\Omega} \big(\min\{C u_z(x), \hat{u}(x)\} \big)$$

is a positive supersolution to $-\Delta_\infty v = 1 + \lambda v$ that vanishes on $\partial\Omega$.

Next we fix a ball $B \subset\subset \Omega$ of radius $r > 0$ and let u_B be a positive radial solution, obtained in the previous section, to

$$\begin{cases} -\Delta_\infty v = \lambda_1(B) v & \text{in } B, \\ v = 0 & \text{on } \partial B, \end{cases}$$

normalized so that $\lambda_1(B) u_B(x) \leq 1$ and $u_B(x) \leq U(x)$ for all $x \in B$. In fact, we have $\lambda_1(B) = \left(\frac{\pi}{2r}\right)^2$ and $u_B(x) = C\cos(\frac{\pi}{2r}|x - x_0|)$, where x_0 is the center of the ball B. It is clear that the zero extension of u_B is a subsolution of $-\Delta_\infty v = 1 + \lambda v$. Now the existence of the asserted solution w follows from the standard Perron method, see [13, Section 4], and its positivity in Ω from the Harnack inequality in Lemma 5.1. ☐

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\lambda = \lambda_1(\Omega)$. Then there exists $w \in C(\overline{\Omega})$ such that $w > 0$ in Ω, $w = 0$ on $\partial\Omega$, and $-\Delta_\infty w = \lambda w$. In particular, $\lambda_1(\Omega)$ is an eigenvalue.*

Proof. Let λ_k be an increasing sequence of numbers converging to λ_1, and let w_k be a positive solution to $-\Delta_\infty w_k = 1+\lambda_k w_k$ with $w_k = 0$ on $\partial\Omega$, provided by Lemma 5.2. We first claim that the sequence $\sup_\Omega w_k$ is unbounded. Indeed, if this is not the case, then Lemma 5.1 implies that the sequence (w_k) is locally equicontinuous and thus converges (up to a subsequence) locally uniformly to a positive viscosity solution w of

$$\begin{cases} -\Delta_\infty w = 1 + \lambda_1 w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega; \end{cases}$$

the fact that $w = 0$ on $\partial\Omega$ can be seen by using the (uniform) barriers of the form $x \mapsto C|x - z|^{1/2}$, where $z \in \partial\Omega$. Then $w_\varepsilon = w + \varepsilon$ is positive in $\overline{\Omega}$ and it satisfies $-\Delta_\infty w_\varepsilon = (1 - \varepsilon\lambda_1) + \lambda_1 w_\varepsilon$. In particular, $-\Delta_\infty w_\varepsilon \geq \mu w_\varepsilon$ for all $\mu \leq \lambda_1 + \frac{1-\varepsilon\lambda_1}{\sup_\Omega w}$, thus contradicting the definition of λ_1 if ε is chosen so that $\lambda_1\varepsilon < 1$.

Let us now denote $v_k = \frac{w_k}{\sup_\Omega w_k}$ and note that

$$-\Delta_\infty v_k = \lambda_k v_k + \frac{1}{\sup_\Omega w_k}$$

in the viscosity sense. Since $\sup_\Omega v_k = 1$ for all k, we deduce from Lemma 5.1 that (v_k) converges (up to a subsequence) locally uniformly to a positive function v satisfying

$$-\Delta_\infty v = \lambda_1 v.$$

Here the fact that $\sup_\Omega w_k \to \infty$ as $k \to \infty$ (at least up to a subsequence) was used. By applying the same barrier argument as above, we see that $v = 0$ on $\partial\Omega$ and that $v \not\equiv 0$ in Ω. Hence we have found our eigenfunction and the proof is complete. □

A slight variation of the reasoning used in the proof of Lemma 5.2 yields existence results for more general non-homogeneous equations:

Theorem 5.4. *Let $0 \leq \lambda < \lambda_1$ and suppose that $f : \overline{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are non-negative continuous functions. Then the Dirichlet problem*

$$\begin{cases} -\Delta_\infty \phi(x) = \lambda\phi(x) + f(x) & \text{in } \Omega, \\ \phi(x) = g(x) & \text{on } \Omega, \end{cases} \tag{5.3}$$

has at least one non-negative solution.

Proof. The proof is again based on the Perron method, and it suffices to find a subsolution and a supersolution of (5.3) attaining the right boundary values. Since the unique solution (see e.g. [21]) to $-\Delta_\infty u = 0$ satisfying $u = g$ on $\partial\Omega$ is non-negative, it qualifies as the subsolution. To construct the desired supersolution, we first recall that in course of proving Lemma 5.2 it was observed that there exists a positive function $w \in C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta_\infty w(x) \geq \lambda w(x) + 1 & \text{in } \Omega, \\ w(x) \geq \eta > 0 & \text{on } \Omega. \end{cases}$$

Choosing a constant $C > 0$ such that $C \geq \max\{\sup_\Omega f, \frac{1}{\eta}\sup_{\partial\Omega} g\}$, we see that $w_C(x) := Cw(x)$ satisfies $-\Delta_\infty w_C \geq \lambda w_C + C \geq \lambda w_C + f(x)$ in Ω and $w_C(x) \geq g(x)$ on $\partial\Omega$. In order to make sure that the right boundary values are attained, we use barriers of the form $h_z(x) = g(z) + C'|x - z|^{1/2}$, where $z \in \partial\Omega$ and $C' \geq 1$. Since

$$-\Delta_\infty h_z(x) = \frac{C'}{4|x-z|^{3/2}} \to \infty \qquad \text{as } x \to z,$$

it follows that $-\Delta_\infty h_z \geq \lambda h_z + \sup f$ in $B_\rho(z) \cap \Omega$ for some $\rho > 0$ depending on λ, $\sup g$ and $\sup f$, but independent of z and $C' \geq 1$. After choosing C' so large that $h_z(x) \geq w_C(x)$ outside $B_{\rho/2}(z) \cap \Omega$, it is easy to see that

$$v(x) = \inf_{z \in \partial\Omega} \big(\min\{h_z(x), w_C(x)\} \big)$$

is the kind of supersolution that we were looking for. □

6. An application: Decay estimates for the evolution equation

Let $h \in C(\overline{\Omega} \times [0, \infty))$ be a viscosity solution to the parabolic equation

$$\begin{cases} h_t = \Delta_\infty h & \text{in } \Omega \times (0, \infty), \\ h(x, 0) = h_0(x) & \text{on } \Omega \times \{0\}, \\ h(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{6.1}$$

This evolution problem (with more general boundary conditions) has been recently studied in [22] and it appears in several applications, for example in differential games, see [3] and [22].

In this section we are interested in the asymptotic behavior, as $t \to \infty$, of the solution $h(x, t)$ of (6.1). Based on the well-known results for the solutions of the ordinary heat equation, one expects h to decay to zero exponentially and that the rate of decay and the extinction profile are somehow connected with the principal eigenvalue and the eigenfunction of the infinity Laplacian, respectively. Since the problem is non-linear and very badly degenerate, precise estimates are much harder to obtain than in the case of the heat equation, where one can for example use the fact that the normalized eigenfunctions of the Laplacian form an orthonormal basis for L^2.

Nevertheless, we attempt to shed some light on the issue and at least do manage to establish the exponential decay with (almost) the right decay rate. The question of extinction profile seems harder to grasp, mainly because it is not known what condition should replace the orthogonality requirement in our non-linear setting. So, roughly speaking, instead of obtaining precise estimates for the difference $|h(x, t)e^{\lambda_1 t} - \varphi_1(x)|$, where φ_1 is a first eigenfunction, we are only able to bound the logarithmic difference

$$\log \Big(h(x, t)e^{\lambda_1 t} \Big) - \log \varphi_1(x) = \log \left(\frac{h(x, t)e^{\lambda t}}{\varphi_1(x)} \right).$$

For the purposes of our first result, suppose that $\Omega \subset\subset \hat{\Omega}$ and let $v \in C(\hat{\Omega})$ be a positive principal eigenfunction in $\hat{\Omega}$, i.e.,

$$\begin{cases} -\Delta_\infty v(x) = \lambda v(x) & \text{in } \hat{\Omega}, \\ \qquad v(x) = 0 & \text{on } \partial\hat{\Omega}; \end{cases}$$

here $\lambda = \lambda_1(\hat{\Omega})$.

Proposition 6.1. *Let h, v and λ be as above. We have*

$$\sup_{\Omega\times(0,\infty)} \frac{h(x,t)e^{\lambda t}}{v(x)} \le \sup_{\Omega} \frac{h_0^+(x)}{v(x)},$$

where $h_0^+ = \max\{h_0, 0\}$ denotes the positive part of h_0.

Proof. Let us denote $H(x,t) = h(x,t)e^{\lambda t}$. A straightforward calculation shows that H satisfies

$$\begin{cases} H_t = \Delta_\infty H + \lambda H & \text{in } \Omega\times(0,\infty),\\ H(x,0) = h_0(x) & \text{on } \Omega\times\{0\},\\ H(x,t) = 0 & \text{on } \partial\Omega\times(0,\infty). \end{cases} \tag{6.2}$$

Indeed, if $\varphi(x,t)$ is a test-function for H, then $\varphi(x,t)e^{-\lambda t}$ is a test-function for h and (6.2) easily follows from (6.1). Moreover, by replacing h_0 with its positive part if necessary, we may assume that the initial data h_0 is non-negative.

It clearly suffices to show that

$$\sup_{\Omega\times(0,T)} \frac{H}{v} = \max\Big\{\sup_{\Omega}\frac{h_0}{v}, 0\Big\}$$

for any $T > 0$. We argue by contradiction and suppose that

$$0 < \frac{H(\hat x,\hat t)}{v(\hat x)} = \sup_{(x,t)\in\Omega\times(0,T)} \frac{H(x,t)}{v(x)} \tag{6.3}$$

for some $\hat x \in \Omega$, $0 < \hat t \le T$. Notice that here we need the fact that $v > 0$ in $\overline{\Omega}$. Let $w = \log v$ and $\theta = \log H$, and observe that

$$\theta_t = \Delta_\infty\theta + \lambda + |D\theta|^2$$

and

$$\Delta_\infty w(x) + \lambda + |Dw|^2 = 0$$

in the viscosity sense in a neighborhood Q of $(\hat x,\hat t)$ where H is positive. Finally, if $w_\varepsilon(x,t) = w(x) + \frac{\varepsilon}{T-t}$ for $\varepsilon > 0$, we see that w_ε is a strict supersolution of

$$u_t = \Delta_\infty u + \lambda + |Du|^2. \tag{6.4}$$

Moreover, $w_\varepsilon(x,t) \to \infty$ uniformly in x as $t \to T$ and $\theta - w_\varepsilon$ has a local maximum in Q for $\varepsilon > 0$ small enough. For simplicity of notation, we denote this maximum point also by $(\hat x,\hat t)$ and notice that $\hat t < T$.

The rest of the proof is now a quite standard application of the maximum principle for semicontinuous functions [13]. We maximize

$$\psi_j(x,t,y,s) = \theta(x,t) - w_\varepsilon(y,s) - \frac{j}{4}|x-y|^4 - \frac{j}{2}(t-s)^2$$

over $\overline{Q}\times\overline{Q}$ and conclude that for j large enough, the maximum is attained at some point $(x_j,t_j,y_j,s_j) \in Q\times Q$ for which $(x_j,t_j) \to (\hat x,\hat t)$, $(y_j,s_j) \to (\hat x,\hat t)$ as $j \to \infty$, and there exist symmetric $n\times n$ matrices X_j, Y_j such that $Y_j - X_j$ is positive semidefinite and

$$(j(t_j-s_j), j|x_j-y_j|^2(x_j-y_j), X_j) \in \overline{\mathcal{P}}^{2,+}\theta(x_j,t_j),$$
$$(j(t_j-s_j), j|x_j-y_j|^2(x_j-y_j), Y_j) \in \overline{\mathcal{P}}^{2,-}w_\varepsilon(y_j,s_j).$$

See [13] for the notation and the relevant definitions. Using the facts that θ is a subsolution and w_ε a strict supersolution of (6.4), this implies in the case $x_j \neq y_j$ that

$$\begin{aligned} 0 &< j(t_j - s_j) - \Big(Y_j \frac{(x_j - y_j)}{|x_j - y_j|}\Big) \cdot \frac{(x_j - y_j)}{|x_j - y_j|} - \lambda - j^2|x_j - y_j|^6 \\ &\quad - j(t_j - s_j) + \Big(X_j \frac{(x_j - y_j)}{|x_j - y_j|}\Big) \cdot \frac{(x_j - y_j)}{|x_j - y_j|} + \lambda + j^2|x_j - y_j|^6 \\ &= -\Big((Y_j - X_j)\frac{(x_j - y_j)}{|x_j - y_j|}\Big) \cdot \frac{(x_j - y_j)}{|x_j - y_j|} \\ &\leq 0, \end{aligned}$$

a contradiction. Since a similar conclusion holds in the case $x_j = y_j$ due to the inequality $X_j \leq 0 \leq Y_j$ (cf. the proof of Proposition 3.2) we see that (6.3) does not hold and hence we are done. □

Corollary 6.2. *Let $h \in C(\overline{\Omega} \times [0, \infty))$ satisfy (6.1) with $h_0 \in C(\overline{\Omega})$. Then*

$$\sup_{\Omega} |h(x, t)| = o(e^{-\lambda t}) \qquad \textit{for all } \lambda < \lambda_1(\Omega).$$

Proof. It is enough to notice that we may run the proof of Proposition 6.1 precisely as it is if the function v in it is any positive (in $\overline{\Omega}$) function that satisfies $-\Delta_\infty v \geq \lambda v$ in Ω. Such a function exists for every $\lambda < \lambda_1(\Omega)$ by the definition of $\lambda_1(\Omega)$ and hence our claim follows. □

Proposition 6.1 gives a kind of upper estimate on the decay of the solutions to the evolution equation (6.1). Our next result shows that at least locally we have a lower estimate as well. To this end, let us suppose $\Omega_1 \subset\subset \Omega$ and let $u \in C(\Omega_1)$ be a positive principal eigenfunction in Ω_1, i.e.,

$$\begin{cases} \Delta_\infty u(x) + \mu u(x) = 0 & \text{in } \Omega_1, \\ u(x) = 0 & \text{on } \partial\Omega_1; \end{cases}$$

here $\mu = \lambda_1(\Omega_1)$. Notice that if h_0 is positive, $h > 0$ in $\Omega \times (0, \infty)$ by the Harnack inequality, Theorem 6.1 in [22].

Proposition 6.3. *Let h, u and μ be as above, and suppose that h_0 is positive in Ω. We have*

$$\sup_{\Omega_1 \times (0,\infty)} \frac{u(x)}{h(x, t)e^{\mu t}} = \sup_{\Omega_1} \frac{u(x)}{h_0(x)}.$$

Proof. We can apply the same argument as in the proof of Proposition 6.1 with some minor changes. Instead of (6.3), we assume that

$$0 < \frac{u(\hat{x})}{h(\hat{x}, \hat{t})e^{\mu \hat{t}}} = \sup_{(x,t) \in \Omega_1 \times (0,T)} \frac{u(x)}{h(x, t)e^{\mu t}} \tag{6.5}$$

for some $(\hat{x}, \hat{t}) \in \Omega_1 \times (0, T]$. By defining $w = \log u$ and $\theta_\varepsilon = \log(h(x, t)e^{\mu t}) + \frac{\varepsilon}{T - t}$, we have that w and θ_ε are a solution and a strict supersolution, respectively, to (6.4) (with λ replaced by μ), and $w - \theta_\varepsilon$ has a local maximum at some point in $\Omega_1 \times (0, T)$. As in the proof of Proposition 6.1, the desired contradiction now follows from the maximum principle for semicontinuous functions. □

Finally, we observe that the estimates in Propositions 6.1 and 6.3 can be made explicit by using the estimate (4.1) for the principal eigenvalue.

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