QUASIMINIMA OF THE LIPSCHITZ EXTENSION PROBLEM

PETRI JUUTINEN

Abstract. In this paper, we extend the notion of quasiminimum to the framework of supremum functionals by studying the model case

$$S(u,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} |Du|,$$

which governs the real analysis problem of finding optimal Lipschitz extensions. Using a characterization involving the concept of comparison with cones, we obtain a Harnack inequality, Lipschitz estimates and various convergence and stability properties for the quasiminima. Several examples of quasiminima are also given.

Introduction

The notion of quasiminimum, introduced by Giaquinta and Giusti [16], [17], can be viewed as a concept that unifies much of the regularity theory of the minima of functionals in the calculus of variations, and of solutions of elliptic partial differential equations and systems in divergence form. It embodies the notion of minima of variational integrals

$$I(u,\Omega) = \int_\Omega F(x,u,Du)\,dx,$$

but it is substantially more general. By definition, a quasiminimum u minimizes the functional I only up to a multiplicative constant, that is, there exists $K \geq 1$ such that

$$I(u,V) \leq K\, I(u+\varphi, V)$$

for each subdomain V that is compactly contained in Ω and for each $\varphi \in C_0^\infty(V)$. Regularity properties such as Hölder continuity, the Harnack inequality and L^p-estimates can be derived directly from the quasiminimizing property, which in turn can be shown to be satisfied by the solutions of a wide class of elliptic equations and systems. See e.g. [12], [17], [18], [24] and [27] for details.

The purpose of this paper is to extend the notion of quasiminimum to the setting of supremum functionals[1] of the form

$$S(u,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} F(x,u,Du). \tag{0.1}$$

These functionals provide often the most realistic framework in applications, cf. [1], [7], [14], [19], and partly because of that they have recently attracted considerable attention (see for example [5], [6], [8], [20] and their bibliographies). For simplicity, we restrict our attention to the archetypal model case $F(x,u,Du) = |Du|$, in case

Date: April 27, 2004.

2000 *Mathematics Subject Classification.* 47J20, 49N60, 35B65.

Key words and phrases. quasiminima, absolute minimizers, comparison with cones.

The author is supported by the Academy of Finland, project #80566.

[1]The terms “supremal functionals” and “L^∞ functionals” have also been used in this connection.

of which we already encounter the basic features of the theory but avoid some of the technicalities that could draw the attention away from the main ideas. Another reason for sticking to this special case is its interesting geometric interpretation; it corresponds to the real analysis problem of finding optimal Lipschitz extensions, see [4], [5], [20]. A reader interested in generalizing the results of this paper to more general functionals of the form (0.1) should consult [8] and [26].

A fundamental difference between the integral functionals and the supremum functionals is that the latter are not set additive. This means, in particular, that while a function u that minimizes $I(\cdot,\Omega)$ with given boundary data also minimizes $I(\cdot,V)$, subject to its own boundary values, for every subdomain $V\subset\Omega$, the same is not true for $S(\cdot,\Omega)$. Since it further turns out, see [5], [6], [20], that only imposing this "locality" leads to a satisfactory class of solutions for the supremum functionals, we are led to the concept of absolute minimizer, introduced by Aronsson [2]-[4]: a locally Lipschitz continuous function $u:\Omega\to\mathbb{R}$ is called an *absolute minimizer*[2] of $S(\cdot,\Omega)$, if $S(u,V)\le S(v,V)$ for every $V\subset\subset\Omega$ and $v\in W^{1,\infty}(\Omega)\cap C(\overline{V})$ such that $v|_{\partial V}=u|_{\partial V}$.

Using this concept, Jensen [20] showed that in the case $F(x,u,Du)=|Du|$ the Dirichlet problem is well-posed and that minimizers are characterized as viscosity solutions of the infinity Laplace equation

$$\Delta_\infty u := \sum_{i,j=1}^{n} u_i u_j u_{ij} = 0. \tag{0.2}$$

These results have been generalized in many ways, see e.g. [5], [6], [9], [21], [26] and the references therein.

It is important to observe that in the case of a supremum functional, the derivation of the Euler equation as well as any regularity properties for the solutions require genuinely new techniques. The known estimates for the absolute minimizers are obtained either by using an approximation scheme involving suitable integral functionals or by using the notion of comparison with cones, first introduced in [10]. As shown in [5], [10], comparison with cones provides a new, more geometric, view to the theory of absolutely minimizers and solutions of (0.2). It has also proved to be very useful in various related problems. For example, it extends to the framework of metric spaces and to the supremum functionals of the form $F(x,u,Du)=H(x,Du)$, see [5], [8], [15], [25], [26].

In this paper, we show that the notion of comparison with cones, suitably adapted, can be used to obtain estimates for the quasiminima of the Lipschitz extension problem governed by the functional $\operatorname{ess\,sup}_\Omega |Du|$. In particular, we prove a Harnack inequality, a Lipschitz continuity estimate and various stability and convergence results. These are all obtained quite easily once a characterization of quasiminima in terms of the comparison with cones property is established. Since the class of quasiminima also includes minima of the functionals of the form $\operatorname{ess\,sup} H(x,Du)$ and viscosity solutions of certain elliptic equations, the situation seems quite analogous to the case of integral functionals as far as results are concerned. However, there are some interesting open questions, regarding, for example, the exact relationship of quasiminimum and some related concepts. These are discussed in more detail below.

[2]In the case $F(x,u,Du)=|Du|$, "absolute minimizers" are often called "absolutely minimizing Lipschitz extensions" due to their geometric interpretation.

This paper is organized as follows. After this introduction, in Section 1, we define precisely what we mean by a quasiminimum of $S(\cdot,\Omega)$ and discuss some immediate consequences of the definition. The notion of comparison with cones is introduced in Section 2, where we also study the equivalence of the different concepts. The main regularity estimates are proved in Section 3, and we end the paper with a set of examples displayed in Section 4.

1. Definition of quasiminima

The usual definition of quasiminimum can formally be extended to the framework of supremum functionals without much difficulties.

Definition 1.1. We say that a locally Lipschitz continuous function $u:\Omega\to\mathbb{R}$ is a K-*quasiminimum* if

$$\operatorname*{ess\,sup}_{x\in V}|Du(x)|\le K\operatorname*{ess\,sup}_{x\in V}|Dv(x)|$$

for every open $V\subset\subset\Omega$ and for every Lipschitz continuous function $v:\overline{V}\to\mathbb{R}$ such that $u=v$ on ∂V.

The definition above is perhaps the most natural but not the only possible way to define quasiminima in our situation. Namely, it can be shown [5] that a locally Lipschitz continuous function u is an absolute minimizer (i.e., a 1-quasiminimum) in Ω if and only if

$$Lip(u,V)=Lip(u,\partial V)\qquad\text{for every open } V\subset\subset\Omega. \tag{1.1}$$

Here $Lip(w,E)$ denotes the least constant $L\in[0,+\infty]$ for which

$$|w(x)-w(y)|\le L|x-y|\quad\text{for all } x,y\in E;$$

in particular, $Lip(w,E)$ is defined to be $+\infty$ if w is not Lipschitz on the set E. Now (1.1) clearly offers an alternative way to define quasiminima.

Definition 1.2. We say that a locally Lipschitz continuous function u is a K-*quasiamle* if

$$Lip(u,V)\le K Lip(u,\partial V)$$

for every open $V\subset\subset\Omega$.

The terminology used above needs a short explanation. If $K=1$, then a function u satisfying (1.1) is often called an absolutely minimizing Lipschitz extension, or an AMLE for short. We have adopted this name here, although there will at no point be any reference to a function that is to be extended.

It readily follows from the definitions that quasiminima and quasiamles satisfy maximum and minimum principles. Since the proof is essentially the same for both classes, it is enough to present it for one of them.

Lemma 1.3. *Suppose that u is a K-quasiamle in Ω. Then u satisfies the minimum and maximum principles, i.e.,*

$$\sup_{x\in V}u(x)=\sup_{x\in\partial V}u(x),\qquad\textit{and}\qquad\inf_{x\in V}u(x)=\inf_{x\in\partial V}u(x)$$

for every open $V\subset\subset\Omega$.

Proof. We prove only the maximum principle. Suppose that for some $V \subset\subset \Omega$ and $x_0 \in V$ we have $u(x_0) > M > \sup_{x\in\partial V} u(x)$. Denoting

$$W = \{x \in V : u(x) > M\},$$

we have that

$$Lip(u, W) \leq K Lip(u, \partial W) = 0.$$

This contradicts the fact that $u(x_0) > M$. □

In [5], Proposition 4.5, it is shown that the property of being an absolute minimizer of the functional $v \mapsto \operatorname{ess\,sup}_\Omega |Dv|$ is completely local. However, if $K > 1$, then being a K-quasiminimum is not a local property. Indeed, given $1 < K < \infty$, one can easily check that each point $x \in]0, 1[$ has a neighborhood $U_{x,K}$ in which $u(x) = \sqrt{x}$ is a K-quasiminimum, but u is not a K'-quasiminimum in $]0, 1[$ for any $1 \leq K' < \infty$.

On the other hand, a Lipschitz function $w \in C(\overline{\Omega})$ that minimizes the sup-norm of the gradient globally, i.e.,

$$\operatorname*{ess\,sup}_{x\in\Omega} |Dw(x)| \leq \operatorname*{ess\,sup}_{x\in\Omega} |Dv(x)|$$

for any Lipschitz function $v \in C(\overline{\Omega})$ such that $v = w$ on $\partial\Omega$, need not be a K-quasiminimum for any $1 \leq K < \infty$. The reason is that such a function w does not, in general, satisfy the maximum and minimum principles.

2. Quasiminima and comparison with cones

As mentioned already in the introduction, a key step towards obtaining the regularity estimates is the characterization of the quasiminima in terms of a variant of the "comparison with cones" property introduced in [10].

Definition 2.1. We say that a function $u \in C(\Omega)$ enjoys *K-quasi-comparison with cones from above* if, whenever $V \subset\subset \Omega$ is open, $x_0 \notin V$, $a, b \in \mathbb{R}$, $a > 0$, are such that

$$u(x) \leq C(x) := a|x - x_0| + b \qquad \text{on } \partial V,$$

we have

$$u(x) \leq C_K(x) := Ka|x - x_0| + b \qquad \text{in } V.$$

Moreover, a function $u \in C(\Omega)$ is said to enjoy *K-quasi-comparison with cones from below* if $-u$ enjoys K-quasi-comparison with cones from above. Finally, a function $u \in C(\Omega)$ that enjoys K-quasi-comparison with cones both from above and from below is said to enjoy *K-quasi-comparison with cones*, and in that case we denote $u \in QCC_K(\Omega)$.

The reader should notice that in the definition of K-quasi-comparison with cones from above it is required that the slope a of the cone function $C(x)$ is positive. This is in accordance with the original definition of comparison with cones in [10], but differs from the formulation used for example in [5], where any slope $a \in \mathbb{R}$ is allowed. In the case $K = 1$ these two versions of the definition are equivalent, but for $K > 1$ they clearly are not. In this respect, the theory of quasiminima has similarities with the theory of absolute minimizers in metric spaces, cf. [8].

At this moment, we have three concepts whose mutual relations are of interest: quasiminima, quasiamle and quasi-comparison with cones. Our first result states that the last two are equivalent. To some extent the proof mimics that of the case $K = 1$ in [5], but some new ideas are needed as well.

Theorem 2.2. *A continuous function u is a K-quasiamle in $\Omega \subset \mathbb{R}^n$ if and only if $u \in QCC_K(\Omega)$.*

Proof. Let us first assume that u is a K-quasiamle in Ω, and fix an open set $V \subset\subset \Omega$ and a cone $C(x) = a|x - x_0| + b$, $x_0 \notin V$, $a, b \in \mathbb{R}$, $a > 0$, such that $u(x) \leq C(x)$ on ∂V. We wish to show that the set

$$W := \{x \in V : u(x) > Ka|x - x_0| + b\}$$

is empty. If this is not the case, then we fix a point $y \in W$ and let $\tilde{V}$ be the connected component of $\{x \in V : u(x) > C(x)\}$ containing y. Since $u = C$ on $\partial\tilde{V}$ and $x_0 \notin \tilde{V}$, we have $Lip(u, \partial\tilde{V}) = Lip(C, \partial\tilde{V}) = a$. Next we choose $z \in \partial\tilde{V}$ so that $|x_0 - y| = |x_0 - z| + |z - y|$; that is, z is located on the line segment joining x_0 and y. Then

$$u(y) > Ka|y - x_0| + b = (b + Ka|x_0 - z|) + Ka|z - y| \geq u(z) + Ka|z - y|,$$

which shows that $Lip(u, \tilde{V}) > Ka = KLip(u, \partial\tilde{V})$. This contradicts the fact that u is a K-quasiamle, and hence W must be empty. We have thus shown that u enjoys K-quasi-comparison with cones from above in Ω, and the comparison from below is established by applying the above argument to $-u$.

For the converse direction, we again argue by contradiction and suppose that there exists an open set $V \subset\subset \Omega$ and $x_1, x_2 \in V$ such that

$$u(x_1) - u(x_2) > K\, Lip(u, \partial V)|x_1 - x_2|; \tag{2.1}$$

notice that this forces $Lip(u, \partial V)$ to be finite. Let $w \in C(\overline{V})$ be a minimal Lipschitz extension of $u|_{\partial V}$ to V, i.e., $w = u$ on ∂V and $Lip(w, V) = Lip(u, \partial V)$. We divide the proof into four cases:

Case 1: $u(x_i) \geq w(x_i)$, $i = 1, 2$.
Let

$$C(x) = u(x_2) + \left(\max_{z\in\partial V} \frac{u(z) - u(x_2)}{|z - x_2|}\right) |x - x_2|$$

and notice that the slope of this cone is positive by (2.1) and Lemma 1.3. Since $u \leq C$ on $\partial(V \setminus \{x_2\})$ and $u \in QCC_K(\Omega)$, we have

$$u(x_2) + K\, Lip(u, \partial V)|x_1 - x_2| < u(x_1) \leq C_K(x_1).$$

Using the fact that $w(x_2) \leq u(x_2)$ and the properties of w this yields

$$\begin{aligned} K\, Lip(u, \partial V) &< K \max_{z\in\partial V} \frac{u(z) - u(x_2)}{|z - x_2|} \\ &\leq K \max_{z\in\partial V} \frac{w(z) - w(x_2)}{|z - x_2|} \leq KLip(u, \partial V), \end{aligned}$$

which is clearly a contradiction.

Case 2: $u(x_i) \leq w(x_i)$, $i = 1, 2$.
This case can be handled by applying the argument above to $-u$, $-w$ and using the fact that u enjoys quasi-comparison from below.

Case 3: $u(x_1) < w(x_1)$ and $u(x_2) > w(x_2)$.
This case is also easy to treat. Namely, from (2.1) it follows

$$K\,Lip(u,\partial V)|x_1 - x_2| < u(x_1) - u(x_2) < w(x_1) - w(x_2) \le Lip(u,\partial V)|x_1 - x_2|,$$

an obvious contradiction.

Case 4: $u(x_1) > w(x_1)$ and $u(x_2) < w(x_2)$.
Let us denote

$$V^+ = \{x \in V : u(x) > w(x)\}, \qquad V^- = \{x \in V : u(x) < w(x)\},$$

and let $z^\pm \in \partial V^\pm$ be points on the line segment joining x_1 and x_2 so that $[x_1, z^+[\subset V^+$ and $]z^-, x_2] \subset V^-$. Since $u = w$ on $\partial V^\pm$ and $Lip(w, V) = Lip(u, \partial V)$, we have by (2.1)

$$\begin{aligned} K\,Lip(u,\partial V)|x_1 - x_2| &< u(x_1) - u(x_2) \\ &= u(x_1) - u(z^+) + u(z^+) - u(z^-) + u(z^-) - u(x_2) \\ &\le u(x_1) - u(z^+) + Lip(u,\partial V)|z^+ - z^-| \\ &\quad + u(z^-) - u(x_2). \end{aligned}$$

This together with

$$|x_1 - x_2| = |x_1 - z^+| + |z^+ - z^-| + |z^- - x_2|$$

implies that

$$\max\left\{\frac{u(x_1) - u(z^+)}{|x_1 - z^+|}, \frac{u(z^-) - u(x_2)}{|z^- - x_2|}\right\} > K\,Lip(u,\partial V).$$

Thus we have reduced Case 4 to the Cases 1 and 2, and now also the second implication is proved. □

Next we show that the quasiminima are included in the set of quasiamles. The converse inclusion, however, remains as an interesting open problem. In the case $K = 1$ the equivalence of these two concepts is based on the "increasing slope estimate", see Lemma 2.18 in [5]. Its counterpart for the K-quasiminima says, roughly speaking, that in a similar situation the slope decreases at most by a multiplicative factor $1/K$, and this estimate does not behave well if iterated.

Theorem 2.3. *Every K-quasiminimum u is a K-quasiamle.*

Proof. By Theorem 2.2, it is enough to show that $u \in QCC_K(\Omega)$. To this end, we fix an open set $V \subset\subset \Omega$ and a cone function $C(x) = a|x - x_0| + b$, $a > 0$, $x_0 \notin V$ satisfying $u \le C$ on ∂V. Our aim is to show that $u(x) \le Ka|x - x_0| + b$ for $x \in V$.

If $u \le C$ in V, then we are clearly done, and so we may assume that the set

$$W := \{x \in V : u(x) > C(x)\}$$

is not empty. Observe that it suffices to show $u(x) \le Ka|x - x_0| + b$ for all $x \in W$. As u is a K-quasiminimum and $u = C$ on ∂W, we have

$$\operatorname*{ess\,sup}_{x\in W} |Du(x)| \le K \operatorname*{ess\,sup}_{x\in W} |C(x)| = Ka.$$

Thus, as u is Lipschitz continuous on $\overline{W}$,

$$Lip(u, W) = \max\left\{\operatorname*{ess\,sup}_{x\in W} |Du(x)|, Lip(u, \partial W)\right\} \le Ka.$$

Now for $y \in W$ we have

$$u(y) \le \inf_{z\in\partial W} (u(z) + Ka|y-z|) = \inf_{z\in\partial W} (C(z) + Ka|y-z|).$$

Choosing $z^* \in \partial W$ to be a point on the line segment joining x_0 and y thus gives

$$u(y) \le C(z^*) + Ka|z^* - y| \le Ka(|x_0 - z^*| + |z^* - y|) + b = Ka|x_0 - y| + b.$$

We have now shown that u enjoys quasi-comparison with cones from above, and comparison from below follows by applying the same argument to $-u$. □

Remark 2.4. Following the example of [22] and [5], it would be possible to define the "one-sided" notions of quasi-subminima and quasi-superminima for the supremum functionals, and investigate their relationship with the notions of quasi-comparison with cones from above and from below. We in fact already did this to a certain extent in course of proving Theorems 2.2 and 2.3 above, but for simplicity will not pursue this issue further in this paper.

3. Properties of quasiminima

In this section, we prove some regularity estimates and convergence results that all follow from the quasi-comparison with cones property. We begin by establishing a Harnack inequality and some of its immediate consequences. The proof is a straightforward adaptation of the original argument, due to Crandall [5], to the current setting.

Proposition 3.1. *Let $u \in LSC(\Omega)$ be a non-negative function that enjoys K-quasi-comparison with cones from below in Ω. If $x_0 \in \Omega$ and $0 < r < R \le \operatorname{dist}(x_0, \partial\Omega)$, then*

$$u(y) \le u(z)e^{\frac{K|y-z|}{R-r}} \qquad \text{for all } y, z \in B_r(x_0). \tag{3.1}$$

Moreover,

$$|Du(x)| \le \frac{K\,u(x)}{\operatorname{dist}(x,\partial\Omega)} \qquad \text{for a.e. } x \in \Omega. \tag{3.2}$$

Proof. Fix $y, z \in B_r(x_0)$ and for a positive integer m define

$$y_j = y + j\frac{z-y}{m} \qquad \text{for} \;\; j = 0, 1, \ldots, m.$$

We assume m is so large that $B_{|z-y|/m}(y_j) \subset\subset B_r(x_0)$ for $j = 0, 1, \ldots, m$. Since u is non-negative, the cone function

$$x \mapsto u(y_j)\left(1 - \frac{|x-y_j|}{s}\right), \qquad 0 < s < \operatorname{dist}(y_j, \partial\Omega)$$

is below u on $\partial(B_s(y_j) \setminus \{y_j\})$. Hence, as u enjoys quasi-comparison with cones from below, it follows that

$$u(x) \ge u(y_j)\left(1 - \frac{K|x-y_j|}{\delta(y_j)}\right)$$

for $x \in B_{\delta(y_j)}(y_j)$, where $\delta(y_j) = \operatorname{dist}(y_j, \partial\Omega)$. In particular, since $\delta(y_j) > R - r$, we have

$$u(y_{j+1}) \ge u(y_j)\left(1 - \frac{K|y_{j+1}-y_j|}{\delta(y_j)}\right) \ge u(y_j)\left(1 - \frac{K|y-z|}{m(R-r)}\right).$$

Iterating this relation, we conclude

$$\begin{aligned} u(z) = u(y_m) &\geq u(y_0)\left(1 - \frac{K|y-z|}{m(R-r)}\right)^m \\ &= u(y)\left(1 - \frac{K|y-z|}{m(R-r)}\right)^m . \end{aligned}$$

Letting $m \to \infty$ gives estimate (3.1).

In order to prove (3.2), we first note that if u is positive, then (3.1) implies

$$|\log u(y) - \log u(z)| \leq \frac{K|z-y|}{\operatorname{dist}(y, \partial\Omega) - r}$$

for $y \in \Omega$ and $z \in B_r(y)$, $0 < r < \operatorname{dist}(y, \partial\Omega)$. Thus $\log u$ is locally Lipschitz, as is u itself, and we have

$$|D \log u(x)| \leq \frac{K}{\operatorname{dist}(x, \partial\Omega)}$$

for a.e. $x \in \Omega$. The estimate (3.2) clearly follows. □

As usual, continuity estimates and strong maximum and minimum principles follow from the Harnack inequality. Below these results are stated for the quasiamles, but by Theorem 2.3 they hold for the quasiminima as well.

Corollary 3.2. *If u is a K-quasiamle in a domain Ω, then*

$$|Du(x)| \leq \frac{2K \sup_\Omega |u|}{\operatorname{dist}(x, \partial\Omega)} \qquad \text{for a.e. } x \in \Omega.$$

In particular, every K-quasiamle is locally Lipschitz continuous.

Corollary 3.3. *A non-constant K-quasiamle in a domain Ω cannot attain its supremum or infimum.*

Easy examples show that a K-quasiamle need not be differentiable everywhere in Ω if $K > 1$, and thus the Lipschitz continuity guaranteed by Corollary 3.2 is optimal. In the case $K = 1$ the question of optimal regularity is still largely open, although Savin [25] has recently showed that 1-quasiamles are C^1 in dimension $n = 2$.

As a consequence of the Harnack inequality and the "comparison with cones" -characterization in Theorem 2.2, we obtain various convergence and stability results for the quasiamles. The two model theorems below exhibit in particular the fact that the quasiamle -property is preserved under both uniform and monotone convergence. Moreover, as $K \to 1$, the quasiamles converge towards the genuine absolutely minimizing Lipschitz extensions. A direct proof for these facts seems hard to come by since both $Lip(v, V)$ and $\operatorname{ess\,sup}_V |Dv|$ are merely lower semicontinuous with respect to strong convergence in $L^\infty(V)$.

Theorem 3.4. *Let u_j be a K_j-quasiamle for $j = 1, 2, \ldots$, $K_j \to 1$, and suppose that there exists $C > 0$ such that*

$$\sup_{x \in \Omega} |u_j(x)| \leq C \qquad \text{for all } j = 1, 2, \ldots .$$

Then there is a subsequence (u_{j_k}) that converges locally uniformly to a function u that is a 1-quasiamle.

Proof. By the assumptions and Corollary 3.2, the sequence (u_j) is locally uniformly bounded and equicontinuous. Hence the theorem of Ascoli guarantees the existence of a subsequence, still denoted by (u_j), and a continuous function u such that $u_j \to u$ locally uniformly in Ω.

In order to show that u is a 1-quasiamle, let us fix an open set $V \subset\subset \Omega$ and a cone function $C(x) = a|x - x_0| + b$, $a > 0$, $x_0 \notin V$, so that $u \leq C$ on ∂V. For a given $\epsilon > 0$, since $u_j \to u$ uniformly in $\overline{V}$,

$$u_j(x) - \epsilon \leq u(x) \leq C(x) \qquad \text{for } x \in \partial V,$$

provided j is large enough. In other words, the cone function $x \mapsto a|x - x_0| + (b + \epsilon)$ is above u_j on ∂V. Since $u_j \in QCC_{K_j}(\Omega)$, this implies

$$u_j(x) \leq K_j a|x - x_0| + b + \epsilon \qquad \text{for } x \in V.$$

Letting $j \to \infty$ and then $\epsilon \to 0$ (and recalling that $K_j \to 1$), we obtain $u \leq C$ in V. Hence u enjoys comparison with cones from above and an analogous argument shows that u also enjoys comparison with cones from below. The claim now follows from Theorem 2.2. □

Theorem 3.5. *Suppose that (u_i) is a monotone sequence of K-quasiamles in a domain Ω. If the function $u = \lim_{i\to\infty} u_i$ is finite at some point $x \in \Omega$, then u is a K-quasiamle.*

Proof. If u is finite at some point $x \in \Omega$, it follows from the Harnack inequality, Prop. 3.1, that u is locally bounded in Ω. By Corollary 3.2 this implies that the sequence (u_i) is locally equicontinuous, and thus $u_i \to u$ locally uniformly in Ω. Now one can repeat the argument used in the proof of Theorem 3.4 above to conclude that $u \in QCC_K(\Omega)$. □

4. Examples

4.1. **The one-dimensional case.** In the case $n = 1$, Definition 1.1 can be rewritten in a more appealing way. Let $\Omega =]0, 1[\subset \mathbb{R}$. Then a locally Lipschitz continuous function u is a K-quasiminimum if and only if

$$\operatorname*{ess\,sup}_{a<x<b} |u'(x)| \leq K \frac{|u(b) - u(a)|}{|b - a|} \qquad \text{for all } 0 < a < b < 1. \tag{4.1}$$

This follows by observing that among all Lipschitz functions v that agree with u on the boundary of $]a, b[\subset]0, 1[$, the quantity $\operatorname{ess\,sup}_{a<x<b} |v'(x)|$ is minimized by the affine function whose graph connects the points $(a, u(a))$ and $(b, u(b))$. It is also clear that Definitions 1.1 and 1.2 are equivalent in the one-dimensional case.

The Harnack inequality, Proposition 3.1, readily implies that a one-dimensional K-quasiminimum on a domain Ω is either constant or strictly monotone. However, the derivative of a non-constant quasiminimum may vanish; the function $x \mapsto x^3$, which is a 6-quasiminimum on $]-1, 1[$, is an example.

4.2. **Solutions to certain PDEs are quasiminima.** It is by now well-known (see [5], [10] or [20]) that a continuous function u is an absolutely minimizing Lipschitz extension if and only if it is a viscosity solution to the infinity Laplace equation

$$-\sum_{i,j=1}^{n} u_i u_j u_{ij} = 0.$$

This is a special case of the connection between absolute minimizers of a functional of the form

$$\operatorname*{ess\,sup}_{x\in V} H(x,Du) \tag{4.2}$$

and the viscosity solutions of the partial differential equation

$$-\sum_{i,j}^{n} H_{\xi_i}(x,Du)H_{\xi_j}(x,Du)u_{ij} - H_{\xi_i}(x,Du)H_{x_i}(x,Du) = 0, \tag{4.3}$$

which is often called the Aronsson-Euler equation of (4.2). Conditions ensuring that an absolute minimizer is a solution to the Aronsson-Euler equation have been found in [6] and [9], but here we are mainly interested in the converse implication. In this respect, Yu [26] has recently proved the following

Theorem 4.1. *Let $H(x,\xi) \in C^2(\Omega\times\mathbb{R}^n)$ be convex in ξ and*

$$\lim_{|\xi|\to\infty} H(x,\xi) = +\infty \qquad \textit{uniformly in } \Omega.$$

Then any viscosity solution $u \in C(\Omega)$ of (4.3) in Ω is an absolute minimizer of the functional (4.2).

Now it is evident how to further connect the solutions of the Aronsson-Euler equation with the theory of quasiminima. If, in addition to coercivity and convexity in ξ, the function $H(x,\xi)$ satisfies

$$c_1|\xi|^q \le H(x,\xi) \le c_2|\xi|^q \qquad \text{for all } (x,\xi)\in\Omega\times\mathbb{R}^n$$

for some $c_2 \ge c_1 > 0$ and $q > 0$, then Theorem 4.1 implies that any viscosity solution $u \in C(\Omega)$ of (4.3) is a K-quasiminimum with $K = (c_2/c_1)^{1/q}$.

4.3. **Bilipschitz change of variables.** One way to construct examples of quasiminima is to use a bilipschitz change of variables. More precisely, if u is a K-quasiminimum in Ω and $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is L-bilipschitz, i.e.,

$$\frac{1}{L}|x-y| \le |\psi(x)-\psi(y)| \le L|x-y| \qquad \text{for all } x,y\in\mathbb{R}^n,$$

then it follows immediately that $w = u\circ\psi$ is a $K\cdot L^2$-quasiminimum in the open set $\psi^{-1}(\Omega)$. In particular, the function $x \mapsto |\psi(x)-z|$ is a L^2-quasiminimum in $\mathbb{R}^n \setminus \{\psi^{-1}(z)\}$.

Similarly, if $f : \mathbb{R}\to\mathbb{R}$ is L-bilipschitz and u is a K-quasiminimum in $\Omega\subset\mathbb{R}^n$, then $v = f(u)$ is a $K\cdot L^2$-quasiminimum in Ω. Note that since the identity $u(x) = x$ is an absolutely minimizing Lipschitz extension in $\mathbb{R}$, this means that in one dimension every L-bilipschitz function is an L^2-quasiminimum.

4.4. **Comparison with L^p-quasiminima.** The problem of finding an absolutely minimizing Lipschitz extension can be regarded as an asymptotic limit, as $p\to\infty$, of problems of finding minimizers of the p-Dirichlet integral

$$\int_\Omega |Dw|^p\,dx,$$

see e.g. [4], [5], [20]. Thus it is natural to compare quasiminima of the functional $\operatorname{ess\,sup}|Dw|$ with "L^p-quasiminima": we say that a function $u \in W^{1,p}_{loc}(\Omega)$ is a

(K,p)-*quasiminimum* in Ω, $1 < p < \infty$, if

$$\left(\int_V |Du|^p\,dx\right)^{1/p} \le K \left(\int_V |Dv|^p\,dx\right)^{1/p} \tag{4.4}$$

for every $V \subset\subset \Omega$ and for every v such that $u - v \in W_0^{1,p}(V)$.

In the one-dimensional case, with $\Omega =]0,1[$, the relation (4.4) can be written as

$$\left(\frac{1}{|b-a|}\int_a^b |u'|^p\,dx\right)^{1/p} \le K\frac{|u(b)-u(a)|}{|b-a|} \qquad \text{for all } 0 < a < b < 1. \tag{4.5}$$

Thus it follows from (4.1) that a K-quasiminimum is also a (K,p)-quasiminimum for every $1 < p < \infty$. The converse statement is clearly false. For example, the function $x \mapsto x^\alpha$, $\frac{1}{2} < \alpha < 1$, is not a K-quasiminimum on $]0,1[$, but it is a $(\frac{\alpha}{\sqrt{2\alpha-1}}, 2)$-quasiminimum. In higher dimensions, one can choose as the domain the punctured ball $B(0,1)\setminus\{0\}$ and consider the function $u(x) = |x|^{(p-n)/(p-1)}$. Since u is p-harmonic in Ω, i.e., a (weak) solution to the p-Laplace equation

$$-\operatorname{div}(|Dv|^{p-2}Dv) = 0,$$

it is a $(1,p)$-quasiminimum, but clearly not a K-quasiminimum for any $1 \le K < \infty$. It would be of interest to know whether there exists a K-quasiminimum that is not a (K',p)-quasiminimum for any $1 < p < \infty$, $K' \ge 1$; by the previous discussion, such a function is necessarily a function of more than one variable. This question seems to be connected with the validity of certain Lipschitz estimates for p-harmonic functions.

Finally, as regards the regularity properties, a key estimate that the (K,p)-quasiminima satisfy is the Caccioppoli inequality: there exists $C = C(p,K) > 0$ such that

$$\int_{B_r} |D(u-k)_\pm|^p\,dx \le \frac{C}{(R-r)^p}\int_{B_R} (u-k)_\pm^p\,dx$$

whenever $0 < r < R$, $B_R \subset\subset \Omega$, and $k \in \mathbb{R}$. For the K-quasiminima a completely analogous estimate follows from the proof of (3.2):

$$\operatorname*{ess\,sup}_{B_r} |D(u-k)_\pm| \le \frac{K}{R-r}\operatorname*{ess\,sup}_{B_R}(u-k)_\pm$$

for all $0 < r < R$ such that $B_R \subset\subset \Omega$, and for all $k \in \mathbb{R}$.

Acknowledgments. We would like to thank the anonymous referee for useful suggestions and comments and for pointing out an error in the previous version of the manuscript.

References

[1] Almansa, A., *Échantillonage, interpolation et détection. Applications en imagerie satellitaire*, Ph.D. thesis, E.N.S. de Cachan (2002).

[2] Aronsson, G., *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$, Ark. Mat. 6 (1965), 33–53.

[3] Aronsson, G., *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$. *II.*, Ark. Mat. 6 (1966), 409–431.

[4] Aronsson, G., *Extension of functions satisfying Lipschitz conditions*, Ark. Mat. 6 (1967), 551–561.

[5] Aronsson, G., Crandall, M. G., and Juutinen, P., *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505

[6] Barron, E. N., Jensen, R. R., and Wang, C. Y., *The Euler equation and absolute minimizers of L^∞ functionals*, Arch. Ration. Mech. Anal. 157 (2001), no. 4, 255–283.

[7] Caselles, V., Morel, J.-M.,, and Sbert, C., *An axiomatic approach to image interpolation*, IEEE Trans. Image Process. 7 (1998), no. 3, 376–386.

[8] Champion, T., and De Pascale L., *A principle of comparison with distance functions for absolute minimizers*, preprint (available at http://cvgmt.sns.it/papers/).

[9] Crandall, M. G., *An efficient derivation of the Aronsson equation*, Arch. Ration. Mech. Anal. 167 (2003), no. 4, 271–279.

[10] Crandall, M. G., Evans, L. C. and Gariepy, R. F., *Optimal Lipschitz extensions and the infinity Laplacian*, Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123–139.

[11] Crandall, M. G. and Evans, L. C., *A remark on infinity harmonic functions*, Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viã del Mar-Valparaiso, 2000), 123–129 (electronic), Electron. J. Differ. Equ. Conf., 6, Southwest Texas State Univ., San Marcos, TX, 2001.

[12] DiBenedetto, E., and Trudinger, N. S., *Harnack inequalities for quasiminima of variational integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 4, 295–308.

[13] Evans, L. C., *Estimates for smooth absolutely minimizing Lipschitz extensions*, Electron. J. Differential Equations 1993, No. 03, approx. 9 pp. (electronic only).

[14] Garroni, A., Nesi, V., and Ponsiglione, M., *Dielectric breakdown: optimal bounds*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 457 (2001), no. 2014, 2317–2335.

[15] Gaspari, T., *The infinity Laplacian in infinite dimensions*, Calc. Var. Partial Differential Equations 21 (2004), No. 3, 243–257.

[16] Giaquinta, M., and Giusti, E., *On the regularity of the minima of variational integrals*, Acta Math. 148 (1982), 31–46.

[17] Giaquinta, M., and Giusti, E., *Quasiminima*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 2, 79–107.

[18] Giusti, E., *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[19] Hassani, R., Ionescu, I. R., and Lachand-Robert, T., *Shape optimization and supremal minimization approaches in landslide modeling*, preprint.

[20] Jensen, R., *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Rational Mech. Anal. 123 (1993), no. 1, 51–74.

[21] Juutinen, P., *Minimization problems for Lipschitz functions via viscosity solutions*, Ann. Acad. Sci. Fenn. Math. Diss. No. 115 (1998), 53 pp.

[22] Kinnunen, J., and Martio, O., *Potential theory of quasiminimizers*, Ann. Acad. Sci. Fenn. Math. 28 (2003), no. 2, 459–490.

[23] Lindqvist, P., and Manfredi, J., *The Harnack inequality for ∞-harmonic functions*, Electron. J. Differential Equations 1995, No. 04, approx. 5 pp.

[24] Modica, G., *Quasiminima of some degenerate functionals*, Ann. Mat. Pura Appl. (4) 142 (1985), 121–143.

[25] Savin, O., *C^1 regularity for infinity harmonic functions in two dimensions*, Arch. Rational Mech. Anal. published online DOI: 10.1007/s00205-005-0355-8.

[26] Yu, Y. *Viscosity solutions of Aronsson equation*, preprint.

[27] Ziemer, W. P., *Boundary regularity for quasiminima*, Arch. Rational Mech. Anal. 92 (1986), no. 4, 371–382.

Department of Mathematics and Statistics, P.O.Box 35, FIN-40014 University of Jyväskylä, Finland

E-mail address: peanju@maths.jyu.fi