REMOVABILITY OF A LEVEL SET FOR SOLUTIONS OF QUASILINEAR EQUATIONS

PETRI JUUTINEN AND PETER LINDQVIST

ABSTRACT. In this paper, we study the removability of a level set for the solutions of quasilinear elliptic and parabolic equations of the second order. We show, under rather general assumptions on the coefficients of the equation, that if a function $u \in C^1(\Omega)$ is a viscosity solution to the equation in the set

$$\Omega \setminus \{x : u(x) = 0\},$$

then u is, in fact, a solution in the whole domain Ω. In addition to the linear equations in non-divergence form with Lipschitz coefficients, our results cover for example the p-Laplace equation, the minimal surface equation, the Burgers equation, and the heat equation.

1. INTRODUCTION

For the Laplace equation the following removability property is valid (see [2], [6], [19], [14]): *Suppose that u is continuously differentiable in a domain Ω in the Euclidean space $\mathbb{R}^n$. If the function u is harmonic in the set where $u \neq 0$, then u is harmonic in the whole* Ω. Thus the level set $\{x \in \Omega : u(x) = 0\}$ can be "removed". The object of our work is this phenomenon for a very wide class of quasilinear elliptic and parabolic partial differential equations of the second order.

Special cases for which our results hold are for example the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0,$$

the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad p \geq 2,$$

and the increasingly popular infinity Laplace equation

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

These are all of the general form

$$\sum_{i,j=1}^{n} A_{ij}(x, u, \nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} + B(x, u, \nabla u) = 0, \tag{1.1}$$

where the coefficients A_{ij} and B are continuous in all their variables, and the matrix $A = (A_{ij})$ is symmetric and positive semidefinite. It is obviously necessary to assume that $B(x, 0, 0) = 0$ in order to assure that $u \equiv 0$ is a solution. In the case when also $A_{ij}(x, 0, 0) = 0$ for every $i, j = 1, \ldots, n$, the removability of the zero level set is obtained in a rather straightforward manner. This assumption is not necessary and can be abandoned provided that the coefficients A_{ij} are independent

Date: November 17, 2004.
2000 *Mathematics Subject Classification.* Primary 35B60, 35J70, 35K55.
Key words and phrases. removability, quasilinear equations, viscosity solutions.
First author supported in part by the Academy of Finland, project #80566.

of u and Lipschitz continuous in x. The exact formulation of our results is somewhat involved and will be given in detail in Section 2 and in part of the parabolic case in Section 5.

Since the equations we consider are allowed to be very degenerate, nonlinear and of non-divergence form, the notions of classical solutions and distributional weak solutions are not general enough for the purposes of this paper. We will instead use the concept of viscosity solutions, assuming that the reader is at least somewhat familiar with this framework. It is a typical feature that all other reasonable solutions, whatever their definition, are included in the class of viscosity solutions. In particular, the classical solutions are exactly the smooth viscosity solutions and in the case of equations in divergence form with sufficiently smooth coefficients the distributional weak solutions and the viscosity solutions coincide. Hence we obtain removability results for these classes of solutions as well.

The essential difficulty in proving that a level set is removable lies in the fact that the size of this set is, in general, not known, at least not in advance. At the points where the gradient of u does not vanish, the level set is locally a C^1-hypersurface, and this case can be dealt with quite easily. Instead of using a coordinate transformation as in [13], [16], which is suitable for equations in divergence form, we find an argument similar to the proof of Hopf's maximum principle convenient. Observe that this establishes the removability of the entire level set if it is assumed that $A(x, 0, 0) = 0$, because then the equation is clearly satisfied in the viscosity sense at the critical points. In general, it is a more demanding task to show the removability of the points at which $\nabla u = 0$. However, the somewhat surprising relaxation given in Theorem 4.2 says that no testing at all is needed at these points. They pass for free so that the equation has to be verified only in the set $\{x \in \Omega : \nabla u(x) \neq 0\}$. This is formulated in terms of what we call *feeble viscosity solutions*, for lack of a better name, and the proof is based on a judicious choice of a penalization function in connection with the use of the so-called maximum principle for semicontinuous functions in [5].

Simple, essentially one dimensional examples show that the *a priori* regularity assumption $u \in C^1(\Omega)$ cannot be weakened to Lipschitz continuity. On the other hand, an assumption like $u \in C^2(\Omega)$ leads to total oversimplification. By regularity theory the solutions of many important equations are, actually, of class C^1, indicating that the assumption is natural.

In addition to the case of harmonic functions, the removability of a level set has been previously obtained at least for the solutions of certain linear elliptic equations [19], [15]. In [13] Kilpeläinen proved the corresponding result for the p-harmonic functions in the plane, and his result was later extended by the authors [11] to higher dimensions. We are not aware of any results of this type in the parabolic case.

Acknowledgments. We thank E. Malinnikova for informing us about the reference [19] and H. Hanche-Olsen for detecting a flaw in a previous version of the manuscript.

Notation. For the reader's convenience, we here list some notation that will be used throughout the paper. For vectors $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$,

$$\xi \cdot \eta = \sum_{i=1}^{n} \xi_i \eta_i$$

is the usual inner product, $|\xi| = (\xi \cdot \xi)^{1/2}$, and the tensor product $\xi \otimes \eta$ is the $n \times n$ matrix with the entries

$$(\xi \otimes \eta)_{ij} = \xi_i \eta_j \qquad \text{for } 1 \leq i, j \leq n.$$

By $S_{n\times n}$ we denote the (real) $n \times n$ symmetric matrices. This space is equipped with the inner-product

$$X \cdot Y := \operatorname{trace}(XY) = \sum_{i,j=1}^{n} X_{ij} Y_{ij}$$

for $X, Y \in S_{n\times n}$ and the corresponding norm $\|X\| := (X \cdot X)^{1/2}$. Moreover, in $S_{n\times n}$ there is a partial ordering: $X \leq Y$ if and only if $Y - X$ is positive semidefinite, that is, $(Y - X)\xi \cdot \xi \geq 0$ for every $\xi \in \mathbb{R}^n$.

The gradient of a function u and its Hessian matrix consisting of the second derivatives are denoted by ∇u and $D^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})_{ij}$, respectively.

For a set $A \subset \mathbb{R}^n$, ∂A and $\overline{A}$ denote the boundary and the closure of A, respectively. $B_r(x)$ is the open ball with the center at x and radius r.

2. Definitions and statements of the results

We find it convenient to write equation (1.1) in the form

$$-A(x, u, \nabla u) \cdot D^2 u - B(x, u, \nabla u) = 0. \tag{2.1}$$

The minus sign in front of the equation is a standard convention in the framework of viscosity solutions. We assume that the coefficients at least satisfy the following conditions:

- (I) $A = (A_{ij})$ is a symmetric and positive semidefinite matrix with continuous entries A_{ij},
- (II) B is continuous and $B(x, 0, 0) = 0$ for all $x \in \Omega$.

To be on the safe side, we give the definition of viscosity solutions.

Definition 2.1. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution* to (2.1) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$0 = u(x_0) - \phi(x_0) > u(x) - \phi(x) \qquad \text{for all } x \neq x_0,$$

then

$$-A(x_0, \phi(x_0), \nabla\phi(x_0)) \cdot D^2\phi(x_0) - B(x_0, \phi(x_0), \nabla\phi(x_0)) \leq 0.$$

A lower semicontinuous function $v : \Omega \to \mathbb{R}$ is a *viscosity supersolution* to (2.1) if, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that

$$0 = v(x_0) - \psi(x_0) < v(x) - \psi(x) \qquad \text{for all } x \neq x_0,$$

then

$$-A(x_0, \psi(x_0), \nabla\psi(x_0)) \cdot D^2\psi(x_0) - B(x_0, \psi(x_0), \nabla\psi(x_0)) \geq 0.$$

Finally, $u \in C(\Omega)$ is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

Observe that in the definition we associate to each point $x_0 \in \Omega$ its own family of test-functions touching from above (respectively, below), and that the family may very well be empty.

A standard reference to the general theory of viscosity solutions is [5], see also [4] and [7]. Here we merely record some facts that are relevant in the current study. First, as mentioned already in the introduction, the notion of viscosity solutions is consistent with the notion of classical solutions in the sense that $u \in C^2(\Omega)$ is a viscosity solution to (2.1) if and only if u satisfies the equation pointwise. This follows easily by Calculus because $A(x, u, \nabla u)$ is positive semidefinite. The relationship between viscosity solutions and distributional weak solutions for equations in divergence form is more complicated. In the case of an Euler-Lagrange equation of a variational problem with a convex integrand it is easy to see that distributional weak solutions are viscosity solutions, see e.g. [1], [9]. The reverse inclusion can be obtained, roughly speaking, provided that the standard Dirichlet problem for the divergence form equation admits a unique viscosity solution, see [12].

We now turn to the results of this paper. The statements below are about the removability of the zero level set, but it is clear how one has to adjust the assumptions in order to obtain similar conclusions for a generic level set $\{u = s\}$. Our first theorem deals with the case where the critical points $\{x \in \Omega : \nabla u(x) = 0\}$ are trivially removable due to the fact that the equation is automatically satisfied at these points. The proof is based on an argument similar to the classical Hopf's maximum principle, cf. [18], and is given in Section 3 below.

Theorem 2.2. *Suppose, in addition to (I) and (II), that* $A(x, 0, 0) = 0$ *for every* $x \in \Omega$. *If* $u \in C^1(\Omega)$ *is a viscosity solution to (2.1) in*

$$\Omega \setminus \{x \in \Omega : u(x) = 0\},$$

then it is a viscosity solution in the whole Ω.

Observe that the only regularity assumption on the coefficients is continuity. This theorem applies for example to the well-known p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in the range $2 < p < \infty$, in which case

$$A(x, s, \xi) = |\xi|^{p-2} I + (p-2)|\xi|^{p-4}\xi \otimes \xi,$$

and to its limit equation, the infinity Laplace equation $-\Delta_\infty u = 0$, for which

$$A(x, s, \xi) = \xi \otimes \xi.$$

However, the ordinary Laplace equation is not covered by this theorem.

The rather restrictive condition $A(x, 0, 0) = 0$, which is not valid even for the linear equations, can easily be relaxed in the case when $A(x, u, \nabla u)$ does not depend explicitly on u. In other words, $A = A(x, \xi)$. We need to assume some local regularity for the coefficient matrix A near $\xi = 0$. Let us denote by $\mathbb{A}$ the symmetric positive semidefinite square root of A, that is, $\mathbb{A}\mathbb{A} = A$. We assume that

for every $x_0 \in \Omega$ there exist $\delta > 0$ and $C > 0$ such that

$$\|\mathbb{A}(x, \xi) - \mathbb{A}(y, \xi)\|^2 \le C|x - y| \tag{2.2}$$

whenever $x, y \in B_\delta(x_0)$ and $|\xi| < \delta$.

Observe that since the mapping $A \mapsto \mathbb{A}$ is Hölder continuous with exponent $1/2$, (2.2) holds if

$$\|A(x, \xi) - A(y, \xi)\| \le \tilde{C}|x - y|$$

whenever $x, y \in B_\delta(x_0)$ and $|\xi| < \delta$.

Theorem 2.3. *Suppose, in addition to (I) and (II), that the square root* $\mathbb{A}(x, \xi)$ *satisfies condition (2.2). If* $u \in C^1(\Omega)$ *is a viscosity solution to (2.1) in*

$$\Omega \setminus \{x \in \Omega : u(x) = 0\},$$

then it is a viscosity solution in the whole Ω.

Theorem 2.3 follows from the proof of Theorem 2.2 and from a removability result for the critical points. The latter is formulated and proved in Section 4 below.

Let us now analyze condition (2.2) in certain special cases. First, in the linear case $A(x, \xi) = A(x)$, (2.2) is satisfied if the coefficients of the equation (2.1) are locally Lipschitz continuous. Also linear equations in divergence form like

$$-\operatorname{div}(A(x)\nabla u(x)) = 0$$

are within the scope of our results if $A \in C^1(\Omega)$, because then they can be written as

$$-A(x) \cdot D^2u(x) - \sum_{i=1}^{n} \big(\frac{\partial A^i(x)}{\partial x_i} \cdot \nabla u(x) \big) = 0,$$

where A^i denotes the i^{th} row of A. Observe that the lower order term vanishes at the points where $\nabla u(x) = 0$.

Another interesting special class is $A(x, \xi) = A(\xi)$, in which case (2.2) is trivially true. Thus Theorem 2.3 applies for example to the minimal surface equation

$$-\Delta u + \frac{(\nabla u \otimes \nabla u) \cdot D^2 u}{1 + |\nabla u|^2} = 0,$$

as well as to all other equations of the form

$$-\operatorname{div} F(\nabla u) = 0$$

provided $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and the matrix $(\frac{\partial F_i}{\partial \xi_j})_{ij}$ is positive semidefinite. Levi's equation $-A(\nabla u) \cdot D^2 u = 0$ with

$$A(\xi) = \begin{pmatrix} 1+\xi_3^2 & 0 & \xi_3\xi_1 - \xi_2 \\ 0 & 1+\xi_3^2 & \xi_3\xi_2 + \xi_1 \\ \xi_3\xi_1 - \xi_2 & \xi_3\xi_2 + \xi_1 & \xi_1^2 + \xi_2^2 \end{pmatrix},$$

see e.g. [3], provides an example that is of non-divergence form.

Finally, we want to point out also that our arguments can be even applied to equations that are singular at the points where $\nabla u = 0$. The notion of feeble viscosity solution that we define and utilize in Section 4 adapts naturally to these equations and it follows from the proof of Theorem 2.2 that a level set of any continuously differentiable feeble viscosity solution of (2.1) is removable, even if (2.1) is singular. This observation is significant, because for some singular equations in divergence form it can be shown that feeble viscosity solutions and distributional weak solutions coincide. A primary example of this is the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad 1 < p < \infty,$$

which is singular for $1 < p < 2$. In this case the proof of equivalence is given in [12], and the argument therein can be adjusted to cover various other equations as well. See also [8] and [17].

Remark 2.4. The a priori regularity assumption $u \in C^1(\Omega)$ is optimal in both theorems above in the sense that the conclusion is not true in general for Lipschitz functions. To illustrate this, let us suppose $B(x, s, \xi) \equiv 0$ and that there exists $\xi_0 \neq 0$ such that

$$A(x, 0, \xi_0)\xi_0 \cdot \xi_0 > 0$$

for all $x \in \mathbb{R}^n$ for which $x \cdot \xi_0 = 0$; this condition holds for example in the case of the Laplacian for any $\xi_0 \neq 0$. Let

$$u(x) = 2|x \cdot \xi_0|.$$

Then $D^2u(x) = 0$ in $\mathbb{R}^n \setminus \{x : x \cdot \xi_0 = 0\}$, which implies that u is a viscosity solution to (2.1) in $\mathbb{R}^n \setminus \{x : u(x) = 0\}$. However, if

$$\phi(x) = x \cdot \xi_0 + \frac{1}{2}(\xi_0 \otimes \xi_0)x \cdot x,$$

then $u(x) = \phi(x) = 0$ whenever $x \cdot \xi_0 = 0$ and $u \geq \phi$ in some open neighborhood of $\{x \in \mathbb{R}^n : x \cdot \xi_0 = 0\}$, but

$$-A(x, \phi(x), \nabla\phi(x)) \cdot D^2\phi(x) = -A(x, 0, \xi_0)\xi_0 \cdot \xi_0 < 0$$

if $x \cdot \xi_0 = 0$. Thus u is not a viscosity supersolution of (2.1) in the whole $\mathbb{R}^n$.

Another example displaying the same feature can be constructed using the eigenfunctions of the Laplace operator. Let λ_2 denote the second eigenvalue of a bounded domain Ω and let $\varphi_2 \not\equiv 0$ be any eigenfunction associated to λ_2. For $\alpha, \beta > 0$ we set

$$v = \alpha\varphi_2^+ - \beta\varphi_2^-,$$

where $\varphi_2^+ = \max\{\varphi_2, 0\}$ and $\varphi_2^- = -\min\{0, \varphi_2\}$ are the positive and negative part of φ_2. Then clearly $-\Delta v = \lambda_2 v$ in the set $\{x \in \Omega : v(x) \neq 0\}$, but, in general, v does not satisfy the equation in the whole Ω. Indeed, v is orthogonal to the first eigenfunction φ_1 if and only if $\alpha = \beta$. Observe that in the case of this eigenvalue equation the zero level set has a special role because $u \equiv 0$ is the only constant functions that satisfies the equation.

Remark 2.5. Although Theorems 2.2 and 2.3 are formulated for the viscosity solutions of (2.1), we actually prove the removability also for the viscosity subsolutions and the viscosity supersolutions. More precisely, it follows from our proofs that under the assumptions of either theorem, if $u \in C^1(\Omega)$ is a viscosity subsolution (respectively, supersolution) to (2.1) in

$$\Omega \setminus \{x \in \Omega : u(x) = 0\},$$

then it is a viscosity subsolution (supersolution) to (2.1) in the whole Ω.

3. The non-critical case

In this section, we prove Theorem 2.2. The argument below establishes the removability of the set

$$\{x \in \Omega : u(x) = 0 \text{ and } \nabla u(x) \neq 0\},$$

and by the assumption

$$\begin{cases} A(x,0,0) = 0 \\ B(x,0,0) = 0 \end{cases} \quad \text{for every } x \in \Omega, \tag{3.1}$$

this is enough for the conclusion.

Proof of Theorem 2.2. The proof is indirect. Suppose the conclusion of the theorem is not true. Then u cannot be both a viscosity subsolution and a viscosity supersolution. We may assume, without loss of generality, that u is not a viscosity subsolution of (2.1). This means that there exist a point $\hat{x} \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u(\hat{x}) = \phi(\hat{x}) = 0$, $u \leq \phi$ in Ω, and

$$-A(\hat{x}, 0, \nabla\phi(\hat{x})) \cdot D^2\phi(\hat{x}) - B(\hat{x}, 0, \nabla\phi(\hat{x})) > 0. \tag{3.2}$$

By (3.1), this implies that $\nabla\phi(\hat{x}) \neq 0$. Using continuity, the assumption that $u \in C^1(\Omega)$, and the fact that $\nabla(u - \phi)$ vanishes at $\hat{x}$, we conclude that there exist a small ball $B_r(\hat{x})$ of radius $r > 0$ and center at $\hat{x}$ and a constant $c > 0$ such that

$$\begin{cases} |\nabla\phi(x)| \geq c \\ |\nabla u(x)| \geq c \\ -A(x,\phi,\nabla\phi) \cdot D^2\phi - B(x,\phi,\nabla\phi) \geq c \end{cases} \quad \text{in } B_r(\hat{x}). \tag{3.3}$$

Next we define a linear differential operator L by setting

$$Lv(x) := -\mathcal{A}(x) \cdot D^2 v(x) - f(x),$$

where

$$\mathcal{A}(x) = A(x, u(x), \nabla u(x))$$

and

$$f(x) = B(x, u(x), \nabla u(x))$$

are continuous in Ω. Then $Lu = 0$ in $\Omega \setminus \{u = 0\}$ in the viscosity sense. Indeed, if $\psi \in C^2(\Omega)$ touches u from above (below) at a point $x \in \Omega \setminus \{u = 0\}$, then

$$L\psi(x) = -A(x, \psi, \nabla\psi) \cdot D^2\psi(x) - B(x, \psi, \nabla\psi) \leq 0, \quad \text{(respectively, } \geq 0)$$

since $u(x) = \psi(x)$ and $\nabla u(x) = \nabla\psi(x)$. Moreover, because

$$\begin{cases} \|\mathcal{A}(x) - A(x, \phi, \nabla\phi)\| \to 0 \\ |f(x) - B(x, \phi, \nabla\phi)| \to 0 \end{cases} \quad \text{as } x \to \hat{x}$$

and (3.3) holds, there exists a small $\varepsilon > 0$ such that $L\phi \geq \frac{c}{2}$ in $B_\varepsilon(\hat{x})$. Since the equation is degenerate elliptic, this holds also in the viscosity sense.

Let $\theta = \phi - u$. Then

$$\begin{cases} -\mathcal{A}(x) \cdot D^2\theta \geq \frac{c}{2} > 0 & \text{in } B_\varepsilon(\hat{x}) \setminus \{u = 0\} \text{ in the viscosity sense,} \\ \theta(\hat{x}) = 0, & \\ \theta(x) \geq 0 & \text{in } B_\varepsilon(\hat{x}). \end{cases} \tag{3.4}$$

In particular, $\nabla\theta(\hat{x}) = 0$. Since $\nabla\phi(\hat{x}) \neq 0$ and $\phi \in C^2(\Omega)$, the implicit function theorem implies that the level set $\{x : \phi(x) = 0\}$ is locally a graph of a C^2 function. This in turn implies that there exists a ball $B_\rho(z)$ contained in the set $\{\phi < 0\}$ tangent to the level set $\{\phi = 0\}$ at the point $\hat{x}$. More precisely, we have

$$B_\rho(z) \subset \big(\{\phi < 0\} \cap B_\varepsilon(\hat{x})\big) \subset \big(\{u < 0\} \cap B_\varepsilon(\hat{x})\big)$$

and

$$\partial B_\rho(z) \cap \{\phi = 0\} \; = \; \partial B_\rho(z) \cap \{u = 0\} \; = \; \{\hat{x}\},$$

see Figure 3.1.

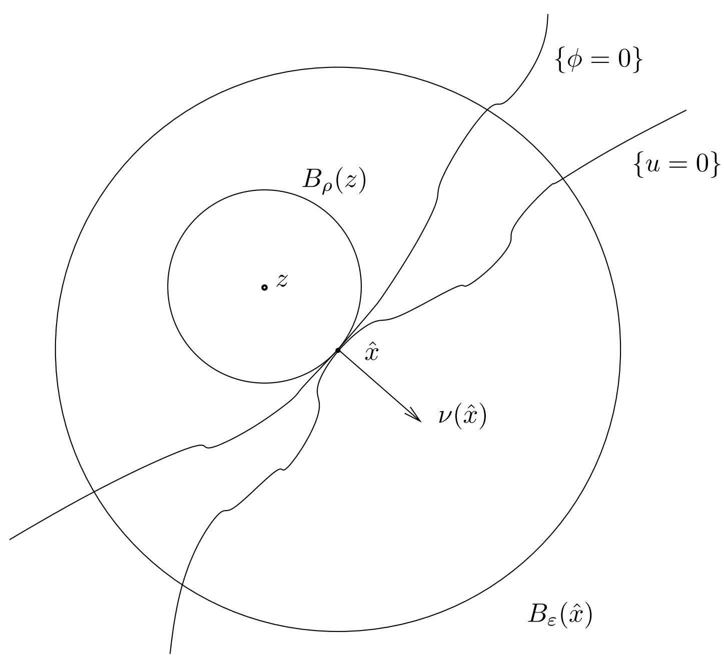

FIGURE 3.1

Let

$$w(x) = \frac{\sigma}{2}\left(\rho^2 - |x - z|^2\right), \qquad \sigma > 0.$$

Then

$$-\mathcal{A}(x) \cdot D^2 w(x) = \sigma \ \mathrm{trace}(\mathcal{A}(x)) \leq \frac{c}{4} \qquad \text{in } B_\rho(z) \tag{3.5}$$

if σ is chosen small enough. Notice that $w = 0 \leq \theta$ on $\partial B_\rho(z)$. We claim that $w \leq \theta$ also in $B_\rho(z)$. Indeed, if

$$\theta(y_0) - w(y_0) = \min_{y \in B_\rho(z)} \big(\theta(y) - w(y)\big) < 0$$

for some $y_0 \in B_\rho(z)$, i.e., $w + (\theta(y_0) - w(y_0))$ touches θ from below at y_0, then by (3.4) we have

$$-\mathcal{A}(y_0) \cdot D^2 w(y_0) \geq \frac{c}{2},$$

a contradiction with (3.5). Hence $w \leq \theta$ in $B_\rho(z)$. Since $\theta(\hat{x}) = w(\hat{x}) = 0$, we must have

$$\frac{\partial \theta}{\partial \nu}(\hat{x}) \leq \frac{\partial w}{\partial \nu}(\hat{x}) = -\sigma\rho < 0,$$

where ν denotes the exterior normal to $B_\rho(z)$ at $\hat{x}$. But this is impossible, because $\nabla\theta(\hat{x}) = 0$ by (3.4). We conclude that u is a solution to (2.1) in Ω. □

Remark 3.1. A quick look at the proof above reveals that our argument relies to a great extent on the linearization of (2.1) near $\hat{x}$. Due to the quasilinear structure, the coefficients of the linearized equation are independent of the second order derivatives of u (which, of course, are not even known to exist under the assumptions of the theorem). The situation is quite different for a general fully nonlinear equation.

4. The critical case

In the proof of Theorem 2.2, we used the assumption (3.1), that is, $A_{ij}(x, 0, 0) = B(x, 0, 0) = 0$, only to deduce from (3.2) that the gradient of the test-function ϕ does not vanish at the point $\hat{x}$, where $u - \phi$ has a local maximum. Next we will show that a similar conclusion can be made even without (3.1) provided the coefficients A_{ij} of the equation are sufficiently nice.

So the task at hand is to show that under suitable assumptions, viscosity subsolutions and supersolutions of (2.1) can be detected using only test-functions whose gradient does not vanish. To formalize this, we introduce the notions of feeble viscosity subsolutions and supersolutions.

Definition 4.1. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a *feeble viscosity subsolution* to (2.1) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

(i) $0 = u(x_0) - \phi(x_0) > u(x) - \phi(x)$ for all $x \neq x_0$,
(ii) $\nabla\phi(x_0) \neq 0$,

then

$$-A(x_0, \phi(x_0), \nabla\phi(x_0)) \cdot D^2\phi(x_0) - B(x_0, \phi(x_0), \nabla\phi(x_0)) \leq 0. \tag{4.1}$$

A lower semicontinuous function $v : \Omega \to \mathbb{R}$ is a *feeble viscosity supersolution* to (2.1) if, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that

(i) $0 = v(x_0) - \psi(x_0) < v(x) - \psi(x)$ for all $x \neq x_0$,
(ii) $\nabla\psi(x_0) \neq 0$,

then

$$-A(x_0, \psi(x_0), \nabla\psi(x_0)) \cdot D^2\psi(x_0) - B(x_0, \psi(x_0), \nabla\psi(x_0)) \geq 0. \tag{4.2}$$

Finally, $u \in C(\Omega)$ is a *feeble viscosity solution* if it is both a feeble viscosity subsolution and a feeble viscosity supersolution.

Observe that the only difference to the standard definition given in Definition 2.1 is that nothing is required if it so happens that $\nabla\phi(x_0) = 0$ (respectively, $\nabla\psi(x_0) = 0$). In order to avoid any misinterpretations, we will sometimes talk about *ordinary viscosity solutions* when we refer to Definition 2.1. Our aim is to show that under suitable assumptions on the coefficients, the feeble and the ordinary viscosity solutions coincide. Notice that this is evidently the case if $A_{ij}(x, s, 0) = B(x, s, 0) = 0$ for all $(x, s) \in \Omega \times \mathbb{R}$. On the other hand, since every constant function is always a feeble viscosity solution, the two definitions can be equivalent only if $B(x, s, 0) = 0$ for all x and s.

For the equivalence of the two notions, we require that A is independent of u and the structural assumption from Section 2 is valid, that is, for every $x_0 \in \Omega$ there exist $\delta > 0$ and $C > 0$ such that

$$\|\mathbb{A}(x, \xi) - \mathbb{A}(y, \xi)\|^2 \leq C|x - y| \tag{4.3}$$

whenever $x, y \in B_\delta(x_0)$ and $|\xi| < \delta$. We remind the reader that $\mathbb{A}$ denotes the positive semidefinite square root of A.

Theorem 4.2. *Suppose that $A(x, \xi)$ satisfies (4.3) and that $B(x, s, 0) = 0$ for all $x \in \Omega$ and $s \in \mathbb{R}$. Then every continuous feeble viscosity subsolution to (2.1) is an ordinary viscosity subsolution.*

Proof. We argue by contradiction. If the conclusion is false for a feeble viscosity subsolution $u \in C(\Omega)$, then there exist $\hat{x} \in \Omega$ and $\phi \in C^2(\Omega)$ so that $u(\hat{x}) = \phi(\hat{x})$, $u - \phi$ has a strict (global) maximum at $\hat{x}$, and

$$-A(\hat{x}, \nabla\phi(\hat{x})) \cdot D^2\phi(\hat{x}) - B(\hat{x}, \phi(\hat{x}), \nabla\phi(\hat{x})) = \mu > 0. \tag{4.4}$$

Observe that necessarily $\nabla\phi(\hat{x}) = 0$, because u is assumed to be a feeble viscosity subsolution. The strategy of the proof is, roughly speaking, to produce for a suitable perturbation of $u - \phi$ a local maximum point that is near $\hat{x}$ and at which there is a test-function with a non-vanishing gradient, and then to derive a contradiction.

We begin by doubling the variables and adding an adequate penalization. Consider the functions

$$w_j(x, y) = u(x) - \phi(y) - \psi_j(x, y), \quad j = 1, 2, \ldots,$$

where

$$\psi_j(x, y) = \tfrac{j}{q}|x - y|^q, \quad q > 2,$$

and let (x_j, y_j) be a point where w_j achieves its maximum in $\overline{\Omega} \times \overline{\Omega}$. By standard arguments, see [5] or [4],

$$\psi_j(x_j, y_j) \to 0 \ \text{ and } \ (x_j, y_j) \to (\hat{x}, \hat{x}) \ \text{ as } \ j \to \infty.$$

In particular, for j sufficiently large, x_j and y_j are both interior points of Ω. Since

$$u(x) - \phi(y) - \psi_j(x, y) \leq u(x_j) - \phi(y_j) - \psi_j(x_j, y_j) \tag{4.5}$$

for all $x, y \in \Omega$, we obtain by choosing $x = x_j$ that

$$\phi(y) \geq -\psi_j(x_j, y) + \phi(y_j) + \psi_j(x_j, y_j)$$

for all $y \in \Omega$. Let us denote the right-hand side of the above inequality as θ_j, that is,

$$\theta_j(y) = -\psi_j(x_j, y) + \phi(y_j) + \psi_j(x_j, y_j).$$

Since $\phi - \theta_j$ has a (global) minimum at y_j,

$$\begin{aligned} \nabla\phi(y_j) &= \nabla\theta_j(y_j) = j|z_j|^{q-2}z_j, \\ D^2\phi(y_j) \geq D^2\theta_j(y_j) &= -j|z_j|^{q-2}I - j(q-2)|z_j|^{q-4}z_j \otimes z_j, \end{aligned}$$

where we have denoted $z_j = x_j - y_j$. Now, if we had $x_j = y_j$, then $\nabla\phi(y_j) = 0$ and $D^2\phi(y_j) \geq 0$, which would imply

$$-A(y_j, \nabla\phi(y_j)) \cdot D^2\phi(y_j) - B(y_j, \phi(y_j), \nabla\phi(y_j)) = -A(y_j, 0) \cdot D^2\phi(y_j) \leq 0.$$

Since $y_j \to \hat{x}$ as $j \to \infty$ and $\phi \in C^2(\Omega)$, this contradicts (4.4) for j large enough. Thus $z_j \neq 0$, and consequently

$$\eta_j := \nabla_x w_j(x_j, y_j) = -\nabla_y w_j(x_j, y_j) = j|z_j|^{q-2} z_j$$

is a non-zero vector, at least for all j sufficiently large. On the other hand, since $\eta_j = \nabla\phi(y_j) \to \nabla\phi(\hat{x}) = 0$ as $j \to \infty$, we have

$$j|z_j|^{q-1} \to 0 \qquad \text{as } j \to \infty, \tag{4.6}$$

which improves the initial estimate on the rate of convergence for the difference $z_j = x_j - y_j$.

Next we will apply "the maximum principle for semicontinuous functions"[1] from [5]. For that purpose, we need to introduce some notation. Given a function $v : \Omega \to \mathbb{R}$ and $z \in \Omega$, we define the "semijets" of v at z by setting

$$J^{2,+}v(z) = \big\{(\nabla\psi(z), D^2\psi(z)) : \psi \in C^2(\Omega),\ v - \psi \text{ has a local maximum at } z\big\},$$

and

$$J^{2,-}v(z) = \big\{(\nabla\psi(z), D^2\psi(z)) : \psi \in C^2(\Omega),\ v - \psi \text{ has a local minimum at } z\big\}.$$

Note that this is not the standard way to define the semijets. Nevertheless, it yields precisely the same set-valued functions as the usual definition in [5], [4], see e.g. Proposition 1 in [7]. In addition to jets, we need their closures. We say that $(p, X) \in \overline{J}^{2,\pm}v(z)$ if there exists $z_k \in \Omega$ and $(p_k, X_k) \in J^{2,\pm}v(z_k)$ such that $(z_k, v(z_k), p_k, X_k) \to (z, v(z), p, X)$ as $k \to \infty$. The relevance of these concepts in the theory of viscosity solutions is quite clear. Indeed, it is easy to see that an upper semicontinuous function v is an ordinary viscosity subsolution to (2.1) if and only if

$$-A(x, v(x), p) \cdot X - B(x, v(x), p) \leq 0 \qquad \text{for all } x \in \Omega \text{ and } (p, X) \in \overline{J}^{2,+}v(x).$$

An analogous remark naturally holds for the supersolutions. We refer the reader to [4] and [5] for further properties of jets.

Now we are ready to employ the maximum principle for semicontinuous functions. Since (x_j, y_j) is a local maximum point of $w_j(x, y)$, this maximum principle implies that there exist $n \times n$ symmetric matrices X_j, Y_j such that

$$\begin{aligned} (\eta_j, X_j) &\in \overline{J}^{2,+}u(x_j), \\ (\eta_j, Y_j) &\in \overline{J}^{2,-}\phi(y_j), \end{aligned} \tag{4.7}$$

and

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq D^2\psi_j(x_j, y_j) + \frac{1}{j}\left[D^2\psi_j(x_j, y_j)\right]^2. \tag{4.8}$$

Recalling the definition of ψ_j, (4.8) can be rewritten as

$$\begin{aligned} \begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq{}& j(|z_j|^{q-2} + 2|z_j|^{2q-4}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\ &+ j(q-2)(|z_j|^{q-4} + 2q|z_j|^{2q-6}) \begin{pmatrix} z_j \otimes z_j & -z_j \otimes z_j \\ -z_j \otimes z_j & z_j \otimes z_j \end{pmatrix}. \end{aligned}$$

[1]This is also known as "the Theorem on Sums".

By evaluating the corresponding quadratic forms at $\begin{pmatrix}\xi\\ \zeta\end{pmatrix} \in \mathbb{R}^{2n}$, we obtain

$$
\begin{aligned}
X_j\xi\cdot\xi - Y_j\zeta\cdot\zeta \le\ & j\left[|z_j|^{q-2} + 2|z_j|^{2q-4}\right]|\xi-\zeta|^2 \\
& + j(q-2)\left[|z_j|^{q-4} + 2q|z_j|^{2q-6}\right](z_j\cdot(\xi-\zeta))^2 \\
\le\ & j\left[(q-1)|z_j|^{q-2} + 2(q-1)^2|z_j|^{2(q-2)}\right]|\xi-\zeta|^2
\end{aligned}
\tag{4.9}
$$

for all $\xi,\zeta\in\mathbb{R}^n$.

Next we use (4.7). Recalling that $\eta_j \neq 0$ and the assumption that u is a feeble viscosity subsolution of (2.1), we have

$$
\begin{aligned}
0 &\le A(x_j,\eta_j)\cdot X_j + B(x_j,u(x_j),\eta_j) \\
&= \sum_{i=1}^{n}\left(X_j\mathbb{A}^i(x_j,\eta_j)\cdot\mathbb{A}^i(x_j,\eta_j)\right) + o(1),
\end{aligned}
$$

where $\mathbb{A}^i(x_j,\eta_j)$ denotes the i^{th} column of the symmetric square root of $A(x_j,\eta_j)$. The estimate on the lower order term $B(x_j,u(x_j),\eta_j)$ follows from the fact that $(x_j,u(x_j),\eta_j)\to(\hat{x},u(\hat{x}),0)$ as $j\to\infty$ and the assumption $B(x,t,0)=0$. On the other hand, by (4.4),

$$
\begin{aligned}
0 < \mu/2 &\le\ -A(y_j,\eta_j)\cdot Y_j - B(y_j,\phi(y_j),\eta_j) \\
&=\ -\sum_{i=1}^{n}\left(Y_j\mathbb{A}^i(y_j,\eta_j)\cdot\mathbb{A}^i(y_j,\eta_j)\right) + o(1)
\end{aligned}
$$

for all j large enough. Hence, by using the structural assumption (4.3) together with (4.9), we obtain

$$
\begin{aligned}
0 < \frac{\mu}{2} &\le \sum_{i=1}^{n}\left(X_j\mathbb{A}^i(x_j,\eta_j)\cdot\mathbb{A}^i(x_j,\eta_j) - Y_j\mathbb{A}^i(y_j,\eta_j)\cdot\mathbb{A}^i(y_j,\eta_j)\right) + o(1) \\
&\le j\left[(q-1)|z_j|^{q-2} + 2(q-1)^2|z_j|^{2(q-2)}\right]\|\mathbb{A}(x_j,\eta_j)-\mathbb{A}(y_j,\eta_j)\|^2 + o(1) \\
&\le Cj|z_j|^{q-1} + o(1)
\end{aligned}
\tag{4.10}
$$

for some constant C independent of j. By (4.6), the right hand side tends to zero as $j\to\infty$, and thus we have finally reached a contradiction. $\square$

Remark 4.3. Although Theorem 4.2 above is formulated for continuous feeble viscosity subsolutions, the continuity assumption does not play a really significant role in the proof. In fact, the fact that $u\in C(\Omega)$ was used only to guarantee that $u(x_j)\to u(\hat{x})=\phi(\hat{x})$ as $j\to\infty$, which in turn is needed in order to conclude that $B(x_j,u(x_j),\eta_j)$ tends to zero as $j\to\infty$. Thus the conclusion of the theorem holds for any feeble viscosity subsolution if we assume, for example, that the lower order term $B(x,s,\xi)$ does not depend on the variable s.

For continuously differentiable solutions of (2.1), Theorem 4.2 can be reformulated as a removability result:

Corollary 4.4. *Suppose that the coefficient matrix $A(x,\xi)$ satisfies (4.3) and that $B(x,s,0)=0$ for all $x\in\Omega$ and $s\in\mathbb{R}$. If $u\in C^1(\Omega)$ is a viscosity solution to (2.1) in the set*

$$\Omega\setminus\{x\in\Omega : \nabla u(x)=0\},$$

then u is a viscosity solution in the whole Ω.

Proof. Since u is an ordinary viscosity solution to (2.1) in $\{x \in \Omega : \nabla u(x) \neq 0\}$, it is a feeble viscosity solution in Ω. Indeed, if a test-function $\phi \in C^2(\Omega)$ has a non-vanishing gradient at a local maximum (respectively, minimum) point $\hat{x}$ of $u - \phi$, then

$$\nabla u(\hat{x}) = \nabla \phi(\hat{x}) \neq 0,$$

and thus $\hat{x}$ belongs to the open subset of Ω where (4.1) (respectively, (4.2)) is already known to hold. The claim then follows immediately from Theorem 4.2 and its natural counterpart for supersolutions. □

By combining Corollary 4.4 with the proof of Theorem 2.2 we obtain Theorem 2.3. Indeed, the non-critical points in the level set are removable by the argument in Section 3, and Corollary 4.4 takes care of the critical points. Observe that if it is assumed that u is a viscosity subsolution in $\Omega \setminus \{x : u(x) = 0\}$, then the point $\hat{x}$ in (4.4) necessarily lies on the zero level set. Thus, by continuity, it suffices to assume $B(x, 0, 0) = 0$ for all $x \in \Omega$, and this is precisely what was assumed in Theorem 2.3.

5. Parabolic equations

It should be noted that certain parabolic equations, for example the heat equation $u_t - \Delta u = 0$, are included in Theorems 2.2 and 2.3, provided that we interpret the concept of a parabolic viscosity solution correctly. Consider the equation

$$u_t = A(x, t, u, \nabla u) \cdot D_x^2 u + B(x, t, u, \nabla u) \tag{5.1}$$

in an open set $\mathcal{O} \subset \mathbb{R}^{n+1}$. Denoting $Du = (\nabla u, u_t) \in \mathbb{R}^{n+1}$ and $z = (x, t)$, (5.1) can be written as

$$-\tilde{A}(z, u, Du) \cdot D_{x,t}^2 u - \tilde{B}(z, u, Du) = 0, \tag{5.2}$$

where $\tilde{B}(z, u, Du) = B(x, t, u, \nabla u) - u_t$ and

$$\tilde{A}_{ij}(z, u, Du) = \begin{cases} A_{ij}(x, t, u, \nabla u), & \text{if } 1 \leq i, j \leq n \\ 0, & \text{if } i = n+1 \text{ or } j = n+1. \end{cases}$$

If the coefficients A and B satisfy (3.1) (or (2.2) and $B(x, t, 0, 0) = 0$ for all $(x, t) \in \mathcal{O}$), then Theorem 2.2 (respectively, Theorem 2.3) clearly applies to (5.2) and we may conclude that the level set $\{u = 0\}$ is removable for $u \in C^1(\mathcal{O})$. We want to emphasize that the C^1 assumption is on both x and t; in particular, the time derivative u_t is assumed to be continuous.

The only problem in this argument is the interplay between parabolic and elliptic viscosity solutions. More precisely, when dealing with the parabolic equations it is customary to emphasize the special role of the time variable t either by allowing test-functions $\phi(x, t)$ that are only C^1 with respect to t (cf. [4], [5]) or by determining the admissibility of a test-function at $(\hat{x}, \hat{t})$ based only on what happens prior to the time $\hat{t}$ (cf. [10], [12]). In practical terms this means, in particular, that every parabolic viscosity solution of (5.1) is automatically an elliptic viscosity solution to (5.2) since the number of admissible test-functions is smaller in the elliptic case. However, for the removability result we need also the converse implication. But that is easily obtained by using a mollification argument, and thus we conclude

Theorem 5.1. *Suppose that the coefficient matrix $A(x, t, s, \xi)$ satisfies either (3.1) or (2.2) and that $B(x, t, 0, 0) = 0$ for all $(x, t) \in \mathcal{O}$. If $u \in C^1(\mathcal{O})$ is a parabolic viscosity solution (as defined in* [4], [5]*) to (5.1) in*

$$\mathcal{O} \setminus \{x \in \mathcal{O} : u(x) = 0\},$$

then it is a parabolic viscosity solution in $\mathcal{O}$.

Theorem 5.1 applies, for example, to the heat equation

$$u_t = \Delta u$$

and Burgers's equation

$$u_t = uu_x + u_{xx}.$$

Observe also that the remarks made above in Section 2 concerning equations in divergence form, singular equations and distributional solutions have their natural counterparts in the parabolic case (cf. [12]).

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Department of Mathematics and Statistics, P.O.Box 35, FIN-40014 University of Jyväskylä, Finland

E-mail address: peanju@maths.jyu.fi

Department of Mathematics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail address: lqvist@math.ntnu.no