MINIMIZATION PROBLEMS FOR LIPSCHITZ FUNCTIONS VIA VISCOSITY SOLUTIONS

Petri Juutinen

Acknowledgements

I wish to express my sincere gratitude to my teacher, Docent Tero Kilpeläinen, for his support and guidance during the preparation of this thesis. I am also grateful to Dr. Stephen Buckley and Professor Olli Martio for carefully reading the manuscript, and to Professor Peter Lindqvist and Professor Juan J. Manfredi for several discussions and valuable comments on the work presented here.

For linguistic comments my thanks are due to Mrs. Tuula Blåfield, and for technical help to Docent Ari Lehtonen. For financial support I am indebted to the Academy of Finland (project # 8597), the Graduate School on Mathematical Analysis and Logic, and the foundation Yrjö, Vilho ja Kalle Väisälän rahasto.

Finally, I would like to thank my family and all my friends for their support and encouragement.

Jyväskylä, April 1998

Petri Juutinen

Contents

Introduction

The origin of the theory we are about to present in this thesis lies in the problem of finding a minimal Lipschitz extension of a continuous function $g\colon \partial\Omega \to \mathbb{R}$ to a domain $\Omega \subset \mathbb{R}^n$, that is, finding a function $u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ with $u|_{\partial\Omega} = g$ such that

$$\|\nabla u\|_{\infty,\Omega} \leq \|\nabla v\|_{\infty,\Omega}$$

for all $v \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ with $v|_{\partial\Omega} = g$. If g is Lipschitz continuous with respect to the internal distance relative to Ω, then this problem is known to have a solution which is, however, in general nonunique. In his papers [1], [2], G. Aronsson suggested that the minimal Lipschitz extension problem should be considered as a limit, as $p \to \infty$, of problems of finding a minimal "p-extension", that is, a function $u_p \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $u_p|_{\partial\Omega} = g$ such that

$$\|\nabla u_p\|_{p,\Omega} \leq \|\nabla v\|_{p,\Omega}$$

for all $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $v|_{\partial\Omega} = g$. Minimal p-extensions are exactly weak solutions of the p-Laplace equation and hence unique. An equivalent way to characterize a minimal p-extension is to require that

$$\|\nabla u_p\|_{p,D} \leq \|\nabla v\|_{p,D},$$

whenever $D \subset \Omega$ is open and $v \in W^{1,p}(D) \cap C(\overline{D})$ is such that $v|_{\partial D} = u_p|_{\partial D}$. Aronsson reasoned that this should hold also when $p = \infty$, and accordingly defined an *absolutely minimizing Lipschitz extension* to be a function $u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that

$$\|\nabla u\|_{\infty,D} \leq \|\nabla v\|_{\infty,D},$$

whenever $D \subset \Omega$ is open and $v \in W^{1,\infty}(D) \cap C(\overline{D})$ is such that $v|_{\partial D} = u|_{\partial D}$. For this subclass of minimal Lipschitz extensions, he derived a candidate for the Euler equation by taking a formal limit of p-Laplacians, and was in fact able to prove that smooth absolute minimizers are solutions of this equation in the classical sense, and vice versa. However, Aronsson also demonstrated that the Euler equation does not always have a classical solution, and, consequently, that absolutely minimizing Lipschitz extensions are in general nonsmooth. Aronsson's conjecture was proved to be correct in full by R. Jensen in his remarkable paper [17]. Instead of classical solutions, Jensen used the concept of viscosity solutions introduced by M. G. Crandall and P.-L. Lions, and showed that every absolutely minimizing Lipschitz extension is a viscosity solution of the Euler equation. Furthermore, he proved the comparison principle for the Euler equation and, as a consequence, obtained uniqueness. The fundamental work of Aronsson and Jensen has been expanded on by a number of other authors. Especially, we should mention the papers by T. Bhattacharya, E. DiBenedetto, and J. Manfredi [4], which contains a rigorous proof for the existence of an absolute minimizer, and by P. Lindqvist and J. Manfredi [21], where the Harnack inequality was obtained. In addition, in [6], V. Caselles, J.-M. Morel, and C. Sbert have found applications of the absolute minimizers in image processing.

In this work we seek to generalize the theory outlined above to a wider class of minimization problems in L^∞. A basic example that we have in mind is a measurable perturbation of the minimal Lipschitz extension problem, namely, the problem of minimizing

$$||\theta(x)\nabla u(x)\cdot\nabla u(x)||_{\infty,\Omega}$$

among all the functions $u\in W^{1,\infty}(\Omega)$ having prescribed boundary values. Here θ is a measurable function with values in the space of $n\times n$ -symmetric matrices satisfying

$$\alpha|\xi|^2\le\theta(x)\xi\cdot\xi\le\beta|\xi|^2$$

for some constants $0<\alpha\le\beta<\infty$. However, our theory applies to even more general problems of the form

$$\min||F(x,\nabla u(x))||_{\infty,\Omega}, \tag{1}$$

where $F(x,\xi)\approx|\xi|^2$ satisfies conditions that we specify in Section 1. These conditions, in particular, imply that the corresponding L^p -problem has a unique solution, and hence we have an initial setting similar to the one in the case of minimal Lipschitz extensions. For the problem (1), we adapt in a natural way the notion of absolute minimizers, and, by following Aronsson's example, obtain a fully nonlinear partial differential equation

$$-\nabla\big(F(x,\nabla u(x))\big)\cdot\nabla_\xi F(x,\nabla u(x))=0 \tag{2}$$

that we regard as the Euler equation of (1). Our results include the existence of an absolute minimizer and, under some regularity assumptions, the existence and uniqueness of a viscosity solution of (2) with continuous boundary values. Moreover, we show that the unique viscosity solution is an absolute minimizer and a limit of solutions of the corresponding L^p -problems. The only thing that makes our theory incomplete is the absence of an uniqueness theorem for absolute minimizers in the general case. On the other hand, some of our results, in particular those related to viscosity supersolutions of (2), are new even in the case of minimal Lipschitz extensions. This is an outcome of the fact that we also consider problems involving obstacles. The techniques used in obtaining the abovementioned results are readily seen to be applicable to various other minimization problems. As an example of this, we consider a problem that can be regarded as an eigenvalue problem associated with the Euler equation (2).

Our presentation begins in Section 1 with a brief review of some facts about the calculus of variations in L^p -spaces. We give all the relevant definitions and record without proofs the results needed in later sections. Additionally, we define viscosity solutions for a large class of second order partial differential equations, and in a few words describe their properties. In Section 2, we turn to the L^∞ -theory and consider the minimization problem (1). We establish the existence of an absolute minimizer and the Harnack inequality, both of which follow from the analogous results to the corresponding L^p -problems. In Section 3, we continue to exploit the L^p -theory and show that in the case of a sufficiently smooth kernel F, the absolute minimizers obtained in the previous section are viscosity solutions of the Euler equation (2). We also introduce two auxiliary equations involving gradient constraints, properties of which play a vital role in Section 4, where we prove the comparison principle for equation (2). The latter part of Section 4 includes uniqueness theorems and a basic interior regularity result obtained as a consequence of the comparison principle. After that, in Section 5, we study more closely viscosity supersolutions of (2) and sharpen some results of the preceding sections. In the final section we turn our attention to the eigenvalue problem and, as a main result, prove the existence of a maximal solution.

Note added after the completion of the manuscript: It has come to our attention that similar questions have been studied from a slightly different perspective by Y. Wu in [23].

Notation

For the reader's convenience we here list some notation that will be used throughout this text:

Ω	an open and bounded subset of $\mathbb{R}^n$, $n \geq 2$.
$\overline{E}$, ∂E	the closure and the boundary of a set $E \subset \mathbb{R}^n$, respectively.
$\lvert E\rvert$	the Lebesgue n-measure of a measurable set $E \subset \mathbb{R}^n$.
$B(x,r)$	an open ball of $\mathbb{R}^n$ with center at x and with radius $r > 0$.

$E \Subset \Omega$	means that $\overline{E}$ is a compact subset of Ω.
$C(E)$	$\{f\colon E \to \mathbb{R}\colon\ f$ is continuous$\}$.
$C_0(\Omega)$	$\{f \in C(\Omega)\colon \lim_{x\to z} f(x) = 0$ for every $z \in \partial\Omega\}$.
$C^k(\Omega)$	$\{f\colon \Omega \to \mathbb{R}\colon\ f$ is k-times continuously differentiable $\}$.
$C^\infty(\Omega)$	$\{f\colon \Omega \to \mathbb{R}\colon\ f$ is infinitely many times differentiable $\}$.
$C_c^\infty(\Omega)$	$\{f \in C^\infty(\Omega)\colon\ f$ vanishes outside a compact subset of Ω $\}$.
$W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$	standard Sobolev spaces, $1 \le p \le \infty$, for the definition see [24, 2.1.1].
∇f	the (weak) gradient of f, $\nabla f = \big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\big)$.
$D^2 f$	the Hessian matrix of f, $(D^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.
A^*	the transpose of an $n \times n$-matrix A.
$\Vert A\Vert$	$\sup\{\lvert Ax\rvert\colon x \in \mathbb{R}^n,\ \lvert x\rvert = 1\}$; the norm of the matrix A.
S_n	the space of symmetric $n \times n$ matrices with real coefficients.
$p \otimes q$	the tensor product of the vectors $p, q \in \mathbb{R}^n$.

Furthermore, if $g\colon E \to \mathbb{R}$ is measurable, we denote by

$$\fint_E g(x)\,dx := \frac{1}{|E|}\int_E g(x)\,dx$$

the integral average of g over a measurable set E, $0 < |E| < \infty$.

1. Minimization problems in L^p and viscosity solutions

As explained in the introduction, a basic tool in examining minimization problems in L^∞ is the use of the theory of related L^p -problems for $n < p < \infty$. Therefore, it is sensible to recall first some results concerning the existence and uniqueness of minimizers in the L^p -case and to discuss the corresponding Euler equations. Although our exposition is far from being complete, it gives sufficient background for the later sections. For a deeper treatment of this topic, see for example the monograph [15].

Let Ω be a bounded open set in the Euclidean space $\mathbb{R}^n$, $n \ge 2$, and let $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a mapping satisfying the following assumptions for some constants $0 < \alpha \le \beta < \infty$:

(1.1) the mapping $x \mapsto F(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^n$;

for a.e. $x \in \Omega$ (with respect to the Lebesgue measure)

$$\alpha|\xi|^2 \le F(x,\xi) \le \beta|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n, \tag{1.2}$$

(1.3) the mapping $\xi \mapsto F(x,\xi)$ is strictly convex and differentiable, and

$$F(x,\lambda\xi) = \lambda^2 F(x,\xi), \quad \xi \in \mathbb{R}^n,\ \lambda \in \mathbb{R}. \tag{1.4}$$

By the *strict convexity* of $\xi \mapsto F(x,\xi)$ we mean that

$$F(x, t\xi_1 + (1-t)\xi_2) < tF(x,\xi_1) + (1-t)F(x,\xi_2),$$

whenever $\xi_1 \neq \xi_2$ and $0 < t < 1$. The conditions (1.2) – (1.4) imply the following useful facts: For a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$,

$$|\nabla_\xi F(x,\xi)| \le 4\beta|\xi|, \tag{1.5}$$

$$\alpha|\xi|^2 \le \nabla_\xi F(x,\xi)\cdot\xi, \tag{1.6}$$

and

$$\nabla_\xi F(x,\lambda\xi) = \lambda\nabla_\xi F(x,\xi) \qquad \text{for all } \lambda \in \mathbb{R}, \tag{1.7}$$

where $\nabla_\xi F(x,\xi)$ is the gradient of the mapping $\xi \mapsto F(x,\xi)$. For the proof of these, see [15, 5.9].

Suppose that $\vartheta \in W^{1,p}(\Omega)$ and that $\psi\colon \Omega \to [-\infty,\infty]$ is an arbitrary function. Let

$$\mathcal{K}^p_{\psi,\vartheta}(\Omega) = \left\{ v \in W^{1,p}(\Omega)\colon\ v \ge \psi \text{ a.e. in } \Omega,\ v - \vartheta \in W^{1,p}_0(\Omega) \right\}$$

and consider the following minimization problem: Find a function $u \in \mathcal{K}^p_{\psi,\vartheta}(\Omega)$ such that

$$I_F(u) = \inf\left\{ I_F(v)\colon v \in \mathcal{K}^p_{\psi,\vartheta}(\Omega) \right\}, \tag{1.8}$$

where

$$I_F(v) = \int_\Omega F(x,\nabla v(x))^{p/2}\,dx. \tag{1.9}$$

The function ψ above is called an *obstacle*, and we say that $u \in \mathcal{K}^p_{\psi,\vartheta}(\Omega)$ is a *solution to the obstacle problem in* $\mathcal{K}^p_{\psi,\vartheta}(\Omega)$ if it satisfies (1.8). Further, if $u \in W^{1,p}_{loc}(\Omega)$ is a solution to the obstacle problem in $\mathcal{K}^p_{u,u}(D)$ for every open $D \Subset \Omega$, then u is called an (F,p)-*superextremal*. In the special case when ψ is identically $-\infty$, the infimum in (1.8) is taken over all functions v for which $v - \vartheta \in W^{1,p}_0(\Omega)$, and we use different terminology. We call a function $u \in W^{1,p}(\Omega)$

satisfying (1.8) with $\psi \equiv -\infty$ an (F,p)*-minimizer with boundary values* ϑ, and say that $u \in W^{1,p}_{loc}(\Omega)$ is an (F,p)*-extremal*, if it is an (F,p)-minimizer (with boundary values u) in D for each open $D \Subset \Omega$. Note that since $p > n$, every function $u \in W^{1,p}_{loc}(\Omega)$ has a real-valued and continuous representative, and thus we may assume that u itself is real-valued and continuous.

A basic example of a kernel satisfying the assumptions (1.1) – (1.4) above is $F(x,\xi) = \theta(x)\xi \cdot \xi$, where $\theta\colon \Omega \to S_n$ is measurable and for some constants $0 < \alpha \le \beta < \infty$ satisfies

$$\alpha|\xi|^2 \le \theta(x)\xi \cdot \xi \le \beta|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and for a.e. $x \in \Omega$. In the simplest case, when $\theta(x)$ equals to the identity matrix for all $x \in \Omega$, the (F,p)-extremals are nothing but local minimizers of the p*-Dirichlet integral*

$$\int_\Omega |\nabla v(x)|^p \, dx.$$

We recall the following two well-known theorems, see [15, 5.27, 5.13].

1.10. Theorem. *If* $\mathcal{K}^p_{\psi,\vartheta}(\Omega) \neq \emptyset$, *there exists a unique solution to the obstacle problem in* $\mathcal{K}^p_{\psi,\vartheta}(\Omega)$.

1.11. Theorem. *A function* $u \in \mathcal{K}^p_{\psi,\vartheta}(\Omega)$ *satisfies* (1.8) *if and only if*

$$\int_\Omega F(x,\nabla u)^{(p-2)/2} \nabla_\xi F(x,\nabla u) \cdot \nabla(v-u) \, dx \ge 0 \tag{1.12}$$

for all $v \in \mathcal{K}^p_{\psi,\vartheta}(\Omega)$.

From (1.12) it is clear that u is a solution to the obstacle problem in $\mathcal{K}^p_{\psi,\vartheta}(\Omega)$ if and only if it is a solution to the obstacle problem in $\mathcal{K}^p_{\psi,u}(D)$ for every open $D \subset \Omega$. Furthermore, Theorem 1.11 implies that there is a connection between the minimization problem and the quasilinear partial differential equation

$$-\mathrm{div}\left(F(x,\nabla u)^{(p-2)/2} \nabla_\xi F(x,\nabla u)\right) = 0. \tag{1.13}$$

To avoid misunderstandings, we define in detail what we mean by weak solutions of (1.13) before making this connection precise.

1.14. Definition. A function $u \in W^{1,p}_{loc}(\Omega)$ is a *weak solution* of (1.13) in Ω if

$$\int_\Omega F(x,\nabla u)^{(p-2)/2} \nabla_\xi F(x,\nabla u) \cdot \nabla\varphi \, dx = 0$$

for all $\varphi \in C_c^\infty(\Omega)$. Further, $u \in W_{loc}^{1,p}(\Omega)$ is a *weak supersolution* of (1.13) in Ω if

$$\int_\Omega F(x,\nabla u)^{(p-2)/2} \nabla_\xi F(x,\nabla u) \cdot \nabla\varphi \, dx \geq 0$$

for all nonnegative $\varphi \in C_c^\infty(\Omega)$. Finally, $v \in W_{loc}^{1,p}(\Omega)$ is a *weak subsolution* of (1.13) if $-v$ is a weak supersolution.

By Theorem 1.11, every solution to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}^p(\Omega)$ is a weak supersolution of (1.13), and conversely, a supersolution u is always an (F,p)-superextremal. In the later sections we will frequently use a refinement of this correspondence given by Theorem 1.15 below. For the proof, see [15, 5.29, 3.67].

1.15. Theorem. *Let $\psi\colon \Omega \to [-\infty,\infty)$ be continuous. If u is a solution to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}^p(\Omega)$, then u is a weak solution of (1.13) in the open set $\{x \in \Omega\colon u(x) > \psi(x)\}$.*

In the case of $\psi \equiv -\infty$, the above result implies that $u \in W_{loc}^{1,p}(\Omega)$ is an (F,p)-extremal if and only if it is a weak solution of (1.13) in Ω. Equation (1.13) is called the *Euler equation* of the minimization problem (1.8).

We will also need the following lemma, which allows us to compare solutions to different obstacle problems having the same obstacle.

1.16. Lemma. *Let $u,v \in W^{1,p}(\Omega)$ be solutions to the obstacle problems in $\mathcal{K}_{\psi,u}^p(\Omega)$ and $\mathcal{K}_{\psi,v}^p(\Omega)$, respectively, and suppose that $\|u-v\|_{\infty,\partial\Omega}$ is finite. Then*

$$\|u-v\|_{\infty,\Omega} = \|u-v\|_{\infty,\partial\Omega}.$$

Proof. Denote $w = \min(u,v) \in W^{1,p}(\Omega)$ and let $\hat{w}$ be the unique solution to the obstacle problem in $\mathcal{K}_{\psi,w+c}^p(\Omega)$, where $c = \|u-v\|_{\infty,\partial\Omega}$. Since $\min(u,\hat{w}) \in \mathcal{K}_{\psi,u}^p(\Omega)$ and $\min(v,\hat{w}) \in \mathcal{K}_{\psi,v}^p(\Omega)$, we have by [15, 3.22] that $u,v \leq \hat{w}$ in Ω. Moreover, since $w+c$ is also a supersolution of (1.13), we have again by [15, 3.22] that $\hat{w} \leq w+c$ in Ω. Combining these inequalities yields

$$\min(u,v) \leq u,v \leq \min(u,v)+c,$$

which proves the lemma. □

We finish this discussion about L^p-minimization problems by stating the following *comparison principle* for sub- and supersolutions of (1.13). The proof can be found in [15, 3.18].

1.17. Lemma. *Let $u \in W^{1,p}(\Omega)$ be a supersolution and $v \in W^{1,p}(\Omega)$ a subsolution of* (1.13) *in Ω. If* $\min(u-v,0) \in W_0^{1,p}(\Omega)$, *then $u \geq v$ in Ω.*

The latter part of this section deals with the notion of viscosity solutions introduced by Crandall and Lions in [9]. We first give the necessary definitions and discuss the properties of viscosity solutions in a quite general situation, and then conclude the section by showing that weak solutions of the Euler equation (1.13) are solutions in the viscosity sense as well. A standard reference to the general theory of viscosity solutions is [8], see also [5], [7] and [16].

We begin with the definition of viscosity solutions. Assume that $G\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \times S_n \to \mathbb{R}$ is a continuous mapping satisfying

$$G(x,r,\xi,X) \leq G(x,r,\xi,Y), \tag{1.18}$$

whenever $x \in \Omega$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $X, Y \in S_n$ are such that $Y \leq X$, that is, $(X-Y)\eta\cdot\eta \geq 0$ for all $\eta \in \mathbb{R}^n$. Recall that a function $u\colon E \to \mathbb{R}\cup\{\infty\}$, defined on a set $E \subset \mathbb{R}^n$, is *lower semicontinuous* if the set $\{x \in E\colon u(x) > \lambda\}$ is open for every $\lambda \in \mathbb{R}$, or, equivalently, if

$$\liminf_{y\to x} u(y) \geq u(x)$$

for all $x \in E$. A function $u\colon E \to \mathbb{R}\cup\{-\infty\}$ is *upper semicontinuous* if $-u$ is lower semicontinuous.

1.19. Definition. An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a *viscosity subsolution* of the equation $G = 0$ in Ω if

$$G(x,\varphi(x),\nabla\varphi(x),D^2\varphi(x)) \leq 0, \tag{1.20}$$

whenever $x \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x) = \varphi(x)$ and $u(y) \leq \varphi(y)$ for all $y \in \Omega$. Similarly, a lower semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a *viscosity supersolution* of $G = 0$ in Ω if

$$G(x,\varphi(x),\nabla\varphi(x),D^2\varphi(x)) \geq 0, \tag{1.21}$$

whenever $x \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x) = \varphi(x)$ and $u(y) \geq \varphi(y)$ for all $y \in \Omega$. Finally, a continuous function $u\colon \Omega \to \mathbb{R}$ is a *viscosity solution* of $G = 0$ if it is both a viscosity subsolution and a viscosity supersolution.

Note that the assumption (1.18) guarantees that classical solutions of the equation $G = 0$ are viscosity solutions. Indeed, if $u \in C^2(\Omega)$ is a classical solution of $G = 0$, $x \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x) = \varphi(x)$ and $u(y) \geq \varphi(y)$ for all $y \in \Omega$, then by calculus $\nabla u(x) = \nabla\varphi(x)$ and $D^2u(x) \geq D^2\varphi(x)$. Combining this with (1.18) shows that u is a viscosity supersolution, and by a similar argument we see that it is also a viscosity subsolution. On the other hand, if u is a viscosity solution of $G = 0$ and we a priori know that $u \in C^2(\Omega)$, then u is a solution in the classical sense. This is a trivial consequence of the fact that in this case u itself will do as a test function. It is also evident that it suffices to test the validity of (1.21) by functions $\varphi \in C^2$ satisfying $u(x) = \varphi(x)$ and $u(y) > \varphi(y)$ for all $y \neq x$. This fact can be seen by replacing an arbitrary test function φ by a function $\varphi - \varepsilon|y-x|^4$ for $\varepsilon > 0$. An analogous remark holds for subsolutions. As a last observation concerning the definition, we note that since (1.20) and (1.21) involve derivatives of φ only at the point x, all the global assumptions about the test functions can be replaced by local ones. In the supersolution case this means that we could as well require (1.21) to be true for all $\varphi \in C^2(U)$ satisfying $u(x) = \varphi(x)$ and $u(y) \geq \varphi(y)$ for all $y \in U$, where U is some neighborhood of x, not necessarily the same for every test function φ.

The next lemma gives an equivalent definition for viscosity solutions that will be used in the formulation of the maximum principle for semicontinuous functions in Section 4. We use the following notation:

$$\begin{aligned} D^{2,+}u(x) = \Big\{(q,X) \in \mathbb{R}^n \times S_n \colon u(y) \leq u(x) + q\cdot(y-x) + \frac{1}{2}X(y-x)\cdot(y-x) \\ + o(|y-x|^2) \quad \text{as } y \to x \Big\} \end{aligned}$$

and

$$\begin{aligned} D^{2,-}u(x) = \Big\{(q,X) \in \mathbb{R}^n \times S_n \colon u(y) \geq u(x) + q\cdot(y-x) + \frac{1}{2}X(y-x)\cdot(y-x) \\ + o(|y-x|^2) \quad \text{as } y \to x \Big\}. \end{aligned}$$

1.22. Lemma. *An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a viscosity subsolution of $G = 0$ if and only if $G(x,u(x),q,X) \leq 0$ for all $x \in \Omega$ and $(q,X) \in D^{2,+}u(x)$. Similarly, a lower semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a viscosity supersolution of $G = 0$ if and only if $G(x,u(x),q,X) \geq 0$ for all $x \in \Omega$ and $(q,X) \in D^{2,-}u(x)$.*

Proof. We prove only the supersolution case. Assume first that (1.21) holds and that $(q,X) \in D^{2,-}u(x)$. Define

$$\varphi_k(y) = u(x) + q\cdot(y-x) + \frac{1}{2}X(y-x)\cdot(y-x) - \frac{1}{k}|y-x|^2$$

for $k = 1,2,\dots$. Then $\varphi_k \in C^2(\Omega)$, $\varphi_k(x) = u(x)$ and $\varphi_k(y) \leq u(y)$ in some neighborhood of x, and so $G(x,u(x),q,X-(2/k)I) \geq 0$. Letting $k \to \infty$ and using the continuity of G, we obtain $G(x,u(x),q,X) \geq 0$.

For the other direction, we let $x \in \Omega$ and $\varphi \in C^2(\Omega)$ be such that $u(x) = \varphi(x)$ and $u(y) \geq \varphi(y)$ for all $y \in \Omega$. By Taylor's theorem,

$$u(y) \geq \varphi(y) = \varphi(x) + \nabla\varphi(x)\cdot(y-x) + \frac{1}{2}D^2\varphi(x)(y-x)\cdot(y-x) + o(|y-x|^2),$$

which shows that $(\nabla\varphi(x), D^2\varphi(x)) \in D^{2,-}u(x)$. This implies that

$$G(x, \varphi(x), \nabla\varphi(x), D^2\varphi(x)) \geq 0,$$

as desired. □

1.23. Remark. It is in fact possible to prove a better result than the one formulated above. Namely, it can be shown that for every pair $(q, X) \in D^{2,+}u(x)$ there exists $\varphi \in C^2(\Omega)$ such that $u(x) = \varphi(x)$, $u(y) \leq \varphi(y)$ for all $y \in \Omega$ and $(q, X) = (\nabla\varphi(x), D^2\varphi(x))$. For the proof, see [16, Prop. 1].

We now consider the equation

$$-\operatorname{div}\Big(F(x, \nabla v(x))^{(p-2)/2}\nabla_\xi F(x, \nabla v(x))\Big) = f(x, v(x)), \tag{1.24}$$

where $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and for some constants $a, b > 0$ satisfies

$$|f(x, u)| \leq a + b|u|^{p-1}$$

for all $x \in \Omega$ and $u \in \mathbb{R}$. We say that a function $u \in W^{1,p}_{loc}(\Omega)$ is a *weak solution* of (1.24) in Ω if

$$\int_\Omega F(x, \nabla u)^{(p-2)/2}\nabla_\xi F(x, \nabla u) \cdot \nabla\varphi \, dx = \int_\Omega f(x, u)\varphi \, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. Correspondingly, a function $u \in W^{1,p}_{loc}(\Omega)$ is a *weak supersolution* of (1.24) in Ω if

$$\int_\Omega F(x, \nabla u)^{(p-2)/2}\nabla_\xi F(x, \nabla u) \cdot \nabla\varphi \, dx \geq \int_\Omega f(x, u)\varphi \, dx$$

for all nonnegative $\varphi \in C_c^\infty(\Omega)$. We are especially interested in two particular choices for f. If $f(x, u) = \lambda|u|^{p-2}u$, $\lambda \in \mathbb{R}$, then (1.24) can be viewed as a nonlinear eigenvalue problem which will be studied more closely in Section 6. The second case we have in mind is simply $f(x, u) = f(x)$. Taking $f \equiv 0$ we recover (1.13), while some other choices of f turn out to be useful in the uniqueness proof of Section 4.

Most of the results mentioned earlier in the homogeneous case can be quite easily extended for equation (1.24) in the case $f(x, u) = f(x)$. If $\vartheta \in W^{1,p}(\Omega)$, there exists a unique weak solution u of (1.24) in Ω such that $u - \vartheta \in W_0^{1,p}(\Omega)$. This can be seen by first showing that weak solutions of (1.24) are exactly (local) minimizers of the functional

$$\int_\Omega F(x, \nabla v(x))^{p/2} - \frac{p}{2}f(x)v(x) \, dx$$

and then proving the existence of a minimizer by using the direct methods in the calculus of variations. The uniqueness follows from the comparison principle, which can be obtained in the same way as for (1.13).

Our aim here is to prove that if $F(x,\xi)$ satisfies certain regularity assumptions, then every weak solution of (1.24) is also a viscosity solution of the same equation. After expansion, (1.24) reads

$$\begin{aligned} -\left(\frac{p-2}{2}\right) & F(x,\nabla v(x))^{(p-4)/2}\nabla\big(F(x,\nabla v(x))\big)\cdot\nabla_\xi F(x,\nabla v(x)) \\ & -F(x,\nabla v(x))^{(p-2)/2}\sum_{j=1}^{n}\frac{\partial^2 F}{\partial x_j\partial\xi_j}(x,\nabla v(x)) \;=\; f(x,v(x)), \end{aligned} \tag{1.25}$$

where $\nabla(F(x,\nabla v(x)))$ is the gradient of the mapping $x\mapsto F(x,\nabla v(x))$, that is,

$$\nabla\big(F(x,\nabla v(x))\big)=\nabla_x F(x,\nabla v(x))+D^2v(x)\nabla_\xi F(x,\nabla v(x)).$$

We assume that (1.25) is pointwise well-defined and continuous in Ω. To be more precise, we require that the following conditions are satisfied:

$$\text{the mapping } (x,\xi)\mapsto\nabla_\xi F(x,\xi) \text{ is continuous,} \tag{1.26}$$

$$\begin{gathered}\text{the mapping } x\mapsto F(x,\xi) \text{ is differentiable for all } \xi\in\mathbb{R}^n \\ \text{and } (x,\xi)\mapsto\nabla_x F(x,\xi) \text{ is continuous,}\end{gathered} \tag{1.27}$$

and

$$\begin{gathered}\text{for every } j=1,\dots,n \text{ the second order partial derivative} \\ \frac{\partial^2 F}{\partial x_j\partial\xi_j}(x,\xi) \text{ exists and is continuous.}\end{gathered} \tag{1.28}$$

1.29. Theorem. *Suppose that* $u\in W^{1,p}_{loc}(\Omega)$ *is a weak supersolution of* (1.24). *Then* u *is also a viscosity supersolution of* (1.24).

Proof. We argue by contradiction and assume that there exists $x_0\in\Omega$ and $\varphi\in C^2(\Omega)$ such that $u(x_0)=\varphi(x_0)$, $u(x)>\varphi(x)$ for all $x\neq x_0$, and that

$$-\operatorname{div}\Big(F(x_0,\nabla\varphi(x_0))^{(p-2)/2}\nabla_\xi F(x_0,\nabla\varphi(x_0))\Big)<f(x_0,u(x_0)). \tag{1.30}$$

By continuity, there exists a radius $r>0$ such that if $|x-x_0|\le r$, then (1.30) holds with x_0 replaced by x. Let

$$m=\inf_{|x-x_0|=r}(u(x)-\varphi(x))>0$$

and define $\tilde\varphi=\varphi+m/2$. Then $\tilde\varphi$ is a classical subsolution of

$$-\operatorname{div}\Big(F(x,\nabla v(x))^{(p-2)/2}\nabla_\xi F(x,\nabla v(x))\Big)=f(x,u(x))$$

in the open set $B(x_0,r)$, $\tilde\varphi<u$ on $\partial B(x_0,r)$ and $\tilde\varphi(x_0)>u(x_0)$. This contradicts the comparison principle and the theorem follows. □

The analogous result for subsolutions easily follows by a similar reasoning.

1.31. Theorem. *Suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a weak subsolution of* (1.24)*. Then u is also a viscosity subsolution of* (1.24)*.*

Combining Theorems 1.15, 1.29 and 1.31, we obtain the following corollary.

1.32. Corollary. *Suppose that $u \in W^{1,p}(\Omega)$ is a solution to the obstacle problem in $\mathcal{K}^p_{\psi,\vartheta}(\Omega)$ with a continuous obstacle ψ. Then u is a viscosity supersolution of* (1.13) *in Ω and a viscosity solution in the open set $\{x \in \Omega\colon u(x) > \psi(x)\}$.*

2. Minimization problems in L^∞

In this section we discuss the obstacle problem in L^∞ and generalize the notion of absolute minimizers to our situation. We prove the existence of a solution to the obstacle problem with Lipschitz-boundary values by following the arguments in [4]. The idea of the proof is to consider our minimization problem in L^∞ as a limit of analogous L^p-problems. This method provides us also with the Harnack inequality for nonnegative solutions obtained by this limiting process.

Assume, as always, that Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be as in Section 1, that is , we assume that F satisfies (1.1)–(1.4) for some constants $0 < \alpha \leq \beta < \infty$. We define F-absolute minimizers as follows.

2.1. Definition. A function $u \in W^{1,\infty}_{loc}(\Omega)$ is an F-*absolute minimizer* in Ω if

$$\|F(x, \nabla u(x))\|_{\infty,D} \leq \|F(x, \nabla v(x))\|_{\infty,D},$$

whenever $D \Subset \Omega$ is open and $v \in W^{1,\infty}(D)$ is such that $u - v \in C_0(D)$.

The obstacle problem now extends to the case $p = \infty$ in an obvious way. Suppose that $\vartheta \in W^{1,\infty}(\Omega)$ and let $\psi\colon \Omega \to [-\infty, \infty]$ be an arbitrary function. We define

$$\mathcal{K}^\infty_{\psi,\vartheta}(\Omega) = \{\, v \in W^{1,\infty}(\Omega)\colon\ \ v \geq \psi \text{ a.e. in } \Omega,\ v - \vartheta \in C_0(\Omega)\,\},$$

and say that a function $u \in \mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$ is a *solution to the obstacle problem in* $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$ if

$$\|F(x, \nabla u(x))\|_{\infty,D} \leq \|F(x, \nabla v(x))\|_{\infty,D},$$

whenever $D \subset \Omega$ is open and $v \in \mathcal{K}^\infty_{\psi,u}(D)$.

2.2. Remark. In the above definitions we did not use the boundary condition $u - \vartheta \in W^{1,\infty}_0(\Omega)$ since it would imply that $u - \vartheta \in C^1(\Omega)$, which we do not want. The condition $u - \vartheta \in C_0(\Omega)$ in turn arises naturally if we look at our problem as a limit problem and recall the fact that $W^{1,p}_0(\Omega) = W^{1,p}(\Omega) \cap C_0(\Omega)$ for $p > n$, see [15, 4.5, 2.11].

In the question of the existence of a solution to the obstacle problem there is a complete analogy to the case $p < \infty$.

2.3. Theorem. *If $\mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega) \neq \emptyset$, there exists a solution to the obstacle problem in $\mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega)$. In particular, for every $\vartheta \in W^{1,\infty}(\Omega)$ there is an F-absolute minimizer $u \in W^{1,\infty}(\Omega)$ such that $u - \vartheta \in C_0(\Omega)$.*

We will in fact prove a somewhat more general result than is necessary for obtaining the theorem above. This is because we want to emphasize the natural connection to variational problems in L^p; see the discussion at the end of this section.

Let $F_p \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$, $1 < p < \infty$, be a family of mappings satisfying (1.1) – (1.4) for some constants $0 < \alpha_p \le \beta_p < \infty$, and assume that $F_p \to F$ uniformly as $p \to \infty$ in the following sense: for every $\varepsilon > 0$ there is $p_\varepsilon < \infty$ such that

$$|F_p(x,\xi) - F(x,\xi)| < \varepsilon |\xi|^2 \tag{2.4}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, whenever $p > p_\varepsilon$. Note that by (2.4), we may assume that $(\alpha_p, \beta_p) \to (\alpha, \beta)$ as $p \to \infty$. Since $\mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega) \neq \emptyset$ and Ω is bounded, it follows that $\mathcal{K}^{p}_{\psi,\vartheta}(\Omega) \neq \emptyset$ for every $p < \infty$, and hence there exists a unique solution $u_p \in W^{1,p}(\Omega)$ to the obstacle problem in $\mathcal{K}^{p}_{\psi,\vartheta}(\Omega)$, associated with the kernel F_p. We now have the following proposition which clearly implies Theorem 2.3.

2.5. Proposition. *There exists an increasing sequence $p_j \nearrow \infty$ and a function $u \in W^{1,\infty}(\Omega)$ such that $u_{p_j} \to u$ uniformly in Ω and u is a solution to the obstacle problem in $\mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega)$.*

Proof. Since $\mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega) \neq \emptyset$, we may assume without loss of generality that $\vartheta \in \mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega)$. By Hölder's inequality and the minimization property of u_p's, we have that

$$\begin{aligned}\left(\fint_\Omega F_p(x,\nabla u_p)^{m/2}\,dx\right)^{2/m} &\le \left(\fint_\Omega F_p(x,\nabla u_p)^{p/2}\,dx\right)^{2/p}\\ &\le \|F_p(x,\nabla\vartheta)\|_{\infty,\Omega}\end{aligned}$$

for $p \ge m$, where $n < m < \infty$ is chosen so that

$$\max\{|\alpha_p - \alpha|, |\beta_p - \beta|\} < \min\{\frac{\alpha}{2}, 1\}.$$

Since $F_p(x,\xi)$ satisfies (1.2), this implies that

$$\|\nabla u_p\|_{m,\Omega} \le C(\alpha,\beta)\|\nabla\vartheta\|_{\infty,\Omega}|\Omega|^{1/m},$$

that is, the family $\{u_p\}_{p\ge m}$ is uniformly bounded in $W^{1,m}(\Omega)$. Thus there exists a sequence $p_j \nearrow \infty$ and a function $u_\infty \in W^{1,m}(\Omega)$ such that

$$u_{p_j} \to u_\infty \qquad \text{weakly in } W^{1,m}(\Omega).$$

Next we use the assumption (2.4) and the calculations above to obtain

$$\begin{aligned}\|F(x,\nabla u_{p_j})\|_{m/2,\Omega} &\le \|F(x,\nabla u_{p_j}) - F_{p_j}(x,\nabla u_{p_j})\|_{m/2,\Omega} + \|F_{p_j}(x,\nabla u_{p_j})\|_{m/2,\Omega} \\ &\le \varepsilon \,\|\nabla u_{p_j}\|_m^2 \,+\, \|F_{p_j}(x,\nabla\vartheta)\|_{\infty,\Omega}\,|\Omega|^{2/m} \\ &\le O(\varepsilon) + \|F(x,\nabla\vartheta)\|_{\infty,\Omega}\,|\Omega|^{2/m}\end{aligned}$$

for $\varepsilon > 0$ and $p_j > p_\varepsilon$. Define

$$\|f\|_{F,m} = \left(\int_\Omega F(x,f(x))^{m/2}\,dx\right)^{1/m}$$

for $f \in L^m(\Omega;\mathbb{R}^n)$. By [15, 5.23], $\|\cdot\|_{F,m}$ is a norm in $L^m(\Omega;\mathbb{R}^n)$, equivalent to the usual L^m- norm, and thus $\nabla u_{p_j} \to \nabla u_\infty$ weakly also in the space $(L^m(\Omega;\mathbb{R}^n);\|\cdot\|_{F,m})$. This combined with the above calculations and the weak lower semicontinuity of norms gives

$$\left(\fint_\Omega F(x,\nabla u_\infty)^{m/2}\,dx\right)^{1/m} \le \|F(x,\nabla\vartheta)\|_{\infty,\Omega}^{1/2}. \tag{2.6}$$

Now notice that since the sequence $(u_{p_j})_{p_j\ge q}$ is bounded in $W^{1,q}(\Omega)$, $m \le q < \infty$, we have that $u_{p_j} \to u_\infty$ weakly in $W^{1,q}(\Omega)$ for every finite q. We conclude that (2.6) holds for any $q < \infty$, and hence

$$\begin{aligned}\|F(x,\nabla u_\infty)\|_{\infty,\Omega}^{1/2} &= \lim_{q\to\infty}\left(\fint_\Omega F(x,\nabla u_\infty)^{q/2}\,dx\right)^{1/q} \\ &\le \|F(x,\nabla\vartheta)\|_{\infty,\Omega}^{1/2}.\end{aligned}$$

Note that since $\vartheta \in \mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$, we have by the comparison principle and Lemma 1.16 that

$$\inf_\Omega \vartheta \le u_{p_j} \le \sup_\Omega \vartheta + \|\vartheta\|_{\infty,\Omega}$$

in Ω, and thus the sequence (u_{p_j}) is uniformly bounded. Further, by Morrey's lemma, see [11, p.143], $u_{p_j} - \vartheta \in C^{0,1-n/m}(\overline{\Omega})$, and, in particular, (u_{p_j}) is equicontinuous in Ω. Hence, using Ascoli's theorem, we may assume that $u_{p_j} \to u_\infty$ uniformly in Ω.

Now let $D \subset \Omega$ be an open set and suppose that $v \in W^{1,\infty}(D)$ is such that $u_\infty - v \in C_0(D)$. Let $v_{p_j} \in W^{1,p_j}(D)$ be the unique solution to the obstacle problem in $\mathcal{K}^{p_j}_{\psi,v}(D)$, corresponding to the kernel F_{p_j}. Repeating the above reasoning, we find a subsequence of (p_j), denoted again by (p_j), and a function $v_\infty \in W^{1,\infty}(D)$ such that $v_{p_j} \to v_\infty$ uniformly in D and

$$\|F(x,\nabla v_\infty)\|_{\infty,D} \le \|F(x,\nabla v)\|_{\infty,D}.$$

By Lemma 1.16,

$$\begin{aligned}\|u_\infty - v_\infty\|_{\infty,D} &\le \|u_\infty - u_{p_j}\|_{\infty,D} + \|u_{p_j} - v_{p_j}\|_{\infty,D} + \|v_{p_j} - v_\infty\|_{\infty,D}\\ &\le \|u_\infty - u_{p_j}\|_{\infty,D} + \|u_{p_j} - u_\infty\|_{\infty,\partial D} + \|v_{p_j} - v_\infty\|_{\infty,D},\end{aligned}$$

which implies that $u_\infty = v_\infty$ in D, and, in particular, that

$$\|F(x,\nabla u_\infty)\|_{\infty,D} \le \|F(x,\nabla v)\|_{\infty,D}.$$

This shows that u_∞ is a solution to the obstacle problem in $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$, and the proposition is thereby proved. $\square$

Following the terminology in [4], we from now on call the solutions obtained by the above limiting process *variational solutions to the obstacle problem* and *variational F-absolute minimizers*, respectively. Jensen has proved in [17] that if $F(x,\xi) = |\xi|^2$, then every F-absolute minimizer is variational. Unfortunately, we have not so far been able to prove this for a general kernel $F(x,\xi)$.

We next establish the Harnack inequality for nonnegative variational solutions to the obstacle problem in $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$. Theorem 2.7 below was first obtained in the case $\psi \equiv -\infty$ and $F(x,\xi) = |\xi|^2$ by Lindqvist and Manfredi in [21], see also [10] and [22].

2.7. Theorem. *Suppose that $u \in W^{1,\infty}(B(x_0,R))$ is a nonnegative variational solution to the obstacle problem in $\mathcal{K}^\infty_{\psi,\vartheta}(B(x_0,R))$. If $0 < r < R$ and $x,y \in B(x_0,r)$, then*

$$u(x) \le e^{C(\alpha,\beta)\frac{|x-y|}{R-r}}u(y).$$

Proof. Denote $B = B(x_0,R)$ and assume first that $u \ge \varepsilon > 0$ in B. Let $p_j \nearrow \infty$ and $u_j \in W^{1,p_j}(B)$ be the unique solution to the obstacle problem in $\mathcal{K}^{p_j}_{\psi,\vartheta}(B)$, corresponding to the kernel F_{p_j}, such that (u_j) is converging to u both uniformly in B and weakly in $W^{1,m}(B)$ for every $m < \infty$. Then for j large enough, we have that $u_j \ge \varepsilon/2 > 0$ in B, and thus $\log u_j \in W^{1,p_j}(B)$. Furthermore,

$$\log u_j \to \log u \qquad \text{weakly in } W^{1,m}(B)$$

for all $m < \infty$.

Recall now that u_j is a weak supersolution of the Euler equation (1.13) in B. Let $\zeta \in C^\infty_c(B)$ be a nonnegative cut-off function satisfying $\zeta = 1$ in $B(x_0,r)$ and $\|\nabla\zeta\|_{\infty,B} \le 2/(R-r)$, and denote $\eta = \zeta^{p_j}{u_j}^{1-p_j} \in W^{1,p_j}_0(B)$. Using η as a test function in the weak formulation of (1.13), we obtain

$$\begin{aligned}&(p_j-1)\int_B \left(\frac{\zeta}{u_j}\right)^{p_j} F_{p_j}(x,\nabla u_j)^{\frac{p_j-2}{2}}\nabla_\xi F_{p_j}(x,\nabla u_j)\cdot\nabla u_j\,dx\\ &\le p_j\int_B \left(\frac{\zeta}{u_j}\right)^{p_j-1} F_{p_j}(x,\nabla u_j)^{\frac{p_j-2}{2}}\nabla_\xi F_{p_j}(x,\nabla u_j)\cdot\nabla\zeta\,dx.\end{aligned} \tag{2.8}$$

Since

$$\nabla_\xi F_p(x,\xi)\cdot\xi \geq \frac{\alpha_p}{\beta_p} F_p(x,\xi)$$

and

$$|\nabla_\xi F_p(x,\xi)| \leq \frac{4\beta_p}{\sqrt{\alpha_p}} F_p(x,\xi)^{1/2}$$

by (1.2), (1.5) and (1.6), we have by (2.8) and Hölder's inequality that

$$\begin{aligned}
&(p_j-1)\frac{\alpha_j}{\beta_j}\int_B \left(\zeta\frac{F_{p_j}(x,\nabla u_j)^{1/2}}{u_j}\right)^{p_j} dx \\
&\leq p_j\frac{4\beta_j}{\sqrt{\alpha_j}}\int_B \left(\zeta\frac{F_{p_j}(x,\nabla u_j)^{1/2}}{u_j}\right)^{p_j-1} |\nabla\zeta|\,dx \\
&\leq p_j\frac{4\beta_j}{\sqrt{\alpha_j}}\left(\int_B \left(\zeta\frac{F_{p_j}(x,\nabla u_j)^{1/2}}{u_j}\right)^{p_j} dx\right)^{(p_j-1)/p_j} \left(\int_B |\nabla\zeta|^{p_j}\,dx\right)^{1/p_j}.
\end{aligned}$$

This in turn implies that

$$\left(\int_B \left(\zeta\frac{F_{p_j}(x,\nabla u_j)^{1/2}}{u_j}\right)^{p_j} dx\right)^{1/p_j} \leq \frac{p_j}{p_j-1}\frac{4\beta_j^2}{\alpha_j\sqrt{\alpha_j}}\left(\int_B |\nabla\zeta|^{p_j}\,dx\right)^{1/p_j},$$

and hence

$$\left(\int_B |\zeta\nabla\log u_j|^{p_j}\,dx\right)^{1/p_j} \leq \left(\frac{2\beta_j}{\alpha_j}\right)^2\left(\frac{p_j}{p_j-1}\right)\left(\int_B |\nabla\zeta|^{p_j}\,dx\right)^{1/p_j}. \tag{2.9}$$

Next fix $m>n$. Using (2.9), we have for $p_j \geq m$ that

$$\begin{aligned}
\left(\fint_B |\zeta\nabla\log u_j|^m\,dx\right)^{1/m} &\leq \left(\fint_B |\zeta\nabla\log u_j|^{p_j}\,dx\right)^{1/p_j} \\
&\leq \left(\frac{2\beta_j}{\alpha_j}\right)^2\left(\frac{p_j}{p_j-1}\right)\|\nabla\zeta\|_{\infty,B}.
\end{aligned}$$

Since $\zeta\log u_j$ converges weakly to $\zeta\log u$ in $W^{1,m}(B)$, we conclude by the weak lower semicontinuity of norms that

$$\left(\fint_B |\zeta\nabla\log u|^m\,dx\right)^{1/m} \leq \left(\frac{2\beta}{\alpha}\right)^2\|\nabla\zeta\|_{\infty,B}.$$

Letting $m \to \infty$ then gives

$$\|\zeta \nabla \log u\|_{\infty,B} \leq \left(\frac{2\beta}{\alpha}\right)^2 \|\nabla \zeta\|_{\infty,B}.$$

In particular, since $\zeta = 1$ in $B(x_0, r)$ and $|\nabla \zeta| \leq 2/(R-r)$ in B, we obtain the estimate

$$\|\nabla \log u\|_{\infty,B(x_0,r)} \leq \frac{8\beta^2}{\alpha^2(R-r)}. \tag{2.10}$$

Now let $x, y \in B(x_0, r)$. By (2.10),

$$\log u(x) - \log u(y) \leq \frac{8\beta^2}{\alpha^2(R-r)}|x-y|,$$

from which we obtain by exponentiating that

$$u(x) \leq e^{C(\alpha,\beta)\frac{|x-y|}{R-r}} u(y). \tag{2.11}$$

This proves the theorem in the case $u \geq \varepsilon > 0$. If we only have that u is nonnegative, we obtain (2.11) for $u + \varepsilon$, which is evidently a variational solution to the obstacle problem in $\mathcal{K}^\infty_{\psi+\varepsilon,\vartheta+\varepsilon}(B)$. The assertion now follows by letting $\varepsilon \to 0$. $\square$

We finish this section by considering an example that motivates the condition (2.4). Let $f \colon \Omega \to \mathbb{R}^n$ be an L-bilipschitz mapping, that is,

$$\frac{1}{L}|x-y| \leq |f(x) - f(y)| \leq L|x-y|$$

for all $x, y \in \Omega$. Then obviously f is injective, and also f^{-1} is a bilipschitz mapping with the same constant L. Furthermore, by Rademacher's theorem, f and f^{-1} are differentiable a.e. and

$$\frac{1}{L} \leq \|f'(x)\| \leq L$$

for a.e. $x \in \Omega$. This implies the double inequality

$$\frac{1}{L^n} \leq |J_f(x)| \leq L^n \tag{2.12}$$

for a.e. $x \in \Omega$, where $J_f(x) = \det f'(x)$ is the Jacobian determinant.

For completeness, we first recall an analogous result for $p < \infty$. We assume that the kernel $F(x,\xi)$ is defined everywhere in $\mathbb{R}^n \times \mathbb{R}^n$, and define the pull-back of F under f to be

$$f^{\#,p}F(x,\xi) = |J_f(x)|^{2/p} F(f(x), f'(x)^{-1^*}\xi),$$

whenever $x \in \Omega$ is such that f is differentiable at x, and

$$f^{\#,p}F(x,\xi) = F(x,\xi)$$

otherwise. Note that $f^{\#,p}F$ satisfies (1.1) – (1.4) with constants $L^{-2-2n/p}\alpha$ and $L^{2+2n/p}\beta$, where α and β are the corresponding constants for F.

2.13. Lemma. *Suppose that $u \in W^{1,p}_{loc}(\Omega')$ is an (F,p)-extremal in Ω' and that $f\colon \Omega \to \mathbb{R}^n$ is a L-bilipschitz mapping. Then $u \circ f \in W^{1,p}_{loc}(f^{-1}(\Omega'))$ is an $(f^{\#,p}F,p)$-extremal in $f^{-1}(\Omega')$.*

Proof. Denote $v = u \circ f$ and fix an open set $D \Subset f^{-1}(\Omega')$. Then $\nabla v(x) = f'(x)^* \nabla u(f(x))$ a.e. in D, and we obtain by the integral transformation formula for bilipschitz mappings, see [11, p.99], that

$$\begin{aligned} \int_D \big(f^{\#,p}F(x,\nabla v(x))\big)^{p/2}\,dx &= \int_D \Big(|J_f(x)|^{2/p} F(f(x),(f'(x)^*)^{-1}\nabla v(x))\Big)^{p/2}\,dx \\ &= \int_D F(f(x),\nabla u(f(x)))^{p/2}|J_f(x)|\,dx \\ &= \int_{f(D)} F(y,\nabla u(y))^{p/2}\,dy. \end{aligned} \tag{2.14}$$

If $w \in W^{1,p}(D)$ is such that $v - w \in W^{1,p}_0(D)$, then the function $\phi = w \circ f^{-1}$ is in $W^{1,p}(f(D))$ and $u - \phi \in W^{1,p}_0(f(D))$. Furthermore, repeating the above reasoning, we have that

$$\int_D \big(f^{\#,p}F(x,\nabla w(x))\big)^{p/2}\,dx = \int_{f(D)} F(y,\nabla\phi(y))^{p/2}\,dy.$$

Combining this with (2.14) and the assumption that u is an (F,p)-extremal proves the lemma. $\square$

For the case $p = \infty$ we define

$$f^{\#,\infty}F(x,\xi) = F(f(x), {f'(x)^{-1}}^{*}\xi),$$

whenever $x \in \Omega$ is such that f is differentiable at x, and

$$f^{\#,\infty}F(x,\xi) = F(x,\xi)$$

otherwise. Again the assumptions (1.1) – (1.4) are satisfied, this time with constants $L^{-2}\alpha$ and $L^2\beta$.

2.15. Lemma. *Suppose that $u \in W^{1,\infty}_{loc}(\Omega')$ is an F-absolute minimizer in Ω' and that $f\colon \Omega \to \mathbb{R}^n$ is a L-bilipschitz mapping. Then $u \circ f \in W^{1,\infty}_{loc}(f^{-1}(\Omega'))$ is an $f^{\#,\infty}F$-absolute minimizer in $f^{-1}(\Omega')$.*

Proof. Since $f\colon f^{-1}(\Omega') \to \Omega' \cap f(\Omega)$ is one-to-one and we have the a.e. pointwise equality $\nabla(v \circ f)(x) = f'(x)^* \nabla v(f(x))$ for all $v \in W^{1,\infty}_{loc}(\Omega' \cap f(\Omega))$, the lemma follows immediately. $\square$

Note that

$$\begin{aligned}|f^{\#,\infty}F(x,\xi) - f^{\#,p}F(x,\xi)| &\le |1-|J_f(x)|^{2/p}||F(f(x), {f'(x)^{-1}}^* \xi)| \\ &\le L^2\beta|1-|J_f(x)|^{2/p}||\xi|^2\end{aligned}$$

for a.e. $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$, and thus, using (2.12), we see that the condition (2.4) is valid for $f^{\#,\infty}F$ and $f^{\#,p}F$. Roughly speaking, the above discussion means that the diagram

$$\begin{tikzcd} (f^{\#,p}F, p) \arrow[r, "p\to\infty"] \arrow[d, "f"'] & (f^{\#,\infty}F, \infty) \arrow[d, "f"] \\ (F, p) \arrow[r, "p\to\infty"] & (F, \infty) \end{tikzcd}$$

commutes as one can expect.

3. Euler equations

In the classical calculus of variations, many important features of the minimizers are obtained by investigating the properties of the Euler equation associated with the minimization problem. If $p < \infty$, then the weak formulation of the Euler equation of the problem

$$\min\left\{\int_\Omega F(x,\nabla v)^{p/2}\,dx \;:\quad v - \vartheta \in W_0^{1,p}(\Omega)\right\}$$

is derived by taking the derivative of the function

$$t \mapsto \int_\Omega F(x, \nabla(u + t\varphi))^{p/2}\,dx,$$

where u is a minimizer and $\varphi \in C_c^\infty(\Omega)$, at the point $t = 0$. For $p = \infty$ this approach does not work, and we have to find another way to determine the Euler equation. As indicated in [4] and [17], the right answer is to take a formal limit of the Euler equations of the corresponding L^p-problems and then use the notion of viscosity solutions. In order to do this successfully, we assume that $F(x,\xi)$ satisfies the following conditions introduced already in Section 1:

(3.1) $$\text{the mapping } (x,\xi) \mapsto \nabla_\xi F(x,\xi) \text{ is continuous,}$$

(3.2) $$\begin{aligned}&\text{the mapping } x \mapsto F(x,\xi) \text{ is differentiable for all } \xi \in \mathbb{R}^n \\ &\text{and } (x,\xi) \mapsto \nabla_x F(x,\xi) \text{ is continuous,}\end{aligned}$$

and

$$\text{for every } j = 1, \dots, n \text{ the second order partial derivative} \tag{3.3}$$
$$\frac{\partial^2 F}{\partial x_j \partial \xi_j}(x, \xi) \text{ exists and is continuous.}$$

These conditions guarantee that the equations appearing below are pointwise well-defined and continuous.

The next theorem is the main result of this section. For simplicity, we consider only the case $F_p = F$ for all $n < p \leq \infty$. In the general case one easily obtains a similar result by assuming that the occurring derivatives of $F_p(x, \xi)$ are converging to the derivatives of $F(x, \xi)$.

3.4. Theorem. *If $u \in W^{1,\infty}(\Omega)$ is a variational solution to the obstacle problem in $\mathcal{K}^{\infty}_{\psi,\vartheta}(\Omega)$, then it is a viscosity supersolution of*

$$-\Delta_{\infty,F}\, u(x) = 0 \tag{3.5}$$

in Ω, where $-\Delta_{\infty,F}\, u(x) = -\nabla(F(x, \nabla u(x))) \cdot \nabla_\xi F(x, \nabla u(x))$. Furthermore, if $\psi\colon \Omega \to [-\infty, \infty)$ is continuous, then u is a viscosity solution of (3.5) in the open set $\{x \in \Omega\colon\ u(x) > \psi(x)\}$.

Proof. Let $p_j \nearrow \infty$ and $u_j \in W^{1,p_j}(\Omega)$ be the unique solution to the obstacle problem in $\mathcal{K}^{p_j}_{\psi,\vartheta}(\Omega)$ such that $u_j \to u$ uniformly in Ω, see the proof of Proposition 2.5. Fix a point $x_0 \in \Omega$ and a function $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) > \varphi(x)$ for all $x \in \Omega \setminus \{x_0\}$. We want to show that

$$-\Delta_{\infty,F}\, \varphi(x_0) \geq 0. \tag{3.6}$$

Let $R > 0$ be such that $B = B(x_0, R) \Subset \Omega$, and note that

$$\inf_{r \leq |y - x_0| \leq R} (u(y) - \varphi(y)) > 0$$

for $0 < r < R$. Since $u_j \to u$ uniformly, there exists an index j_r such that

$$\inf_{r \leq |y - x_0| \leq R} (u_j(y) - \varphi(y)) > u_j(x_0) - \varphi(x_0)$$

for all $j > j_r$. In particular, if we choose a point x_j such that the function $u_j - \varphi$ attains its minimum in $\overline{B}$ at x_j, then $x_j \in B(x_0, r)$ for all $j > j_r$. By letting $r \to 0$, we see that $x_j \to x_0$ as $j \to \infty$.

Next we recall that by Theorems 1.11 and 1.29, u_j is a viscosity supersolution of (1.13) in Ω. This implies that

$$-\left(\frac{p_j - 2}{2}\right) F(x_j, \nabla\varphi)^{\frac{p_j-4}{2}} \Delta_{\infty,F}\, \varphi(x_j) - F(x_j, \nabla\varphi)^{\frac{p_j-2}{2}} \operatorname{div}\big(\nabla_\xi F(x_j, \nabla\varphi)\big) \geq 0 \tag{3.7}$$

for all j large enough. If $\nabla\varphi(x_0)=0$, then by (1.5) also $\nabla_\xi F(x_0,\nabla\varphi(x_0))=0$, and hence (3.6) is satisfied. Thus we may assume that $|\nabla\varphi(x_0)|>\gamma>0$, which implies that $|\nabla\varphi(x_j)|>\gamma/2$ for all sufficiently large j. In particular, by (1.2), $|F(x_j,\nabla\varphi(x_j))|$ is bounded away from zero for j large enough, and thus we may divide in (3.7) by $\left(\frac{p_j-2}{2}\right)F(x_j,\nabla\varphi(x_j))^{(p_j-4)/2}$. This leaves us with

$$-\Delta_{\infty,F}\,\varphi(x_j)\geq\frac{2F(x_j,\nabla\varphi(x_j))\operatorname{div}\bigl(\nabla_\xi F(x_j,\nabla\varphi(x_j))\bigr)}{p_j-2}.$$

Since $\varphi\in C^2(\Omega)$ and $x_j\to x_0$, the numerator

$$2F(x_j,\nabla\varphi(x_j))\operatorname{div}\bigl(\nabla_\xi F(x_j,\nabla\varphi(x_j))\bigr)$$

is bounded independently of j, and hence the quotient on the right-hand side of the above inequality is converging to 0 as $j\to\infty$. We conclude that (3.6) holds, and the first assertion is thereby proved.

In order to prove the second assertion, it suffices to show that u is a viscosity subsolution of (3.5) in B for every ball $B\Subset\{x\in\Omega\colon\ u(x)>\psi(x)\}$. Such a ball B being fixed, we note that since $u_j\to u$ uniformly, $u_j>\psi$ in B for j large enough. Hence by Theorems 1.15 and 1.31, u_j is a viscosity subsolution of (1.13) in B, and we obtain the second assertion by repeating the arguments used in the supersolution case. □

Another way of stating Theorem 3.4 in the case of a continuous obstacle is to say that a variational solution to the obstacle problem in $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$ is a viscosity solution of

$$\min\bigl\{u(x)-\psi(x),-\Delta_{\infty,F}\,u(x)\bigr\}=0$$

in Ω with boundary values ϑ. Hence, if $\psi\equiv-\infty$, we immediately obtain the following corollary.

3.8. Corollary. *If $u\in W^{1,\infty}(\Omega)$ is a variational F-absolute minimizer, then u is a viscosity solution of* (3.5) *in* Ω.

In the case $F(x,\xi)=\theta(x)\xi\cdot\xi$, the assumptions (3.1) – (3.3) are satisfied if $\theta\in C^1(\Omega)$, and the Euler equation then takes the form

$$-\nabla\bigl(\theta(x)\nabla u\cdot\nabla u\bigr)\cdot\theta(x)\nabla u=0.$$

Expanding this gives

$$-D^2u\bigl(\theta(x)\nabla u\bigr)\cdot\theta(x)\nabla u-\frac{1}{2}\nabla_\theta\bigl(\theta(x)\nabla u\cdot\nabla u\bigr)\cdot\theta(x)\nabla u=0,$$

where $\nabla_\theta(\theta(x)\nabla u\cdot\nabla u)$ is a vector in $\mathbb{R}^n$ with

$$\frac{\partial\theta}{\partial x_i}(x)\nabla u(x)\cdot\nabla u(x)$$

as its i^{th} component. In particular, if $\theta(x)=I$ for all $x\in\Omega$, the Euler equation reduces to the ∞-*Laplacian*

$$-\Delta_\infty u(x)=-D^2u(x)\nabla u(x)\cdot\nabla u(x).$$

The viscosity solutions of the ∞-Laplacian are often called ∞-*harmonic functions* and they have a geometric interpretation as absolutely minimizing Lipschitz extensions, see [1], [17]. Jensen [17] proved that in the case $F(x,\xi)=|\xi|^2$ every F-absolute minimizer is a viscosity solution of the Euler equation. We do not know whether this is true or not for a general kernel $F(x,\xi)$, but we still call (3.5) the Euler equation of our minimization problem.

In the remainder of this section, we discuss properties of the following auxiliary equations introduced by Jensen in [17]:

$$\min\left\{F(x,\nabla u(x))-f(x)^2,-\Delta_{\infty,F}\,u(x)\right\}=0 \tag{3.9}$$

$$\max\left\{f(x)^2-F(x,\nabla u(x)),-\Delta_{\infty,F}\,u(x)\right\}=0 \tag{3.10}$$

Here, $f\in C(\Omega)\cap L^\infty(\Omega)$ is a positive and real-valued function. These equations play a crucial role in the next section, where we prove the comparison principle for viscosity solutions of (3.5). Note that a solution of (3.5) is a subsolution of (3.9) and a supersolution of (3.10). Furthermore, a solution of (3.9) is always a supersolution of (3.5), and, correspondingly, a solution of (3.10) is a subsolution of (3.5). Yet the most important feature of each of these equations is that the gradient of a solution is at least formally nonvanishing.

3.11. Theorem. *For each $\vartheta\in W^{1,\infty}(\Omega)$ there exist viscosity solutions $\overline{u}$ and $\underline{u}$ of* (3.9) *and* (3.10)*, respectively, such that $\overline{u}-\vartheta,\underline{u}-\vartheta\in W^{1,\infty}(\Omega)\cap C_0(\Omega)$.*

Proof. We first consider equation (3.9). Let $p>n$ and let $\overline{u}_p\in W^{1,p}(\Omega)$ be the unique weak solution of

$$-\operatorname{div}\left(F(x,\nabla v)^{(p-2)/2}\nabla_\xi F(x,\nabla v)\right)=\frac{1}{p}f(x)^{p-1} \tag{3.12}$$

in Ω with boundary values ϑ. Then $\overline{u}_p$ is a minimizer of the functional

$$\int_\Omega F(x,\nabla v)^{p/2}dx-\frac{1}{2}\int_\Omega f^{p-1}v\,dx,\qquad v-\vartheta\in W_0^{1,p}(\Omega).$$

Arguing as in the proof of Proposition 2.5, we obtain the estimate

$$(3.13)\quad \left(\fint_\Omega |\nabla \overline{u}_p|^m\, dx\right)^{1/m} \le \frac{1}{\sqrt{\alpha}}\left[\beta^{p/2}\|\nabla\vartheta\|_{\infty,\Omega}^p + \frac{1}{2}\|f\|_{\infty,\Omega}^{p-1}\|\vartheta - \overline{u}_p\|_{\infty,\Omega}\right]^{1/p}$$

for $p > m > n$. Since by the Sobolev embedding

$$\|\vartheta - \overline{u}_p\|_{\infty,\Omega} \le C\|\nabla(\vartheta - \overline{u}_p)\|_{m,\Omega},$$

(3.13) implies that

$$\|\nabla\overline{u}_p\|_{m,\Omega} \le C(\Omega,\alpha,\beta,m,n,f)\left(1 + \|\nabla\vartheta\|_{\infty,\Omega} + \|\nabla\overline{u}_p\|_{m,\Omega}^{1/m}\right)$$

for all $p \ge m$. Hence $\{\overline{u}_p\}_{p\ge m}$ is bounded in $W^{1,m}(\Omega)$, and we can use the same kind of reasoning as in the proof of Proposition 2.5 to conclude that there exists a sequence $p_j \nearrow \infty$ and a function $\overline{u} \in W^{1,\infty}(\Omega)$ such that $\overline{u}_{p_j} \to \overline{u}$ uniformly in Ω and weakly in $W^{1,m}(\Omega)$ for all finite m. We prove that this limit function $\overline{u}$ is the desired solution of (3.9).

We begin by showing that $\overline{u}$ is a subsolution. Let $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ be such that $\overline{u}(x_0) = \varphi(x_0)$ and $\overline{u}(x) < \varphi(x)$ for all $x \ne x_0$. We want to show that

$$(3.14)\qquad \min\left\{F(x_0,\nabla\varphi(x_0)) - f(x_0)^2, -\Delta_{\infty,F}\,\varphi(x_0)\right\} \le 0.$$

As in the proof of Theorem 3.4, we find a sequence $x_j \to x_0$ such that x_j is a local maximum point of $\overline{u}_{p_j} - \varphi$. Since by Theorem 1.31 $\overline{u}_{p_j}$ is a viscosity subsolution of (3.12), we have that

$$(3.15)\qquad \begin{aligned} &-\left(\frac{p_j-2}{2}\right)F(x_j,\nabla\varphi(x_j))^{\frac{p_j-4}{2}}\Delta_{\infty,F}\,\varphi(x_j)\\ &-F(x_j,\nabla\varphi(x_j))^{\frac{p_j-2}{2}}\operatorname{div}\big(\nabla_\xi F(x_j,\nabla\varphi(x_j))\big) \le \frac{1}{p_j}f(x_j)^{p_j-1}\end{aligned}$$

for every $j = 1,2,\dots$. If $\nabla\varphi(x_0) = 0$, we see, using (1.2), that (3.14) is clearly satisfied. Hence we may assume that $|\nabla\varphi(x_j)| \ge \gamma > 0$ for some constant γ and for all j large enough. Dividing in (3.15) by $\left(\frac{p_j-2}{2}\right)F(x_j,\nabla\varphi(x_j))^{(p_j-4)/2} \ne 0$, we obtain

(3.16)

$$-\Delta_{\infty,F}\,\varphi(x_j) \le \frac{2F(x_j,\nabla\varphi)\operatorname{div}\big(\nabla_\xi F(x_j,\nabla\varphi)\big)}{p_j-2} + \frac{2f(x_j)^3}{p_j(p_j-2)}\left(\frac{f(x_j)^2}{F(x_j,\nabla\varphi)}\right)^{\frac{p_j-4}{2}}.$$

If $F(x_0,\nabla\varphi(x_0))-f(x_0)^2>0$, then

$$\left(\frac{f(x_j)^2}{F(x_j,\nabla\varphi(x_j))}\right)^{(p_j-4)/2}\to 0$$

as $j\to\infty$, and thus $-\Delta_{\infty,F}\,\varphi(x_0)\le 0$. This proves that (3.14) always holds, and hence $\overline{u}$ is a subsolution of (3.9).

To prove that $\overline{u}$ is a supersolution, we again fix a point $x_0\in\Omega$ and a test function $\phi\in C^2(\Omega)$ such that $\overline{u}(x_0)=\phi(x_0)$ and $\overline{u}(x)>\phi(x)$ for all $x\neq x_0$. Our aim is to show that

$$\min\bigl\{F(x_0,\nabla\phi(x_0))-f(x_0)^2,-\Delta_{\infty,F}\phi(x_0)\bigr\}\ge 0. \tag{3.17}$$

As before, we find a sequence $x_j\to x_0$ such that x_j is a local minimum point of $\overline{u}_{p_j}-\phi$, and then, using Theorem 1.29, we have that

$$\begin{aligned}&-\left(\frac{p_j-2}{2}\right)F(x_j,\nabla\phi(x_j))^{\frac{p_j-4}{2}}\Delta_{\infty,F}\phi(x_j)\\ &-F(x_j,\nabla\phi(x_j))^{\frac{p_j-2}{2}}\operatorname{div}\bigl(\nabla_\xi F(x_j,\nabla\phi(x_j))\bigr)\ge\frac{1}{p_j}f(x_j)^{p_j-1}.\end{aligned} \tag{3.18}$$

If $F(x_j,\nabla\phi(x_j))=0$, then the left-hand side of (3.18) is equal to 0, which is impossible by the positivity of f. Hence we may divide both sides in (3.18) by $\left(\frac{p_j-2}{2}\right)F(x_j,\nabla\phi(x_j))^{(p_j-4)/2}$ to obtain

(3.19)
$$-\Delta_{\infty,F}\phi(x_j)\ge\frac{2F(x_j,\nabla\phi)\operatorname{div}\bigl(\nabla_\xi F(x_j,\nabla\phi)\bigr)}{p_j-2}+\frac{2f(x_j)^3}{p_j(p_j-2)}\left(\frac{f(x_j)^2}{F(x_j,\nabla\phi)}\right)^{\frac{p_j-4}{2}}.$$

If we had $f(x_0)^2>F(x_0,\nabla\phi(x_0))$, then (3.19) would imply by the assumed continuity that $-\Delta_{\infty,F}\phi(x_0)\ge\infty$, which is clearly a contradiction. Therefore $F(x_0,\nabla\phi(x_0))-f(x_0)^2\ge 0$, and we also have by (3.19) that $-\Delta_{\infty,F}\phi(x_0)\ge 0$. This proves that $\overline{u}$ is a supersolution of (3.9), and thus completes the proof of our assertion concerning equation (3.9).

For equation (3.10), we simply note that if v is a viscosity solution of (3.9), then by (1.4) and (1.7) $-v$ is a solution of (3.10). □

3.20. Remark. In the above proof it is not essential that we have precisely $\frac{1}{p}f(x)^{p-1}$ on the right-hand side of (3.12). Indeed, the same arguments can be used to show that equation (3.9) is satisfied by any limit function v_∞ of the family $\{v_p\}$ of solutions to the equation

$$-\operatorname{div}\Bigl(F(x,\nabla v_p)^{(p-2)/2}\nabla_\xi F(x,\nabla v_p)\Bigr)=f_p(x)^p,$$

where $f_p \in C(\Omega) \cap L^\infty(\Omega)$ are positive and satisfy $f_p \to f$ uniformly as $p \to \infty$.

For the next lemma, we assume that $F(x,\xi)$ satisfies the following *strong monotonicity condition*: There exists a constant $\sigma > 0$ and a number $p_0 < \infty$ such that

$$\left(F(x,\xi_1)^{\frac{p-2}{2}} \nabla_\xi F(x,\xi_1) - F(x,\xi_2)^{\frac{p-2}{2}} \nabla_\xi F(x,\xi_2)\right) \cdot (\xi_1 - \xi_2) \geq \sigma^p |\xi_1 - \xi_2|^p \tag{3.21}$$

for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^n$ and $p > p_0$.

3.22. Lemma. *For each $\vartheta \in W^{1,\infty}(\Omega)$ there exist viscosity solutions $\overline{u}$ and $\underline{u}$ of (3.9) and (3.10), respectively, with boundary values ϑ for which*

$$0 \leq \overline{u}(x) - \underline{u}(x) \leq \frac{1}{\sigma} \|f\|_{\infty,\Omega} \, dist(x, \partial\Omega)$$

for all $x \in \Omega$.

Proof. By Theorem 3.11, we can construct a solution $\overline{u}$ of (3.9) as a limit of weak solutions $\overline{u}_{p_j}$ of (3.12) with boundary values ϑ. Furthermore, by the last remark in the proof of Theorem 3.11, $\underline{u}$ can be obtained as a limit function of a subsequence of $(\underline{u}_{p_j})$, denoted again by $(\underline{u}_{p_j})$, where $\underline{u}_{p_j}$ is the unique weak solution of

$$-\operatorname{div}\left(F(x,\nabla\varphi)^{(p_j-2)/2} \nabla_\xi F(x,\nabla\varphi)\right) = -\frac{1}{p_j} f(x)^{p_j-1} \tag{3.23}$$

with boundary values ϑ. Note that since $\overline{u}_{p_j}$ is a weak supersolution and $\underline{u}_{p_j}$ a weak subsolution of (1.13), we have by the comparison principle that $\overline{u}_{p_j} \geq \underline{u}_{p_j}$, and thus $\overline{u} \geq \underline{u}$ in Ω. On the other hand, using $\overline{u}_{p_j} - \underline{u}_{p_j} \in W_0^{1,p_j}(\Omega)$ as a test function in the weak formulations of (3.12) and (3.23) and subtracting the resulting equations, we obtain by applying (3.21) and the Poincaré inequality (see [14, 7.44]) that

$$\begin{aligned}
\fint_\Omega |\nabla\overline{u}_{p_j} - \nabla\underline{u}_{p_j}|^{p_j}\, dx &\leq \frac{2}{p_j \sigma^{p_j}} \fint_\Omega f^{p_j-1} (\overline{u}_{p_j} - \underline{u}_{p_j})\, dx \\
&\leq \frac{2}{p_j\sigma^{p_j}} \left(\fint_\Omega |f|^{p_j}\, dx\right)^{(p_j-1)/p_j} \left(\fint_\Omega |\overline{u}_{p_j} - \underline{u}_{p_j}|^{p_j}\, dx\right)^{1/p_j} \\
&\leq \frac{2}{p_j\sigma^{p_j}} \|f\|_{\infty,\Omega}^{p_j-1} \left(\frac{|\Omega|}{\omega_n}\right)^{1/n} \left(\fint_\Omega |\nabla\overline{u}_{p_j} - \nabla\underline{u}_{p_j}|^{p_j}\, dx\right)^{1/p_j},
\end{aligned}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. This implies that

$$\begin{aligned}
\left(\fint_\Omega |\nabla\overline{u}_{p_j} - \nabla\underline{u}_{p_j}|^{q}\, dx\right)^{1/q} &\leq \left(\fint_\Omega |\nabla\overline{u}_{p_j} - \nabla\underline{u}_{p_j}|^{p_j}\, dx\right)^{1/p_j} \\
&\leq \left(\frac{c(n,\Omega)}{p_j}\right)^{1/(p_j-1)} \sigma^{-p_j/(p_j-1)} \|f\|_{\infty,\Omega},
\end{aligned}$$

whenever $p_j \geq q$, and hence we have by the weak lower semicontinuity of norms that

$$\left(\fint_\Omega |\nabla \overline{u} - \nabla \underline{u}|^q \, dx \right)^{1/q} \leq \frac{1}{\sigma} \|f\|_{\infty,\Omega}$$

for any $q < \infty$. Consequently,

$$\|\nabla \overline{u} - \nabla \underline{u}\|_{\infty,\Omega} \leq \frac{1}{\sigma} \|f\|_{\infty,\Omega},$$

which clearly implies the claim because $\overline{u} - \underline{u} \in C_0(\Omega)$. $\square$

4. The comparison principle and uniqueness

In this section, we prove the comparison principle for the viscosity solutions of (3.5), and consequently obtain that the Dirichlet problem

$$\begin{cases} -\Delta_{\infty,F}\, v(x) = 0 & \text{in } \Omega \\ \qquad\qquad v = g & \text{on } \partial\Omega \end{cases}$$

has a unique viscosity solution for every $g \in C(\partial\Omega)$. The idea of the proof is to show first that the comparison principle is true for equations (3.9) and (3.10), and then, by the results obtained at the end of Section 3, to conclude that it is indeed true also for equation (3.5). This indirect method is due to Jensen, who in [17] proved the result in the case of the ∞-Laplacian. Our proof is a rather straightforward generalization of Jensen's proof, although we have simplified his arguments by using the maximum principle for semicontinuous functions. In order that the proof would work, we have to assume that the kernel $F(x,\xi)$ satisfies some additional regularity conditions that will be specified later.

We begin by constructing "strict" supersolutions of (3.9).

4.1. Lemma. *Suppose that w is a bounded viscosity supersolution of* (3.9) *in Ω. Then for every $\gamma > 0$ there is a function $\hat{w}$ such that $\|w - \hat{w}\|_{\infty,\Omega} < \gamma$ and $\hat{w}$ is a viscosity supersolution of*

$$\min\left\{F(x, \nabla v(x)) - f(x)^2, -\Delta_{\infty,F}\, v(x)\right\} = \mu \min\{1, f(x)^4\}$$

for some $\mu = \mu(w,\alpha,\beta,\gamma) > 0$.

Proof. We look for the function $\hat{w}$ in a form $\hat{w} = g(w)$, where $g\colon \mathbb{R} \to \mathbb{R}$ is a smooth and increasing function such that g^{-1} is also smooth. Let $x_0 \in \Omega$ and suppose that $\varphi \in C^2(\Omega)$ is such that $\hat{w}(x_0) = \varphi(x_0)$ and $\hat{w}(x) \geq \varphi(x)$ for all $x \in \Omega$. Then for $\phi = g^{-1}(\varphi)$ we have that $\phi \in C^2(\Omega)$, $w(x_0) = \phi(x_0)$ and $w(x) \geq \phi(x)$ for all $x \in \Omega$, and hence

$$\min\left\{F(x_0, \nabla\phi(x_0)) - f(x_0)^2, -\Delta_{\infty,F}\,\phi(x_0)\right\} \geq 0. \tag{4.2}$$

Therefore, since

$$\nabla\varphi = g'(\phi)\nabla\phi$$

and

$$D^2\varphi = g'(\phi)D^2\phi + g''(\phi)(\nabla\phi \otimes \nabla\phi),$$

we have, using (4.2) together with the homogeneity assumption, that

$$F(x_0, \nabla\varphi(x_0)) - f(x_0)^2 \geq \left(g'(\phi)^2 - 1\right) f(x_0)^2 \tag{4.3}$$

and

$$\begin{aligned} -\Delta_{\infty,F}\,\varphi(x_0) &= -\nabla_x F(x_0, \nabla\varphi) \cdot \nabla_\xi F(x_0, \nabla\varphi) \\ &\quad - D^2\varphi(x_0)\nabla_\xi F(x_0, \nabla\varphi) \cdot \nabla_\xi F(x_0, \nabla\varphi) \\ &= -g'(\phi)^3\Delta_{\infty,F}\,\phi(x_0) \\ &\quad - g'(\phi)^2 g''(\phi)(\nabla\phi \otimes \nabla\phi)\nabla_\xi F(x_0, \nabla\phi) \cdot \nabla_\xi F(x_0, \nabla\phi) \\ &= -g'(\phi)^3\Delta_{\infty,F}\,\phi(x_0) - g'(\phi)^2 g''(\phi)\left[\nabla_\xi F(x_0, \nabla\phi) \cdot \nabla\phi\right]^2. \end{aligned} \tag{4.4}$$

Since w is bounded, we may assume without loss of generality that $\|\phi\|_{\infty,\Omega} \leq C_0$ for some constant $C_0 > 0$. For $\varepsilon > 0$ we now define

$$g(t) = (1+\varepsilon)t - \frac{\varepsilon}{4C_0}t^2$$

if $|t| < 2C_0$, and then extend this function to an increasing function defined on the whole real-line. Because $g'(t) \geq 1 + \varepsilon/2$ and $g''(t) = -\frac{\varepsilon}{2C_0}$ whenever $|t| \leq C_0$, we infer from (4.3) that

$$F(x_0, \nabla\varphi(x_0)) - f(x_0)^2 \geq \varepsilon f(x_0)^2 \tag{4.5}$$

and from (4.2) and (4.4) that

$$\begin{aligned} -\Delta_{\infty,F}\,\varphi(x_0) &\geq \frac{\varepsilon}{2C_0}\alpha^2|\nabla\phi(x_0)|^4 \geq \frac{\varepsilon}{2C_0}\left(\frac{\alpha}{\beta}\right)^2 F(x_0, \nabla\phi(x_0))^2 \\ &\geq \frac{\varepsilon}{2C_0}\left(\frac{\alpha}{\beta}\right)^2 f(x_0)^4. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\min\left\{F(x_0,\nabla\varphi(x_0))-f(x_0)^2,-\Delta_{\infty,F}\,\varphi(x_0)\right\}\geq\mu\min\{1,f(x_0)^4\}$$

for some constant $\mu=\mu(\alpha,\beta,\varepsilon,C_0)>0$. Since $|g(t)-t|\leq\frac{5C_0}{4}\varepsilon$ for every $|t|\leq C_0$, $\hat{w}=g(w)$ has all the desired properties provided that we choose $\varepsilon>0$ to be sufficiently small. □

Next we recall the *maximum principle for semicontinuous functions.* We first define the closures of the second order semidifferentials introduced in Section 1.

4.7. Definition. A pair $(q,X)\in\mathbb{R}^n\times S_n$ belongs to $\overline{D}^{2,+}u(x)$ if there is a sequence $(x_j,q_j,X_j)\in\Omega\times\mathbb{R}^n\times S_n$ such that $(q_j,X_j)\in D^{2,+}u(x_j)$ and $(x_j,u(x_j),q_j,X_j)\to(x,u(x),q,X)$ as $j\to\infty$.
Correspondingly, a pair $(s,Y)\in\mathbb{R}^n\times S_n$ belongs to $\overline{D}^{2,-}u(y)$ if there is a sequence $(y_j,s_j,Y_j)\in\Omega\times\mathbb{R}^n\times S_n$ such that $(s_j,Y_j)\in D^{2,-}u(y_j)$ and $(y_j,u(y_j),s_j,Y_j)\to(y,u(y),s,Y)$ as $j\to\infty$.

Note that if u is a viscosity subsolution of the equation $G=0$, then by the continuity of G we have that $G(x,u(x),q,X)\leq 0$ for all $(q,X)\in\overline{D}^{2,+}u(x)$. An analogous remark holds for supersolutions.

Theorem 4.8 below is a simplified version of the maximum principle for semicontinuous functions. For the proof and further information concerning the result, see [7], [8, §3] or [16, §2].

4.8. Theorem. *Suppose that u and $-v$ are real-valued and upper semicontinuous in Ω and that $(x_\tau,y_\tau)\in\Omega\times\Omega$ is a local maximum point of the function $u(x)-v(y)-(\tau/2)|x-y|^2$ for $\tau>0$. Then there exist $X_\tau,Y_\tau\in S_n$ such that*

$$(\tau(x_\tau-y_\tau),X_\tau)\in\overline{D}^{2,+}u(x_\tau), \tag{4.9}$$

$$(\tau(x_\tau-y_\tau),Y_\tau)\in\overline{D}^{2,-}v(y_\tau) \tag{4.10}$$

and

$$-3\tau\begin{pmatrix}I&0\\0&I\end{pmatrix}\leq\begin{pmatrix}X_\tau&0\\0&-Y_\tau\end{pmatrix}\leq 3\tau\begin{pmatrix}I&-I\\-I&I\end{pmatrix}. \tag{4.11}$$

Note in particular that by (4.11)

$$\begin{aligned}X_\tau\xi\cdot\xi-Y_\tau\eta\cdot\eta&=\begin{pmatrix}X_\tau&0\\0&-Y_\tau\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}\cdot(\xi\quad\eta)\\&\leq 3\tau\begin{pmatrix}I&-I\\-I&I\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}\cdot(\xi\quad\eta)\\&=3\tau|\xi-\eta|^2\end{aligned} \tag{4.12}$$

for all $\xi,\eta\in\mathbb{R}^n$, and hence, choosing $\xi=\eta$, we see that $X_\tau\leq Y_\tau$. The next simple lemma is also taken from [8, Prop. 3.7].

4.13. Lemma. *Suppose that u and $-v$ are real-valued and upper semicontinuous in $\overline{\Omega}$ and denote*

$$M_\tau = \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(u(x) - v(y) - \frac{\tau}{2}|x-y|^2 \right)$$

for $\tau > 0$. If $(x_\tau, y_\tau) \in \overline{\Omega}\times\overline{\Omega}$ is such that

$$M_\tau = u(x_\tau) - v(y_\tau) - \frac{\tau}{2}|x_\tau - y_\tau|^2,$$

then

(i) $\lim_{\tau\to\infty} \tau |x_\tau - y_\tau|^2 = 0$*;*

(ii) $\lim_{\tau\to\infty} M_\tau = u(\hat{x}) - v(\hat{x}) = \sup_{x\in\overline{\Omega}}(u(x) - v(x))$
whenever $\hat{x}$ is a limit point of x_τ as $\tau\to\infty$.

Now we state the regularity assumptions needed to obtain the comparison principle. These rather technical conditions will help us to overcome the fact that we are dealing with functions that are only semicontinuous. We assume that the following holds :

For every ball $B \Subset \Omega$, there exist constants $C > 0$ and $0 < \kappa \le 1$ such that for all $\xi\in\mathbb{R}^n$ and for all $x, y \in B$

$$|\nabla_\xi F(x,\xi) - \nabla_\xi F(y,\xi)| \le C|\xi||x-y|^{1/2+\kappa}, \tag{4.14}$$

$$|\nabla_x F(x,\xi)| \le C|\xi|^2 \tag{4.15}$$

and

$$|\nabla_x F(x,\xi) - \nabla_x F(y,\xi)| \le C|\xi|^2|x-y|^\kappa. \tag{4.16}$$

Note that by the mean-value theorem and (4.15), we further have

$$|F(x,\xi) - F(y,\xi)| \le C|\xi|^2|x-y| \tag{4.17}$$

for all $\xi\in\mathbb{R}^n$ and $x, y\in B$.

4.18. Theorem. *Suppose that $u, v\colon \overline{\Omega}\to\mathbb{R}$ are bounded, u is upper semicontinuous and v is lower semicontinuous in $\overline{\Omega}$. If u is a viscosity subsolution and v is a supersolution of (3.9) in Ω such that at least one of them is locally Lipschitz continuous, then*

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) = \sup_{x\in\partial\Omega}(u(x) - v(x)). \tag{4.19}$$

Proof. We argue by contradiction and assume that (4.19) fails. By Lemma 4.1, there is a function $\hat{v}$ and a constant $\mu > 0$ such that

$$\sup_{x\in\Omega}(u(x) - \hat{v}(x)) > \sup_{x\in\partial\Omega}(u(x) - \hat{v}(x)) \tag{4.20}$$

and

$$\min\left\{F(x, \nabla\hat{v}(x)) - f(x)^2, -\Delta_{\infty,F}\,\hat{v}(x)\right\} \geq \mu\min\{1, f(x)^4\}$$

in the viscosity sense. Note that if $v \in W^{1,\infty}_{loc}(\Omega)$, then by the construction in Lemma 4.1 also $\hat{v} \in W^{1,\infty}_{loc}(\Omega)$.

Let (x_τ, y_τ) be a maximum point of $u(x) - \hat{v}(y) - (\tau/2)|x-y|^2$ in $\overline{\Omega}\times\overline{\Omega}$. Because $\overline{\Omega}$ is compact, we can always find a sequence $\tau_j \to \infty$ and a point $x_0 \in \overline{\Omega}$ such that $x_{\tau_j} \to x_0$. By Lemma 4.13, x_0 is a maximum point of the function $u - \hat{v}$ in $\overline{\Omega}$, and by (4.20), it is an interior point of Ω. In particular, for j large enough, x_{τ_j} and y_{τ_j} belong to a fixed ball $B \Subset \Omega$. In order to simplify notation, we denote $\tau_j = \tau$ and from now on drop the subscript τ.

Next we apply Theorem 4.8 and conclude that there exist matrices $X, Y \in S_n$ such that (4.9) – (4.11) hold. Using (4.12), we have that

$$\begin{aligned}
&G(y, \tau(x-y), Y) - G(x, \tau(x-y), X)\\
&\quad= \tau^2\big[X\,\nabla_\xi F(x, x-y)\cdot\nabla_\xi F(x, x-y) - Y\,\nabla_\xi F(y, x-y)\cdot\nabla_\xi F(y, x-y)\big]\\
&\qquad+ \tau^3\big[\nabla_x F(x, x-y)\cdot\nabla_\xi F(x, x-y) - \nabla_x F(y, x-y)\cdot\nabla_\xi F(y, x-y)\big]\\
&\quad\leq 3\tau^3|\nabla_\xi F(x, x-y) - \nabla_\xi F(y, x-y)|^2\\
&\qquad+ \tau^3\big[\big(\nabla_x F(x, x-y) - \nabla_x F(y, x-y)\big)\cdot\nabla_\xi F(x, x-y)\\
&\qquad\qquad- \nabla_x F(y, x-y)\cdot\big(\nabla_\xi F(y, x-y) - \nabla_\xi F(x, x-y)\big)\big],
\end{aligned}\tag{4.21}$$

where G is the expanded form of (3.5), that is,

$$G(z, q, M) = -M\,\nabla_\xi F(z, q)\cdot\nabla_\xi F(z, q) - \nabla_x F(z, q)\cdot\nabla_\xi F(z, q).$$

Combining (4.21) with the assumptions (4.14) – (4.16) then gives

$$G(y, \tau(x-y), Y) - G(x, \tau(x-y), X) \leq C\tau^3|x-y|^{3+\kappa}$$

for $\kappa > 0$. Hence we have that

$$\begin{aligned}
0 < \mu\min\{1, f(y)^4\} &\leq \min\left\{F(y, \tau(x-y)) - f(y)^2, G(y, \tau(x-y), Y)\right\}\\
&\quad- \min\left\{F(x, \tau(x-y)) - f(x)^2, G(x, \tau(x-y), X)\right\}\\
&\leq \max\left\{|F(y, \tau(x-y)) - F(x, \tau(x-y))| + |f(x)^2 - f(y)^2|,\right.\\
&\qquad\left. G(y, \tau(x-y), Y) - G(x, \tau(x-y), X)\right\}\\
&\leq C\max\left\{\tau^2|x-y|^3 + |f(x)^2 - f(y)^2|, \tau^3|x-y|^{3+\kappa}\right\},
\end{aligned}\tag{4.22}$$

where we also used (4.17).

Recall now that at least one of the functions u and $\hat{v}$ is locally Lipschitz and assume that this function is u for example. Because

$$u(z_1) - \hat{v}(z_2) - (\tau/2)|z_1 - z_2|^2 \leq u(x) - \hat{v}(y) - (\tau/2)|x - y|^2$$

for all $z_1, z_2 \in \Omega$, we have, by putting $z_1 = z_2 = y$ and using the Lipschitz continuity, that

$$\tau|x - y|^2 \leq 2(u(x) - u(y)) \leq C|x - y|.$$

This implies that $\tau|x - y|^\gamma \to 0$ as $\tau \to \infty$ for all $\gamma > 1$, and, in particular, that the right-hand side of (4.22) tends to zero as $\tau \to \infty$. Since the left-hand side remains positive, we have a contradiction and the theorem follows. □

4.23. Remark. In the case of the ∞-Laplacian, an equivalent way to write equation (3.9) is

$$\min\left\{|\nabla u(x)| - f(x), -\Delta_\infty u(x)\right\} = 0.$$

Hence (3.9) can be viewed as a gradient constraint problem and results concerning it are of independent interest. Theorems 3.11 and 4.18 imply that for given $\vartheta \in W^{1,\infty}(\Omega)$ this problem has a unique viscosity solution $u \in W^{1,\infty}(\Omega)$ such that $u - \vartheta \in C_0(\Omega)$.

For equation (3.10) the comparison principle now follows easily.

4.24. Theorem. *Suppose that $u, v\colon \overline{\Omega} \to \mathbb{R}$ are bounded, u is upper semicontinuous and v is lower semicontinuous in $\overline{\Omega}$. If u is a viscosity subsolution and v is a supersolution of* (3.10) *in Ω such that at least one of them is locally Lipschitz continuous, then*

$$\sup_{x \in \overline{\Omega}}(u(x) - v(x)) = \sup_{x \in \partial\Omega}(u(x) - v(x)).$$

Proof. We simply note that by the homogeneity of F, $-u$ is a supersolution and $-v$ is a subsolution of (3.9). The claim now follows immediately from Theorem 4.18. □

Now we are ready to prove the comparison principle for viscosity solutions of (3.5) with Lipschitz boundary values.

4.25. Theorem. *Suppose that $F(x, \xi)$ satisfies* (3.21) *and* (4.14) – (4.16). *Let u be a bounded subsolution and v be a bounded supersolution of* (3.5) *in Ω, and assume that there exist $\vartheta_1, \vartheta_2 \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that $u|_{\partial\Omega} = \vartheta_1|_{\partial\Omega}$ and $v|_{\partial\Omega} = \vartheta_2|_{\partial\Omega}$. Then*

$$\sup_{x \in \overline{\Omega}}(u(x) - v(x)) = \sup_{x \in \partial\Omega}(u(x) - v(x)).$$

Proof. Let $\varepsilon > 0$ and choose $f \equiv \varepsilon$ in (3.9) and (3.10). By Lemma 3.22, there exist viscosity solutions $\overline{u}_\varepsilon$ and $\underline{u}_\varepsilon$ of (3.9) and (3.10), respectively, such that $\overline{u}_\varepsilon - \vartheta_1, \underline{u}_\varepsilon - \vartheta_1 \in C_0(\Omega)$ and

$$\sup_{x\in\Omega}(\overline{u}_\varepsilon(x) - \underline{u}_\varepsilon(x)) \le C(\sigma,\Omega)\varepsilon.$$

Further, by Theorem 3.11, there exists a viscosity solution $\underline{v}_\varepsilon$ of (3.10) such that $\underline{v}_\varepsilon - \vartheta_2 \in C_0(\Omega)$. Since u is a subsolution of (3.9) and v is a supersolution of (3.10), we have by Theorems 4.18 and 4.24 that

$$u - v \le \overline{u}_\varepsilon - \underline{v}_\varepsilon = (\overline{u}_\varepsilon - \underline{u}_\varepsilon) + (\underline{u}_\varepsilon - \underline{v}_\varepsilon).$$

Using Theorems 4.18 and 4.24 again, we then obtain

$$\begin{aligned}\sup_{x\in\Omega}(u(x) - v(x)) &\le \sup_{x\in\Omega}(\overline{u}_\varepsilon(x) - \underline{u}_\varepsilon(x)) + \sup_{x\in\Omega}(\underline{u}_\varepsilon(x) - \underline{v}_\varepsilon(x))\\ &\le C\varepsilon + \sup_{x\in\partial\Omega}(\underline{u}_\varepsilon(x) - \underline{v}_\varepsilon(x))\\ &= C\varepsilon + \sup_{x\in\partial\Omega}(u(x) - v(x)).\end{aligned}$$

Our assertion follows by letting $\varepsilon \to 0$. □

4.26. Remark. We used the assumption (3.21) only in the proof of Lemma 3.22. Hence Theorem 4.25 above is true if F satisfies (4.14)-(4.16), and we know a priori that $\overline{u}_\varepsilon - \underline{u}_\varepsilon \to 0$ uniformly in Ω as $\varepsilon \to 0$.

4.27. Corollary. *If $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega) \ne \emptyset$, there exists a unique variational solution to the obstacle problem in $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$. In particular, if u_p is the unique (F,p)-minimizer with boundary values $\vartheta \in W^{1,\infty}(\Omega)$, then the whole family $\{u_p\}_{p>n}$ is converging to an F-absolute minimizer $u_\infty \in W^{1,\infty}(\Omega)$.*

Proof. The existence of a variational solution to the obstacle problem in $\mathcal{K}^\infty_{\psi,\vartheta}(\Omega)$ was already established in Theorem 2.3. Suppose now that we have two variational solutions, u and v, and consider the open set

$$D = \{\, x \in \Omega\colon u(x) > v(x)\,\}.$$

Since v is continuous and u is clearly a solution to the obstacle problem in $\mathcal{K}^\infty_{v,v}(D)$, Theorem 3.4 implies that u is a solution of (3.5) in D. Because v is a supersolution, we conclude by Theorem 4.25 that $u \le v$ in D, and hence $D = \emptyset$. By switching the roles of u and v we see that $u = v$ as desired. □

Consider again the special case $F(x,\xi) = \theta(x)\xi\cdot\xi$. It is clear that the assumptions (4.14) – (4.16) are satisfied if $\theta \in C^{1,\kappa}_{\mathrm{loc}}(\Omega)$ for some $\kappa > 0$. To see that also (3.21) holds, we first note that there exists a function $\Psi\colon \Omega \to S_n$ such that $\Psi^2 = \theta$, see e.g. [13, 6.6.4]. Hence we have that

$$\begin{aligned}
&\big((\theta\xi_1\cdot\xi_1)^{\frac{p-2}{2}}\theta\xi_1 - (\theta\xi_2\cdot\xi_2)^{\frac{p-2}{2}}\theta\xi_2\big)\cdot(\xi_1-\xi_2)\\
&\quad= \big(|\Psi\xi_1|^{p-2}\Psi\xi_1 - |\Psi\xi_2|^{p-2}\Psi\xi_2\big)\cdot(\Psi\xi_1-\Psi\xi_2)\\
&\quad= \frac{1}{2}\big(|\Psi\xi_1|^{p-2}+|\Psi\xi_2|^{p-2}\big)|\Psi(\xi_1-\xi_2)|^2\\
&\qquad+\frac{1}{2}\big(|\Psi\xi_1|^{p-2}-|\Psi\xi_2|^{p-2}\big)\big(|\Psi\xi_1|^2-|\Psi\xi_2|^2\big)\\
&\quad\ge \frac{1}{2}\alpha^{p/2}\big(|\xi_1|^{p-2}+|\xi_2|^{p-2}\big)|\xi_1-\xi_2|^2\\
&\quad\ge \left(\frac{\sqrt{\alpha}}{2}\right)^p|\xi_1-\xi_2|^p,
\end{aligned}$$

which is the desired inequality. We conclude by Theorem 4.25 that if θ has locally Hölder continuous partial derivatives, then the comparison principle is valid.

Using the comparison principle, we obtain the following important interior regularity result.

4.28. Lemma. *If u is a locally bounded viscosity subsolution of* (3.5)*, then $u \in W^{1,\infty}_{loc}(\Omega)$. Similarly, if v is a locally bounded viscosity supersolution of* (3.5)*, then $v \in W^{1,\infty}_{loc}(\Omega)$.*

Proof. We prove only the subsolution case. Let $0<\delta<1$ and denote

$$\Omega_\delta = \{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega)>\delta\}.$$

By the assumptions, $M_\delta = \|u\|_{\infty,\Omega_\delta}$ is finite for every $\delta>0$. Let $x_1,x_2\in\Omega_{2\delta}$. If $|x_1-x_2|\ge\delta$, then

$$|u(x_1)-u(x_2)| \le 2M_{2\delta} \le \frac{2M_{2\delta}}{\delta}|x_1-x_2|. \tag{4.29}$$

Now assume that $|x_1-x_2|<\delta$. We then have by (4.29) that

$$u(x)\le u(x_1)+\frac{2M_\delta}{\delta}|x_1-x|$$

for all $x\in\partial B(x_1,\delta)$. Denote $B' = B(x_1,\delta)\setminus\{x_1\}$ and let $w\in W^{1,\infty}(B')$ be a solution of (3.5) in B' with boundary values $\varphi(x) = (2M_\delta/\delta)|x_1-x|$. Since w is an F-absolute minimizer, we obtain

$$\|F(x,\nabla w(x))\|_{\infty,B'} \le \|F(x,\nabla\varphi(x))\|_{\infty,B'} \le \beta\left(\frac{2M_\delta}{\delta}\right)^2,$$

and thus

$$\|\nabla w\|_{\infty,B'} \le \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{2M_\delta}{\delta}.$$

Now let $\overline{w}_\varepsilon$ and $\underline{w}_\varepsilon$ be solutions of (3.9) and (3.10), respectively, with boundary values φ. By Theorems 4.18 and 4.24, $\underline{w}_\varepsilon \le w \le \overline{w}_\varepsilon$ in B', and thus $\overline{w}_\varepsilon \to w$ uniformly as $\varepsilon \to 0$ by Lemma 3.22. Since u is a subsolution of (3.9), we have by Theorem 4.18 that $u \le u(x_1) + \overline{w}_\varepsilon$ for every $\varepsilon > 0$, and thus $u \le u(x_1) + w$ in B'. This combined with the gradient estimate above gives

$$u(x) \le u(x_1) + w(x) \le u(x_1) + \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{2M_\delta}{\delta}|x_1 - x|$$

for all $x \in B(x_1, \delta)$. In particular,

$$u(x_2) - u(x_1) \le \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{2M_\delta}{\delta}|x_1 - x_2|,$$

and hence, by changing the roles of x_1 and x_2, we have

$$|u(x_2) - u(x_1)| \le \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{2M_\delta}{\delta}|x_1 - x_2|.$$

This together with (4.29) proves the assertion in the subsolution case. □

4.30. Example. For a general kernel F, the above lemma is essentially everything that is known about the regularity of viscosity solutions and supersolutions of (3.5). In the case $F(x,\xi) = |\xi|^2$, it follows from the results of Aronsson in [1] and [3] that viscosity solutions are not in general solutions in the classical sense. A particularly interesting example considered by Aronsson is the function

$$u(x) = |x_1|^{4/3} + |x_2|^{4/3} + \cdots + |x_{n-1}|^{4/3} - \sqrt[3]{n-1}\,|x_n|^{4/3},$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. A direct computation shows that $u \in C^{1,1/3}_{\mathrm{loc}}(\mathbb{R}^n)$ and that u does not have the second order partial derivative $\frac{\partial^2 u}{\partial x_i^2}$ on the hyperplane $\{x \in \mathbb{R}^n : x_i = 0\}$, $i = 1, \ldots, n$. On the other hand, it is not hard to verify directly that u is a viscosity solution of the ∞-Laplacian in $\mathbb{R}^n$. Observe in particular that the singular set of u, that is, the set in which u is not a classical solution, is bigger than the set $\{x \in \mathbb{R}^n : \nabla u(x) = 0\}$. Since for a finite exponent p a weak solution u_p of (1.13) is, in this special case, known to be real analytic away from the set $\{\nabla u_p(x) = 0\}$, this example illustrates the fact that ∞-harmonic functions do not inherit all the nice properties of the solutions of (1.13).

As a consequence of the interior regularity result above, we obtain a more general form of the comparison principle.

4.31. Corollary. *Suppose that u and v are locally bounded, u is a viscosity subsolution and v is a supersolution of* (3.5) *in* Ω. *If*

$$\limsup_{x\to z} u(x) \le \liminf_{x\to z} v(x) \tag{4.32}$$

for all $z\in\partial\Omega$ and if both sides of (4.32) *are not simultaneously ∞ or $-\infty$, then $u\le v$ in Ω.*

Proof. As before, we denote $\Omega_\delta = \{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega) > \delta\}$ for $\delta>0$. By (4.32), for every $\varepsilon>0$ there is $\delta>0$ such that $u<v+\varepsilon$ in $\Omega\setminus\Omega_\delta$. Indeed, if such a δ does not exist, we can find a sequence $x_j\in\Omega$ converging to a point $z\in\partial\Omega$ such that $u(x_j)\ge v(x_j)+\varepsilon$ for all $j=1,2,\dots$. This implies that

$$\limsup_{x\to z} u(x) \ge \liminf_{x\to z} v(x)+\varepsilon,$$

which contradicts (4.32) since both sides are not simultaneously ∞ or $-\infty$. By the previous lemma, $u,v\in W^{1,\infty}(\Omega_\delta)\cap C(\overline{\Omega}_\delta)$, and thus we have by Theorem 4.25 that $u\le v+\varepsilon$ in Ω_δ. Hence $u\le v+\varepsilon$ in Ω, and we obtain the assertion by letting $\varepsilon\to 0$. $\square$

We close this section by improving our existence result concerning F-absolute minimizers.

4.33. Corollary. *If $g\in C(\partial\Omega)$, there exists a unique viscosity solution u of* (3.5) *in Ω such that $u|_{\partial\Omega}=g$. Furthermore, $u\in W^{1,\infty}_{loc}(\Omega)$ and it is also an F-absolute minimizer.*

Proof. We first extend the function g to a continuous function $w\colon\mathbb{R}^n\to\mathbb{R}$ and via mollification find a sequence $w_i\in W^{1,\infty}_{loc}(\mathbb{R}^n)$ such that $\|w_i-w\|_{\infty,\overline{\Omega}}\to 0$ as $i\to\infty$. Let now $u_i\in W^{1,\infty}(\Omega)$ be the solution of (3.5) with boundary values w_i. By the comparison principle, $\{u_i\}$ is a Cauchy sequence in $C(\overline{\Omega})$, and thus we find a function u such that $u|_{\partial\Omega}=g$ and $u_i\to u$ uniformly in Ω.

We claim that u is a solution of (3.5). To prove this, we fix a point $x_0\in\Omega$ and a test function $\varphi\in C^2(\Omega)$ such that $u(x_0)=\varphi(x_0)$ and $u(x)<\varphi(x)$ for all $x\ne x_0$. As in the proof of Theorem 3.4, we find by the uniform convergence a sequence $x_i\to x_0$ such that x_i is a local maximum point of $u_i-\varphi$. Since u_i is a solution of (3.5), we have that $-\Delta_{\infty,F}\,\varphi(x_i)\le 0$ for every $i=1,2,\dots$. By continuity, this implies that u is a viscosity subsolution. Similarly, we obtain that u is a supersolution and thus a solution of (3.5). The uniqueness and the interior regularity of u follow from Corollary 4.31 and Lemma 4.28, respectively.

To prove that u is an F-absolute minimizer, we fix an open set $D\Subset\Omega$. Since $u\in W^{1,\infty}(D)$, we have by the comparison principle and Corollaries 3.8 and 4.27 that u is the unique variational F-absolute minimizer in D with boundary values u. This shows that u is an F-absolute minimizer in Ω and hence completes the proof. $\square$

5. Supersolutions and (F,∞)-superharmonic functions

In this section we study certain properties of viscosity supersolutions of the Euler equation (3.5). We first show that every supersolution is a local limit of a sequence of supersolutions of the Euler equations of the corresponding L^p-problems. This is done by establishing the fact that a supersolution of (3.5) is locally a variational solution to a suitable obstacle problem. After that, we define (F,∞)-superharmonic functions via comparison and prove that supersolutions are exactly superharmonic functions. As a result of all this, we obtain that every supersolution of (3.5) is in $W^{1,\infty}_{loc}(\Omega)$ and that nonnegative supersolutions satisfy Harnack's inequality.

Throughout this section, we assume that viscosity sub- and supersolutions satisfy the comparison principle, that is, Corollary 4.31 holds.

5.1. Theorem. *If u is a locally bounded viscosity supersolution of* (3.5) *in Ω, then it is the unique variational solution to the obstacle problem in $\mathcal{K}^{\infty}_{u,u}(D)$ for each open $D \Subset \Omega$. In particular, for each open $D \Subset \Omega$ there is a sequence (u_p) of viscosity supersolutions of* (1.13) *such that $u_p \to u$ uniformly in D.*

Proof. Fix an open set $D \Subset \Omega$ and observe that by Lemma 4.28 $u \in \mathcal{K}^{\infty}_{u,u}(D)$. Let $v \in W^{1,\infty}(D)$ be the variational solution to the obstacle problem in $\mathcal{K}^{\infty}_{u,u}(D)$ and notice that $v \geq u$ by the definition of $\mathcal{K}^{\infty}_{u,u}(D)$. By Theorem 3.4, v is a solution of (3.5) in the open set $D' = \{v > u\}$. Since $u = v$ on $\partial D'$ and u is a supersolution, the comparison principle implies that $v \leq u$ in D'. Thus $D' = \emptyset$ and $u = v$ in D. $\square$

Theorem 5.1 implies Harnack's inequality for locally bounded nonnegative supersolutions of (3.5).

5.2. Corollary. *Let u be a locally bounded and nonnegative viscosity supersolution of* (3.5) *in a domain Ω, and assume that $K \subset \Omega$ is compact. Then*

$$\sup_K u \leq C \inf_K u$$

for some constant $C = C(\alpha, \beta, \Omega, K)$.

Proof. We may assume without loss of generality that also K is connected. Let $d = \operatorname{dist}(K, \partial\Omega)$. By Theorem 5.1, u is locally a variational solution to the obstacle problem, and thus if we choose $R = d/2$ and $r = d/4$ in Theorem 2.7, we obtain the estimate

$$\sup_B u \leq C(\alpha,\beta) \inf_B u$$

for every ball B with radius $d/4$ and with center in K. Because K is compact and connected, there is a constant N, depending only on Ω and K, such that any two points $x, y \in K$ can be joined by a chain of balls $B_1, \ldots, B_N$ with radii $d/4$. Hence we have that

$$\sup_K u \le C(\alpha,\beta)^N \inf_K u,$$

which proves the corollary. □

Next we define (F,∞)-superharmonic functions. The definition is an obvious modification of the definition of (F,p)-superharmonic functions for the corresponding L^p-problem, see [15, ch. 7].

5.3. Definition. A function $u\colon \Omega \to \mathbb{R} \cup \{\infty\}$ is (F,∞)-*superharmonic* in Ω if

(i) u is lower semicontinuous,
(ii) $u \not\equiv \infty$ in each component of Ω, and
(iii) for each open $D \Subset \Omega$ and each viscosity solution $h \in C(\overline{D})$ of (3.5), the inequality $u \ge h$ on ∂D implies $u \ge h$ in D.

A function v is (F,∞)-*subharmonic* if $-v$ is (F,∞)-superharmonic.

In the nonlinear potential theory the class of (F,p)-superharmonic functions is one of the basic objects to study. In this work we merely note some basic facts about (F,∞)-superharmonic functions and their connection to viscosity supersolutions of (3.5).

5.4. Lemma. *If u is (F,∞)-superharmonic in Ω and $D \Subset \Omega$ is open, then there is an increasing sequence $u_i \in W^{1,\infty}(D)$ of viscosity supersolutions of* (3.5) *such that $u = \lim_{i\to\infty} u_i$ in D.*

Proof. Since u is lower semicontinuous, there is an increasing sequence $\varphi_i \in C^\infty(\mathbb{R}^n)$ such that $\varphi_i \to u$ in $\overline{D}$, see for example [15, pp.75-76]. Let $u_i \in W^{1,\infty}(D)$ be the variational solution to the obstacle problem in $\mathcal{K}^\infty_{\varphi_i,\varphi_i}(D)$. We first show that the sequence (u_i) is increasing. In order to do that, we set $D_i = \{u_{i+1} < u_i\}$ and note that $u_i > u_{i+1} \ge \varphi_{i+1} \ge \varphi_i$ in D_i. By Theorem 3.4, this implies that u_i is a solution of (3.5) in D_i, and hence we have by the comparison principle that $u_i \le u_{i+1}$ in D_i. Thus D_i is empty and the sequence (u_i) is increasing. In particular, the limit $\lim_{i\to\infty} u_i(x)$ exists for all $x \in D$.

Next we look at the set $\{u < u_i\}$ for a fixed i. Again we easily see that u_i is a solution in this set, and thus, by the definition of (F,∞)-superharmonic functions, we have that $u \ge u_i$ in D for every $i = 1, 2, \ldots$. We conclude that

$$u = \lim_{i\to\infty} \varphi_i \le \lim_{i\to\infty} u_i \le u,$$

which shows that (u_i) is the desired sequence. □

5.5. Corollary. *If u is (F,∞)-superharmonic in a domain Ω, then u is locally bounded. In particular, every (F,∞)-superharmonic function is real-valued.*

Proof. We fix an open set $D \Subset \Omega$. By the definition of (F,∞)-superharmonic functions, there is a point $x \in \Omega$ such that $u(x) < \infty$. Since x is not necessarily in D, we let $D' \Subset \Omega$ be an open set containing both x and D. Furthermore, by the lower semicontinuity of u, we may assume that $u \ge \varepsilon > 0$ in D'.

Let $u_i \in W^{1,\infty}(D')$ be an increasing sequence of nonnegative supersolutions of (3.5) converging to u in D'. Corollary 5.2 then implies that $u_i(y) \le Cu_i(x)$ for all $y \in D'$ and for all $i = 1, 2, \dots$ with a constant C independent of i. Hence $u(y) \le Cu(x)$ for all $y \in D'$ and we have that

$$\sup_D u \le Cu(x) < \infty. \tag{5.6}$$

This shows that u is locally bounded above and thus proves the corollary. □

5.7. Remark. In the above proof we obtained the estimate (5.6) for any point $x \in D'$ in which the function u is finite. Once we know that (F,∞)-superharmonic functions are real-valued, this implies Harnack's inequality for nonnegative superharmonic functions, that is,

$$\sup_K u \le C(\alpha, \beta, K, \Omega) \inf_K u$$

for every compact set $K \subset \Omega$ and for every nonnegative (F,∞)-superharmonic function u in a domain Ω.

The fact that (F,∞)-superharmonic functions are real-valued enables us to prove the following interesting result.

5.8. Theorem. *A function u is (F,∞)-superharmonic if and only if it is a viscosity supersolution of* (3.5).

Proof. Assume first that u is a supersolution of (3.5) and fix $D \Subset \Omega$. If $h \in C(\overline{D})$ is a solution of (3.5) in D such that $h \le u$ on ∂D, then by the comparison principle $h \le u$ in D. Since u is by definition real-valued and lower semicontinuous, we have that u is (F,∞)-superharmonic.

For the converse, we argue by contradiction and assume that there exists $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$, $u(x) > \varphi(x)$ for all $x \ne x_0$ and

$$-\Delta_{\infty,F}\, \varphi(x_0) < 0. \tag{5.9}$$

By continuity, there is a ball B centered at x_0 such that if $x \in B$, then (5.9) holds with x_0 replaced by x. Let

$$m = \inf_{x \in \partial B} (u(x) - \varphi(x)),$$

and observe that $m > 0$ because the function $u - \varphi$ is lower semicontinuous. We define $\tilde{\varphi} = \varphi + m/2$ and let $v \in C(\overline{B})$ be the unique viscosity solution of (3.5) with boundary values $\tilde{\varphi}$. Since $\tilde{\varphi}$ is a subsolution of (3.5) in B, we have by the comparison principle that $v \geq \tilde{\varphi}$ in B. On the other hand, $u \geq v$ on ∂B, and thus $u \geq v$ in B by the definition of (F, ∞)-superharmonic functions. In particular, $u \geq \tilde{\varphi}$ in B, which contradicts the fact that $u(x_0) < \tilde{\varphi}(x_0)$. This completes the proof. □

Observe that Theorem 5.8 combined with Corollary 5.5 implies that every viscosity supersolution of (3.5) is locally bounded. Thus the statements of Lemma 4.28 and Corollary 4.31 are valid for an arbitrary supersolution u, and, in particular, we have that $u \in W^{1,\infty}_{loc}(\Omega)$.

Theorem 5.8 offers an easy way to establish the following variant of the fundamental convergence theorem of the classical potential theory.

5.10. Theorem. *Suppose that $\mathcal{F}$ is a nonempty family of viscosity supersolutions of (3.5) in Ω, locally uniformly bounded below. Then*

$$u_*(x) = \inf_{u \in \mathcal{F}} u(x)$$

is also a viscosity supersolution of (3.5).

Proof. It suffices to show that u_* is (F, ∞)-superharmonic in Ω. For that, we fix an open set $D \Subset \Omega$ and let $h \in C(\overline{D})$ be a viscosity solution of (3.5) such that $u_* \geq h$ on ∂D. By Theorem 5.8 and the definition of (F, ∞)-superharmonic functions, $u \geq h$ in D whenever $u \in \mathcal{F}$, and hence also $u_* \geq h$ in D.

Thus it remains to show that u_* is real-valued and lower semicontinuous. First notice that since the family $\mathcal{F}$ is uniformly bounded below in D and $\min\{u_1, u_2\}$ is a supersolution of (3.5) whenever u_1 and u_2 are supersolutions, we may assume without loss of generality that $\|u\|_{\infty, D} \leq C$ for some constant $C > 0$ and for every $u \in \mathcal{F}$. By the proof of Lemma 4.28, this implies that every $u \in \mathcal{F}$ is Lipschitz continuous in D with a uniform Lipschitz constant L. Now take $x, y \in D$ and assume that $u_*(x) > u_*(y)$. For $\varepsilon > 0$ we choose $u_\varepsilon \in \mathcal{F}$ such that $u_*(y) \geq u_\varepsilon(y) - \varepsilon$, and then obtain that

$$|u_*(x) - u_*(y)| \leq u_*(x) - u_\varepsilon(y) + \varepsilon \leq |u_\varepsilon(x) - u_\varepsilon(y)| + \varepsilon \leq L|x - y| + \varepsilon.$$

This shows that $u_* \in W^{1,\infty}_{loc}(\Omega)$, and, in particular, that u_* is real-valued and lower semicontinuous. The theorem now follows. □

We close this section by noting the *strict minimum principle* as an immediate consequence of the Harnack inequality.

5.11. Corollary. *A nonconstant viscosity supersolution of* (3.5) *cannot attain its infimum in a domain Ω.*

6. The eigenvalue problem for the ∞-Laplacian

In this last section, we restrict our attention to the case $F(x,\xi) = |\xi|^2$ and consider a different type of minimization problem, namely the problem of minimizing the quotient

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}} \tag{6.1}$$

among all the functions $u \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$, $u \neq 0$. The motivation to study this problem comes from its interpretation as a limit problem; the quotient (6.1) is a formal limit of the *nonlinear Rayleigh quotient*

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \tag{6.2}$$

the minimizers of which form the eigenspace corresponding to the first eigenvalue of the p-*Laplace* operator

$$-\Delta_p u(x) = -\operatorname{div}\big(|\nabla u(x)|^{p-2}\nabla u(x)\big), \qquad 1 < p < \infty.$$

As seen in the earlier sections, the operator

$$-\Delta_\infty u(x) = -D^2 u(x)\nabla u(x) \cdot \nabla u(x) \tag{6.3}$$

can be justifiably regarded as the correct limit of the p-Laplacians, and it is therefore natural to ask whether it is possible to identify a subclass of minimizers of (6.1) that in some sense forms the eigenspace corresponding to the first eigenvalue of the ∞-Laplacian (6.3).

It is not, however, a priori clear what should be used as the correct formulation of the eigenvalue problem for the ∞-Laplacian. For a finite exponent p, one usually looks for weak solutions of

$$-\Delta_p u = \lambda |u|^{p-2} u \tag{6.4}$$

with the boundary condition $u \in W_0^{1,p}(\Omega)$. Due to the compatible homogeneity on the right-hand side, the set of solutions to this equation for a fixed $\lambda \in \mathbb{R}$ consists of the trivial solution $u \equiv 0$ and of a union of one dimensional linear subspaces of the Sobolev space. If this union is non-void, that is, equation (6.4) has a nontrivial solution, then the number λ is called an *eigenvalue* of the p-Laplacian. It is well-known that there exists a smallest eigenvalue $\lambda_1(p) > 0$ which is characterized as the infimum of the nonlinear Rayleigh quotient (6.2) in the set $W_0^{1,p}(\Omega)\setminus\{0\}$. This first eigenvalue has many special properties. It is simple, which means that the corresponding eigenfunctions are just constant multiples of each other. Moreover, these first eigenfunctions do not change their sign in the domain Ω, and they are the only eigenfunctions with this property. For the proof of these facts and for a more thorough discussion on equation (6.4), see [20] and the references therein. In what follows, we derive the limit equation of (6.4) in the case $\lambda = \lambda_1(p)$ and show that it has certain properties that make it reasonable to call its viscosity solutions eigenfunctions associated with the first eigenvalue of the ∞-Laplacian. As will come evident later on, this equation is intimately connected to the geometry of the domain Ω and is therefore of independent interest. Our approach is closely related to that in [18], where this problem has also been studied.

As the first step, we obtain a candidate for the first eigenvalue of the ∞-Laplacian by looking at the limiting behavior of the numbers $\lambda_1(p)$ as $p \to \infty$.

6.5. Lemma. *Let $\lambda_1(p)$ be the first eigenvalue of the p-Laplacian in Ω. Then*

$$\lim_{p\to\infty} \lambda_1(p)^{1/p} = \frac{1}{d},$$

where $d = \|dist(x, \partial\Omega)\|_{\infty,\Omega}$.

Proof. Recall first that the number $\lambda_1(p)$ is characterized as the infimum of the nonlinear Rayleigh quotient (6.2) in the set $W_0^{1,p}(\Omega) \setminus \{0\}$. Because the distance function $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ belongs to the space $W_0^{1,p}(\Omega)$ for all $p < \infty$ and satisfies $|\nabla\delta(x)| = 1$ for a.e. $x \in \Omega$, we have

$$\limsup_{p\to\infty} \lambda_1(p)^{1/p} \le \frac{1}{d}.$$

On the other hand, since Lipschitz functions are dense in $W_0^{1,p}(\Omega)$, we get

$$\lambda_1(p)^{1/p} \ge \inf_{\substack{\varphi\in W^{1,\infty}(\Omega)\cap C_0(\Omega)\\ \varphi\neq 0}} \frac{\left(\int_\Omega |\nabla\varphi|^p\,dx\right)^{1/p}}{\|\varphi\|_{\infty,\Omega}|\Omega|^{1/p}} = \inf_{\substack{\varphi\in W^{1,\infty}(\Omega)\cap C_0(\Omega)\\ \|\varphi\|_{\infty,\Omega}=1}} \left(\fint_\Omega |\nabla\varphi|^p\,dx\right)^{1/p}.$$

If we denote by I_p the infimum on the right, then, by Hölder's inequality, the sequence (I_p) is non-decreasing, and thus it has a limit when $p \to \infty$. By the same arguments as in the proof of Proposition 2.5, it is not hard to see that this limit equals to

$$I_\infty = \inf_{\substack{\varphi\in W^{1,\infty}(\Omega)\cap C_0(\Omega)\\ \|\varphi\|_{\infty,\Omega}=1}} \|\nabla\varphi\|_{\infty,\Omega}.$$

To complete the proof, we only need to note that $I_\infty = 1/d$. Indeed, if we fix $x \in \Omega$ and choose $y \in \partial\Omega$ such that $\delta(x) = |x - y|$, then

$$|\varphi(x)| = |\varphi(x) - \varphi(y)| \le \|\nabla\varphi\|_{\infty,\Omega}\delta(x).$$

for every $\varphi \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$. Since $\|\nabla\delta\|_{\infty,\Omega} = 1$, this implies that the distance function is a minimizer of the quotient (6.1), and we are done. $\square$

We next turn to the question of the limit equation in the case $\lambda = \lambda_1(p)$. Suppose that $u_p \in W_0^{1,p}(\Omega)$ is the unique nonnegative solution of (6.4) satisfying $\|u_p\|_{p,\Omega} = 1$. By the Harnack inequality, see [15, 3.51], u_p is in fact positive in Ω, and we may rewrite (6.4) as

$$-\Delta_p u = \lambda_1(p) u^{p-1}. \tag{6.6}$$

Using u_p as a test function and applying Hölder's inequality, we get

$$\|\nabla u_p\|_{m,\Omega} \le |\Omega|^{1/m-1/p} \lambda_1(p)^{1/p} \tag{6.7}$$

for $p \ge m$ and $n < m < \infty$. This estimate, together with Morrey's lemma, implies that we can extract a subsequence (u_{p_j}) converging uniformly to a function $u_\infty \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$, $\|u_\infty\|_{\infty,\Omega}=1$, see the proof of Proposition 2.5. Recall now that in Theorem 3.11 we showed that any limit function of the family of solutions to the problem $-\Delta_p v(x) = \frac{1}{p} f(x)^{p-1}$ satisfies

$$\min\big\{|\nabla v(x)| - f(x), -\Delta_\infty v(x)\big\} = 0$$

in the viscosity sense. By slightly modifying the proof of this result, see Remark 3.20, and by using Lemma 6.5 above, we are able to identify the limit equation of (6.6).

6.8. Theorem. *The function $u_\infty = \lim_{j\to\infty} u_{p_j}$ is a viscosity solution of*

$$\min\big\{|\nabla u(x)| - \frac{1}{d} u(x),\ -\Delta_\infty u(x)\big\} = 0 \tag{6.9}$$

in Ω.

Note that since a solution u is a supersolution of the ∞-Laplacian, the minimum principle, Corollary 5.9, implies that either $u \equiv 0$ or $u > 0$ in Ω. It is also easy to see that if u is a viscosity solution of (6.9), then $-u$ satisfies

$$\max\big\{ - \frac{1}{d} v(x) - |\nabla v(x)|, -\Delta_\infty v(x)\big\} = 0.$$

If we accept this dichotomy in the limit equation, it becomes evident that the problem has a feature typical of classical eigenvalue problems: the set of solutions is closed under multiplication by a constant. Moreover, it is shown in [18] that $\lambda = \frac{1}{d}$ is the only value for which the equation

$$\min\big\{|\nabla u(x)| - \lambda u(x),\ -\Delta_\infty u(x)\big\} = 0$$

has a nontrivial solution with zero boundary values. This result should be compared with the fact that, for a finite p, the first eigenfunctions are the only eigenfunctions that do not change their sign in Ω. Another observation is that any viscosity solution of (6.9) is a minimizer of the quotient (6.1). This can be deduced from the estimate (6.7), which in the limit yields

$$\|\nabla u\|_{\infty,\Omega} \leq \frac{1}{d}\|u\|_{\infty,\Omega}.$$

In the remaining part of this section, we look at our problem from a slightly different perspective and construct a maximal solution to (6.9) by using a variant of the so-called supersolution method. This iterative way of getting a solution does not employ the interpretation of our problem as a limit problem, and it also gives reasonably good upper and lower bounds for the maximal solution.

The following simple but interesting lemma gives a starting point for our construction.

6.10. Lemma. *The distance function $\delta(x) = dist(x, \partial\Omega)$ is the unique viscosity solution of the Dirichlet problem*

$$\begin{cases} \min\left\{|\nabla u(x)| - 1, -\Delta_\infty u(x)\right\} = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{6.11}$$

Proof. Let u_p be the unique solution to the problem $-\Delta_p u(x) = 1$ with the boundary condition $u_p \in W_0^{1,p}(\Omega)$. Then by the results in [4], [19], $u_p \to \delta$ uniformly in Ω as $p \to \infty$. On the other hand, Theorems 3.11 and 4.18 together with Remark 3.20 imply that the limit of u_p's is the unique viscosity solution of (6.11), and the lemma is thereby proved. □

Note that if u is a solution of (6.9), normalized so that $\|u\|_{\infty,\Omega} = d$, then it is a subsolution of (6.11). The comparison principle then implies that $u(x) \leq \delta(x)$ in Ω, and thus the distance function $\delta(x)$ is an upper bound for all properly normalized solutions. Now let $\delta_1(x)$ be the unique solution of

$$\min\left\{|\nabla v(x)| - \frac{1}{d}\delta(x), -\Delta_\infty v(x)\right\} = 0$$

with $\delta_1(x) = 0$ on $\partial\Omega$. Since $\frac{1}{d}\delta(x) \leq 1$, we see by the comparison principle that $\delta_1 \leq \delta$ in Ω. If we further define $\delta_k(x)$, $k = 2, 3, \ldots$, inductively to be the unique solution of

$$\min\left\{|\nabla v(x)| - \frac{1}{d}\delta_{k-1}(x), -\Delta_\infty v(x)\right\} = 0$$

with $\delta_k = 0$ on $\partial\Omega$, we get a decreasing sequence

$$\delta(x) \geq \delta_1(x) \geq \delta_2(x) \geq \ldots$$

of positive functions converging to a nonnegative function, say, $\delta_\infty(x)$. Note that since each δ_k is an upper bound for all possible solutions, the same holds for δ_∞. We will show that this limit function δ_∞ is the maximal solution we are looking for. The first step in proving this is the following lemma.

6.12. Lemma. *The function $\delta_\infty(x)$ constructed above is a solution of equation* (6.9).

Proof. We first remark that the convergence $\delta_k \to \delta_\infty$ is in fact uniform. This can be seen by using Ascoli's theorem; since $0 \le \delta_k(x) \le d$ for every $x \in \Omega$ and every $k = 1, 2, \dots$, the sequence is uniformly bounded and hence by the estimates above equicontinuous. Now fix a point $x_0 \in \Omega$ and a test function $\phi \in C^2(\Omega)$ satisfying $\delta_\infty(x_0) = \phi(x_0)$ and $\delta_\infty(x) > \phi(x)$ for all $x \ne x_0$. Arguing as in the proof of Theorem 3.4, we find by the uniform convergence a sequence $x_k \to x_0$ such that $\delta_k - \phi$ has a local minimum at x_k. This implies that

$$\min\left\{ |\nabla\phi(x_k)| - \frac{1}{d}\delta_{k-1}(x_k), -\Delta_\infty\phi(x_k) \right\} \ge 0$$

for every k, and we obtain by letting $k \to \infty$ that δ_∞ is a supersolution of (6.9). Since the proof for being a subsolution is similar, we are now done. □

Although it already follows from Theorem 6.8 that $\delta_\infty \ne 0$, we will give another proof for this fact by constructing a nontrivial lower bound for the maximal solution. For that we need to introduce some notations. The set

$$\begin{aligned}\mathcal{R} &= \{x \in \Omega \colon \delta(x) \text{ is not differentiable at } x\} \\ &= \{x \in \Omega \colon \exists\, x_1, x_2 \in \partial\Omega,\ x_1 \ne x_2, \text{ such that } |x - x_1| = |x - x_2| = \delta(x)\}\end{aligned}$$

is called the *ridge set* of Ω, see [4], [12], and its subset

$$\mathcal{M} = \{x \in \Omega \colon \delta(x) = d\}$$

will be referred to as the *set of maximal distance.* We will also use the notation

$$\Omega_0 = \Omega \setminus \mathcal{M}$$

in what follows. Note that $\mathcal{M}$ does not have any interior points, and hence $\partial\Omega_0 = \partial\Omega \cup \mathcal{M}$. Let $\vartheta \in C(\partial\Omega_0)$ be defined by

$$\vartheta(x) = \begin{cases} 1 & , x \in \mathcal{M}, \\ 0 & , x \in \partial\Omega, \end{cases}$$

and let $v \in C(\Omega_0)$ be the unique viscosity solution of the ∞-Laplacian in Ω_0 with boundary values ϑ. We define

$$\gamma(x) = \begin{cases} 1 & , x \in \mathcal{M}, \\ v(x) & , x \in \Omega_0. \end{cases}$$

The function $\gamma(x)$ can be regarded as the "∞-capacitary function of the set $\mathcal{M}$". We next show that $w(x) = d\gamma(x)$ is a lower bound for $\delta_\infty(x)$. Note that since $w(x) = \delta(x)$ on $\partial\Omega_0$ and $\delta(x)$ is a viscosity supersolution of the ∞-Laplacian in Ω, we already have that $w(x) \leq \delta(x)$ in Ω. The following lemma is a key point of the rest of the proof.

6.13. Lemma. *The function $w(x)$ is a viscosity solution of*

$$\min\left\{|\nabla u(x)| - \chi_{\mathcal{M}}(x), -\Delta_\infty u(x)\right\} = 0, \tag{6.14}$$

where $\chi_{\mathcal{M}}(x)$ is the characteristic function of the set $\mathcal{M}$.

Proof. We first prove that w is a subsolution. For that, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ be such that $w(x_0) = \phi(x_0)$ and $w(x) \leq \phi(x)$ for all $x \in \Omega$. Since w is a solution of the ∞-Laplacian in Ω_0, we may assume that $x_0 \in \mathcal{M}$. Suppose that $|\nabla\phi(x_0)| = 1 + \varepsilon > \chi_{\mathcal{M}}(x_0)$ for some $\varepsilon > 0$ and denote

$$e = \frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|}.$$

Since $\gamma(x)$ is a also minimal Lipschitz extension of ϑ into Ω_0, we know that $\|\nabla w\|_{\infty,\Omega} \leq 1$. This implies that

$$w(x_0 + te) \geq w(x_0) - |t|,$$

and by Taylor's theorem, we have also that

$$\begin{aligned}\phi(x_0 + te) =& \phi(x_0) + \nabla\phi(x_0) \cdot te + o(t) \\ =& \phi(x_0) + t(1 + \varepsilon) + o(t).\end{aligned}$$

By choosing t to be negative and sufficiently close to zero, we then obtain $w(x_0{+}te) > \phi(x_0{+}te)$, which contradicts our assumptions. Hence we can conclude that $|\nabla\phi(x_0)| - \chi_{\mathcal{M}}(x_0) \leq 0$, and, consequently, that w is a subsolution.

To prove that w is a supersolution, we fix $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ such that $\psi(x_0) = w(x_0)$ and $\psi(x) \leq w(x)$ for all $x \in \Omega$. Again we may assume, without loss of generality, that $x_0 \in \mathcal{M}$. Since

$$\psi(x_0) = w(x_0) \geq w(x) \geq \psi(x)$$

for all $x \in \Omega$, we must have $\nabla\psi(x_0) = 0$. This implies that

$$\psi(x) = \psi(x_0) + o(|x - x_0|). \tag{6.15}$$

By the definition of the set $\mathcal{M}$, there is $y_0 \in \partial\Omega$ such that $|x_0 - y_0| = d = w(x_0)$. Since $\|\nabla w\|_{\infty,\Omega} \leq 1$, we obtain

$$w(tx_0 + (1 - t)y_0) = td$$

for all $0 < t \leq 1$. But this combined with (6.15) contradicts the assumption $\psi \leq w$, and shows that there cannot exist such a function ψ. Thus the claim is trivially true at the point x_0, and we are done. □

6.16. Remark. The arguments used in the proof of the above lemma show that $u_c(x) = c\gamma(x)$ is a solution of (6.14) in Ω for any $0 < c \leq d$. This illustrates the fact that one cannot extend the uniqueness result of Theorem 4.18 to the case of an arbitrary constraint function.

We now proceed as in the case of the upper bound and define w_1 to be the unique solution of

$$\min\left\{|\nabla u(x)| - \gamma(x), -\Delta_\infty u(x)\right\} = 0$$

with $w_1 = 0$ on $\partial\Omega$. The lemma above shows that w is a subsolution of this equation, and therefore $w(x) \leq w_1(x)$ in Ω. Further, since $\gamma(x) \leq \frac{1}{d}\delta(x)$, the comparison principle implies that $w_1 \leq \delta_1$ in Ω. We continue inductively, denoting by w_k, $k = 2, 3, \dots$, the unique solution of

$$\min\left\{|\nabla u(x)| - \frac{1}{d}w_{k-1}(x), -\Delta_\infty u(x)\right\} = 0$$

with $w_k = 0$ on $\partial\Omega$, and obtain an increasing sequence

$$w(x) \leq w_1(x) \leq w_2(x) \leq \cdots \leq \delta(x)$$

converging uniformly to some function w_∞. Repeating the argument above at each step, we see that $w_k \leq \delta_k$ in Ω for every $k = 1, 2, \dots$, and thus $w_\infty \leq \delta_\infty$. Furthermore, arguing as in the proof of Lemma 6.12, it is easy to show that w_∞ is a solution to the ∞-eigenvalue problem. We have now shown not only that $\delta_\infty \neq 0$, but also that $\delta_\infty(x) = d$ in the set $\mathcal{M}$. We collect our results in the theorem below.

6.17. Theorem. *For the ∞-eigenvalue problem (6.9) there exists a maximal solution $\delta_\infty \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$ satisfying*

$$d\gamma(x) \leq \delta_\infty(x) \leq \delta(x)$$

for all $x \in \Omega$. In particular, $\delta_\infty(x) = d$ if $x \in \mathcal{M}$.

It is natural to ask: when do we have $\delta_\infty = \delta$ or $\delta_\infty = w$ in the whole domain Ω? Although very little is known so far, we do have the following partial result.

6.18. Lemma. *If $\mathcal{M} = \mathcal{R}$, then $\delta(x) = \delta_\infty(x) = d\gamma(x)$ in Ω.*

Proof. By Theorem 6.17, it is enough to show that $d\gamma = \delta$ in Ω_0. Since $\mathcal{R} \cap \Omega_0 = \emptyset$, $\delta \in C^1(\Omega_0)$ and $|\nabla\delta(x)| = 1$ for all $x \in \Omega_0$. By a result of Aronsson, see [1, Thm 5], [4], this implies that $\delta(x)$ is ∞-harmonic in Ω_0, and the claim then follows from the uniqueness of ∞-harmonic functions. □

It is shown in [18] that if Ω is a square, then $\delta(x)$ is not a solution of (6.9), and, consequently, $\delta \neq \delta_\infty$. The proof of this result in fact suggests that $\delta(x)$ is a solution of (6.9) if and only if $\mathcal{M} = \mathcal{R}$.

In the last theorem of this section we list some properties of an arbitrary solution of (6.9). The non-differentiability result, part (c), should be compared with the fact that for a finite p an eigenfunction u_p is known to be in $C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $0 < \alpha < 1$.

6.19. Theorem. *Let $u \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$ be a viscosity solution of (6.9) satisfying $\|u\|_{\infty,\Omega} = d$. Then*

(a) *dist$(\{u > \lambda\}, \partial\Omega) = \lambda$ for all $0 \le \lambda \le d$.*

(b) $e^{-\frac{\text{diam}(\Omega)}{\delta(x)}}\delta(x) \le u(x) \le \delta(x)$ *for all $x \in \Omega$.*

(c) *There exists a point $x_0 \in \Omega$ such that u is not differentiable at x_0.*

Proof. Since $u(x) \le \delta(x)$ for all $x \in \Omega$ and $\|u\|_{\infty,\Omega} = d$, there exists $x_0 \in \mathcal{M}$ satisfying $u(x_0) = d$. Let $y_0 \in \partial\Omega$ be such that $|x_0 - y_0| = d$. Because $\|\nabla u\|_{\infty,\Omega} = 1$, we must have $u(tx_0 + (1-t)y_0) = td$ for all $0 < t \le 1$. This implies that $\text{dist}(\{u > \lambda\}, \partial\Omega) \le \lambda$ for every $0 \le \lambda \le d$. On the other hand,

$$\text{dist}(\{u > \lambda\}, \partial\Omega) \ge \text{dist}(\{\delta > \lambda\}, \partial\Omega) = \lambda,$$

and thus we obtain (a). To show (c), it suffices to notice that u is not differentiable at x_0. This can be easily seen by using the facts that x_0 is a maximum point of u and that $u \le \delta$ in Ω.

Thus it remains to prove (b). Arguing as in the proof of the Harnack inequality, Theorem 2.7, we get

$$\|\zeta\nabla\log u\|_{\infty,\Omega} \le \|\nabla\zeta\|_{\infty,\Omega}$$

for all nonnegative $\zeta \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$. Choosing

$$\zeta_\lambda(x) = \min\{1, \frac{1}{\lambda}\delta(x)\},$$

we obtain

$$\|\nabla\log u\|_{\infty,\Omega_\lambda} \le \frac{1}{\lambda},$$

where $\Omega_\lambda = \{x \in \Omega \colon \delta(x) > \lambda\}$. This implies

$$u(x) \le e^{\frac{1}{\lambda}|x-y|}u(y)$$

for all $x, y \in \Omega_\lambda$. Now fix a point $\hat{x} \in \Omega$ and let $\lambda = \delta(\hat{x})$. Then

$$u(\hat{x}) \ge \min_{\partial\Omega_\lambda} u \ge e^{-\frac{\mathrm{diam}(\Omega)}{\lambda}} \max_{\partial\Omega_\lambda} u = e^{-\frac{\mathrm{diam}(\Omega)}{\delta(\hat{x})}}\delta(\hat{x}),$$

and we are done. □

The main open problem in this area is the simplicity of the first eigenvalue $\frac{1}{d}$. In [18], some results pointing to this direction have been obtained, but the actual theorem is still to be proved. Another interesting question is the connection between the eigenfunctions and the quotient (6.1); is there some kind of absolutely minimizing -property that characterizes the eigenfunctions? It is easy to see that an obvious generalization of the definition in Section 2 does not work.

References

[1] Aronsson, G.: Extension of functions satisfying Lipschitz conditions. - Ark. Mat. 6, 1967, 551–561.

[2] Aronsson, G.: On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. - Ark. Mat. 7, 1968, 395–425.

[3] Aronsson, G.: On certain singular solutions of the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. - Manuscripta Math. 47, 1984, 133–151.

[4] Bhattacharya, T., E. DiBenedetto, and J. Manfredi: Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems. - Rendiconti del Sem. Mat. Fascicolo Speciale Non Linear PDE's, Univ. di Torino, 1989, 15–68.

[5] Caffarelli, L. and X. Cabré: Fully nonlinear elliptic equations. - AMS Colloquium publications, 1995.

[6] Caselles, V., J.-M. Morel, and C. Sbert: An axiomatic approach to image interpolation. - preprint, 1996.

[7] Crandall, M. G.: Viscosity solutions: a primer. - Viscosity solutions and applications (Montecatini Terme, 1995), Lecture Notes in Math., 1660, Springer, Berlin, 1997, 1–43.

[8] Crandall, M. G., H. Ishii, and P.-L. Lions: User's guide to viscosity solutions of second order partial differential equations. - Bull. Amer. Math. Soc. 27, 1992, 1–67.

[9] Crandall, M. G., and P.-L. Lions: Viscosity solutions of Hamilton-Jacobi equations. - Trans. Amer. Math. Soc. 277, 1983, 1–42.

[10] Evans, L. C.: Estimates for smooth absolutely minimizing Lipschitz extensions. - Electron. J. Differential Equations 1993, No. 3, 1993, 1–9.

[11] Evans, L. C., and R. Gariepy: Measure theory and fine properties of functions. - CRC Press, Boca Raton, Florida, 1992.

[12] Evans, W. D., and D. J. Harris: Sobolev embeddings for generalized ridged domains. - Proc. London Math. Soc. (3) 54, 1987, 141–175.

[13] Friedman, A.,: Foundations of modern analysis. - Dover Publications, Inc., New York, 1982.

[14] Gilbarg, D., and N. S. Trudinger: Elliptic partial differential equations of second order (2nd edn.). - Springer-Verlag, Berlin, 1983.

[15] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear potential theory of degenerate elliptic equations. - Oxford University Press, Oxford, 1993.

[16] Ishii, H.: Viscosity solutions of nonlinear partial differential equations. - Sugaku Expositions 9 No. 2, 1996, 135–152.

[17] Jensen, R.: Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. - Arch. Rational Mech. Anal. 123, 1993, 51-74.

[18] Juutinen, P., P. Lindqvist, and J. Manfredi: The ∞-eigenvalue problem. - Arch. Rational Mech. Anal. (to appear).

[19] Kawohl, B.: On a family of torsional creep problems. - J. Reine Angew. Math. 410, 1990, 1–22.

[20] Lindqvist, P.: On the equation $\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + \lambda|u|^{p-2}u = 0$. - Proc. Amer. Math. Soc. 109 No. 1, 1990, 157–164.

[21] Lindqvist, P., and J. Manfredi: The Harnack inequality for ∞- harmonic functions. - Electron. J. Differential Equations 1995 No. 4, 1995, 1–5.

[22] Lindqvist, P., and J. Manfredi: Note on ∞- harmonic functions. - Rev. Mat. Univ. Complut. Madrid 10, 1997, 471–480.

[23] Wu, Y.: Absolute minimizers in Finsler metrics. - Ph.D. dissertation, UC Berkeley, 1995.

[24] Ziemer, W. P.: Weakly differentiable functions. - Graduate Texts in Mathematics 120. Springer-Verlag, New York, 1989.

University of Jyväskylä,
Department of Mathematics,
P.O. Box 35,
FIN-40351 Jyväskylä,
Finland